ON EXTREMAL SETS WITHOUT COPRIMES

Rudolf Ahlswede and Levon H. Khachatrian

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

Institute of Problems
of Information and Automation
Armenian Academy of Sciences
Erevan–44 P. Sevak str. 1
Armenian
Visiting SFB 343

1. Definitions, formulation of problems and conjectures

We use the following notations:

$\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of all primes. We set

$$Q_k = \prod_{i=1}^{k} p_i. \tag{1.1}$$

For two numbers $u, v \in \mathbb{N}$ we write $(u, v) = 1$, if u and v are coprimes.

We are particularly interested in the sets

$$\mathbb{N}_s = \big\{u \in \mathbb{N} : (u, Q_{s-1}) = 1\big\} \tag{1.2}$$

and

$$\mathbb{N}_s(n) = \mathbb{N}_s \cap [1, n], \tag{1.3}$$

where for $i \leq j$ $[i, j]$ equals $\{i, i+1, \dots, j\}$.

Erdös introduced in [1] (and also in [2], [4], [5]) $f(n, k, s)$ as the largest integer r for which an

$$A_n \subset \mathbb{N}_s(n), \ |A_n| = r \tag{1.4}$$

exists with no $k+1$ numbers of A_n being coprimes.

Certainly the set

$$\mathbb{E}(n, k, s) = \big\{u \in \mathbb{N}_s(n) : u = p_{s+i}v \text{ for some } i = 0, 1, \dots, k-1\big\} \tag{1.5}$$

has no $k+1$ coprimes.

The case $s = 1$, in which we have $\mathbb{N}_1(n) = [1, n]$, is of particular interest.

Conjecture 1:

$$f(n, k, 1) = |\mathbb{E}(n, k, 1)| \quad \text{for all} \quad n, k \in \mathbb{N}.$$

It seems that this conjecture of Erdös appeared for the first time in print in his paper [1] of 1962.

The papers [2] and [3] by Erdös, Sárközy and Szemerdi and the recent paper [7] by Erdös and Sarkozy are centered around this problem. Whereas it is easy to show that the conjecture is true for $k = 1$ and $k = 2$, it was proved for $k = 3$ by Szabo and Toth [6] only in 1985. Conjecture 1 can also be found in Section 3 of the survey [4] of 1973. In the survey [5] of 1980 one finds the

General Conjecture:

$$f(n,k,s) = |\mathbb{E}(n,k,s)| \quad \text{for all} \quad n,k,s \in \mathbb{N}.$$

Erdös mentions in [5] that he did not succeed in settling the case $k=1$. We focus on this special case by calling it

Conjecture 2:

$$f(n,1,s) = |\mathbb{E}(n,1,s)| \quad \text{for all} \quad n,s \in \mathbb{N}.$$

Notice that

$$\mathbb{E}(n,1,s) = \big\{u \in \mathbb{N}_1(n) : p_s|u;\ p_1,\dots,p_{s-1} \nmid u\big\}.$$

We shall also study these extremal problems for the square free natural numbers $\mathbb{N}^*$. Thus we are naturally led to the sets $\mathbb{N}_s^* = \mathbb{N}_s \cap \mathbb{N}^*$, $\mathbb{N}_s^*(n) = \mathbb{N}_s(n) \cap \mathbb{N}^*$, $\mathbb{E}^*(n,k,s) = \mathbb{E}(n,k,s) \cap \mathbb{N}^*$ etc. and to the function $f^*(n,k,s)$.

Remark 1:

Our interest in the conjectures stated above is motivated by an attempt to search for new combinatorial principles in this number theoretic environment. Consequently we are looking for statements, which don't depend on the actual distribution of primes. Especially Theorem 3 below has this flavour.

In another paper we shall make a systematic study of combinatorial extremal theory for lattices which are abstractions of lattices such as $\mathbb{N}_s^*(n)$, $\mathbb{N}^*$ etc.

2. Results

Theorem 1. *For all* $s,n \in \mathbb{N}$

$$f^*(n,1,s) = |\mathbb{E}^*(n,1,s)|.$$

Theorem 2.

For every $s \in \mathbb{N}$ *and* $n \geq \frac{Q_{s+1}}{p_{s+1}-p_s}$

$$f(n,1,s) = |\mathbb{E}(n,1,s)|$$

and the optimal configuration is unique.

Example 1: (Conjecture 1 is false)

The claim is verified in Section 5. There we prove first the following result.

Proposition 1: For any $t \in \mathbb{N}$ with the properties

(H) $p_{t+7}p_{t+8} < p_t \cdot p_{t+9}$, $p_{t+9} < p_t^2$

and every n in the half–open intervall $I_n = [p_{t+7} \cdot p_{t+8}, p_t \cdot p_{t+9})$ we have for $k = t+3$

$$f(n,k,1) > |\mathbb{E}(n,k,1)|.$$

Then we show that (H) holds for $t = 209$.

We think that by known methods ([13], [14]) one can show that actually (H) holds for infinitely many t , and that there are counterexamples for arbitrarily large k .

Remark 2:

Erdös (oral communication) conjectures now that for every $k \in \mathbb{N}$ $f(n,k,1) \neq |\mathbb{E}(n,k,1)|$ occurs only for finitely many n .

Example 2: Even for squarefree numbers "Erdös sets" are not always optimal, that is, $f^*(n,k,1) \neq |\mathbb{E}^*(n,k,1)|$ can occur. We verify in Section 5 that the set $\mathbb{N}^* \cap A_n(t+3)$ (defined in (5.1)) is an example.

Example 3: In the light of the facts that $f(n,k,1) = |\mathbb{E}(n,k,1)|$ holds for $k = 1,2,3$ for all n and that $f^*(n,1,s) = |\mathbb{E}^*(n,1,s)|$ for all s , it is perhaps surprising that we can have

$$f^*(n,2,s) \neq |\mathbb{E}^*(n,2,s)|.$$

We show this in Section 5 for $P_s = 101$ and $n \in [109 \cdot 113, 101 \cdot 127)$.

Finally, we generalize Theorem 2 by considering instead of $\mathbb{N}_s$ the set $\mathbb{N}_{\mathbb{P}'}$, that is the set of those natural numbers, which don't have any prime of the finite set of primes $\mathbb{P}'$ in their prime number decomposition. We put $\mathbb{N}_{\mathbb{P}'}(n) = \mathbb{N}_{\mathbb{P}'} \cap [1,n]$ and consider sets $A \subset \mathbb{N}_{\mathbb{P}'}(n)$ of non–coprimes. We are again interested in cardinalities and therefore introduce

$$f(n,1,\mathbb{P}') = \max\big\{|A| : A \subset \mathbb{N}_{\mathbb{P}'}(n) \text{ has no coprimes } \big\}.$$

In analogy to the set $\mathbb{E}(n,1,s)$ in the case $\mathbb{P}' = \{p_1,\dots,p_{s-1}\}$ we introduce now $\mathbb{E}(n,1,\mathbb{P}') = \big\{u \in \mathbb{N}_{\mathbb{P}'}(n) : q_1 \mid u\big\}$, where $\{q_1,q_2,\dots\} = \{p_1,p_2,\dots\} \smallsetminus \mathbb{P}'$ and $q_1 < q_2 < \dots$ and $Q_{\mathbb{P}'} = \prod_{p \in \mathbb{P}'} p$.

Theorem 3. *For any finite set of primes* $\mathbb{P}'$ *we have for* $n \geq \frac{q_1 \cdot q_2}{q_2 - q_1} Q_{\mathbb{P}'}$

$$f(n,1,\mathbb{P}') = |\mathbb{E}(n,1,\mathbb{P}')|.$$

3. Proof of Theorem 1

Let $\widetilde{A} \subset \mathbb{N}_s^*(n)$ be without coprimes. Every $a \in \widetilde{A}$ has a presentation

$$a = \prod_{t=s}^{n} p_t^{\alpha_t} \quad \text{with} \quad \alpha_t \in \{0,1\}. \tag{3.1}$$

We can identify a with $\alpha = (\alpha_s, \dots, \alpha_n)$ and thus $\widetilde{A}$ with A . For $\widetilde{A}$ to have no coprimes means that for any $\alpha, \alpha' \in A$

$$\alpha \wedge \alpha' \neq (o, \dots, o) = \underline{o}\ , \quad \text{say.} \tag{3.2}$$

Now we write

$$A = A_1 \,\dot{\cup}\, A_0, \tag{3.3}$$

where

$$A_\varepsilon = \big\{\alpha = (\alpha_s, \dots, \alpha_n) \in A : \alpha_s = \varepsilon\big\} \quad \text{for} \quad \varepsilon = 0, 1, \tag{3.4}$$

and make three observations:

(a) The set $B_1 = \big\{\beta_1 = (1, 0, \dots, 0) \vee \beta : \beta = \alpha \setminus \alpha' \in A_0 \setminus A_0\big\}$, where $A_0 \setminus A_0 = \{\alpha \setminus \alpha' : \alpha, \alpha' \in A_0\}$, is *disjoint* from A_1 , because otherwise $\beta_1 \wedge \alpha' = \underline{o}$ in contradiction to (3.2).

(b) $\widetilde{B}_1 \subset \mathbb{N}_s^*(n)$, because

$$\prod_{t=s}^{n} p_t^{\beta_{1t}} = \prod_{t=s+1}^{n} p_s\ p_t^{(\alpha \setminus \alpha')_t} < \prod_{t=s}^{n} p_t^{\alpha_t} = \alpha$$

by (3.2).

(c) By an inequality of Marica/Schönheim [8], which is (as explained in [11], [12]) a very special case of the Ahlswede/Daykin inequality [9],

$$|B_1| = |A_0 \setminus A_0| \geq |A_0|. \tag{3.5}$$

By these observations the set $\widetilde{C}_1 = \widetilde{A}_1 \,\dot{\cup}\, \widetilde{B}_1$ is contained in $\mathbb{N}_s^*(n)$, contains no coprimes, and has a cardinality $|\widetilde{C}_1| = |\widetilde{A}_1| + |\widetilde{B}_1| \geq |\widetilde{A}_1| + |\widetilde{A}_0| = |\widetilde{A}|$.
This shows that $f^*(n, 1, s) \leq |\mathbb{E}^*(n, 1, s)|$ and the reverse inequality is obvious.

4. Proof of Theorem 2

We need auxiliary results. A key tool are the congruence classes of $\mathbb{N}$

$$C(r,s)=\big\{r+\ell\,Q_{s-1}\in\mathbb{N}:\ell\in\mathbb{N}\cup\{0\}\big\}\text{ for }r=1,\dots,Q_{s-1}. \tag{4.1}$$

They partition $\mathbb{N}_s$ into the sets

$$G(r,s)=\mathbb{N}_s\cap C(r,s). \tag{4.2}$$

We can say more.

Lemma 1.

(i) For any $r\in\mathbb{N}_s$

$$C(r,s)\subset\mathbb{N}_s\,,\quad \textit{that is,}\quad G(r,s)=C(r,s).$$

(ii) There exist $r_1,r_2,\dots,r_{R_{s-1}}\in\mathbb{N}_s(Q_{s-1})$ *such that* $R_{s-1}=\prod_{i=1}^{s-1}(p_i-1)$ *and* $\mathbb{N}_s=\bigcup_{i=1}^{R_{s-1}}G(r_i,s)$. *Actually,* $\{r_1,\dots,r_{R_{s-1}}\}=\mathbb{N}_s(Q_{s-1})$.

Proof:

(i) For any $c\in C(r,s)$, $r\in\mathbb{N}_s$, we have for some ℓ $c=r+\ell\,Q_{s-1}$. However, if $c\notin\mathbb{N}_s$, then $(c,Q_{s-1})>1$ and this implies $(r,Q_{s-1})>1$ in contradiction to $r\in\mathbb{N}_s$.

(ii) We consider $\mathbb{N}_s(Q_{s-1})=\mathbb{N}_s\left(\prod_{i=1}^{s-1}p_i\right)$ and observe that for Euler's φ–function

$$|\mathbb{N}_s(Q_{s-1})|=\varphi\left(\prod_{i=1}^{s-1}p_i\right)=\prod_{i=1}^{s-1}(p_i-1)=R_{s-1}.$$

Next we realize that no two elements from $\mathbb{N}_s(Q_{s-1})$ belong to the same class, because they differ by less than Q_{s-1} . Finally, if $u\in\mathbb{N}_s$ and $u>Q_{s-1}$, then $u=r+\ell\,Q_{s-1}$ for some $\ell\in\mathbb{N}$ and $r\in\mathbb{N}_s(Q_{s-1})$. Hence $u\in G(r,s)$.

So, as $r_1,r_2,\dots,r_{R_{s-1}}$ we can take all the elements of $\mathbb{N}_s(Q_{s-1})$ and $G(r_i,s)=\big\{r_i+\ell\,Q_{s-1}:\ell\in\mathbb{N}\cup\{0\}\big\}$.

We need a few definitions. For $A\subset\mathbb{N}_s$ and $1\le n_1<n_2$ set

$$A[n_1,n_2]=A\cap[n_1,n_2] \tag{4.3}$$

and

$$A_j[n_1,n_2]=A[n_1,n_2]\cap G(r_j,s)\quad\text{for}\quad j=1,\dots,R_{s-1}. \tag{4.4}$$

Thus we have $A_j[n_1,n_2]\cap A_{j'}[n_1,n_2]=\varnothing(j\neq j')$ and $A[n_1,n_2]=\bigcup_{j=1}^{R_{s-1}}A_j[n_1,n_2]$.

We also introduce

$$\mathbb{E}_j[n_1,n_2]=\big\{u:u=p_s v,(v,Q_{s-1})=1\big\}\cap[n_1,n_2]\cap G(r_j,s). \tag{4.5}$$

Clearly,

$$\bigcup_{j=1}^{R_{s-1}}\mathbb{E}_j[1,n]=\mathbb{E}(n,s). \tag{4.6}$$

Lemma 2. *Let* m_j *be the smallest and let* M_j *be the largest integer in* $G(r_j,s)\cap[n_1,n_2]$ *. Then for* $A\subset\mathbb{N}_s$ *without coprimes*

(i) $|A_j[n_1,n_2]|\le\left\lceil\frac{|[n_1,n_2]\cap G(r_j,s)|}{p_s}\right\rceil=\left\lceil\frac{(M_j-m_j)Q_{s-1}^{-1}+1}{p_s}\right\rceil$,

(ii) $|\mathbb{E}_j[n_1,n_2]|=\left\lceil\frac{|[n_1,n_2]\cap G(r_j,s)|}{p_s}\right\rceil$ *, if* $p_s|m_j\cdot M_j$ *,*

and

(iii) if both, $p_s|m_j$ *and* $p_s|M_j$ *, hold, then* $|A_j[n_1,n_2]|=|E_j(n_1,n_2)]|$ *exactly if* $A_j[n_1,n_2]=E_j[n_1,n_2]$ *.*

Proof:

(i) Write $m_j=r_j+\ell\,Q_{s-1}$ and $M_j=r_j+L\,Q_{s-1}$. Then clearly

$$M_j=m_j+(L-\ell)Q_{s-1} \tag{4.7}$$

and

$$L-\ell=p_s x+y,\ 0\le y<p_s. \tag{4.8}$$

Also by the definitions of m_j and M_j

$$|[n_1,n_2]\cap G(r_j,s)|=(L-\ell)+1 \tag{4.9}$$

and therefore the equality in (i) holds.

For two elements a_1 and a_2 of $A_j[n_1,n_2]\subset\mathbb{N}_s$ clearly $(a_1,a_2)\ge p_s$ and by definition (4.4) we know that $a_1=r_j+\ell_1\,Q_{s-1}$, $a_2=r_j+\ell_2\,Q_{s-1}$.

Since $(a_1,a_2)\mid(a_1-a_2)$ and $\big((a_1,a_2),Q_{s-1}\big)=1$ we also have that $(a_1,a_2)\mid(\ell_1-\ell_2)$ and hence that

$$|\ell_1-\ell_2|\ge p_s. \tag{4.10}$$

This gives (i) by (4.7) and (4.8).

Actually we can also write

$$|A_j[n_1,n_2]|\le\left\lceil\frac{L-\ell+1}{p_s}\right\rceil=\left\lceil\frac{p_s x+y+1}{p_s}\right\rceil=x+1. \tag{4.11}$$

(ii) As $p_s|m_j$ (or $p_s|M_j$) we have by (4.7) and (4.8)

$$\mathbb{E}_j[n_1,n_2] = \{m_j, m_j + p_s\ Q_{s-1}, \dots, m_j + p_s x\ Q_{s-1}\}$$

(or $\mathbb{E}_j[n_1,n_2] = \big\{m_j + y\ Q_{s-1}, \dots, m_j + (p_s x + y) Q_{s-1}\big\}$
In any case $|\mathbb{E}_j[n_1,n_2]| = x+1$ and we complete the proof with (4.11).

(iii) Since $p_s|m_j$ and $p_s|M_j$ (ii) applies and yields together with (i)

$$|\mathbb{E}_j[n_1,n_2]| = \left\lceil \frac{(M_j - m_j)Q_{s-1}^{-1} + 1}{p_s} \right\rceil = \frac{(M_j - m_j)Q_{s-1}^{-1}}{p_s} + 1.$$

Furthermore we know that

$$A_j[n_1,n_2] = \{a_1, a_1 + \ell_1 Q_{s-1}, a_2 + \ell_2 Q_{s-1}, \dots, a_1 + \ell_{|A_j|-1} Q_{s-1}\},$$

where $a_1 \geq m_1$ and $a_1 + \ell_{|A_j|-1} Q_{s-1} \leq M_j$.

If now $|\mathbb{E}_j[n_1,n_2]| = |A_j[n_1,n_2]|$, then by (4.10) necessarily $\mathbb{E}_j[n_1,n_2] = A_j[n_1,n_2]$.

Proposition 2: For all $s,n \in \mathbb{N}$

$$|\mathbb{E}(n,s)| \geq f(n,s) - R_{s-1}.$$

Proof: Let $A \subset \mathbb{N}_s(n)$ satisfy $|A| = f(n,s)$.
Specify Lemma 2 to the case $[n_1,n_2] = [1,n]$ and recall (4.6). By (i) of the lemma

$$|A_j[1,n]| \leq \left\lceil \frac{|[1,n] \cap G(r_j,s)|}{p_s} \right\rceil$$

and

$$|A| = \sum_{j=1}^{R_{s-1}} |A_j[1,n]| \leq \sum_{j=1}^{R_{s-1}} \left\lceil \frac{|[1,n] \cap G(r_j,s)|}{p_s} \right\rceil . \tag{4.12}$$

On the other hand, since $(p_s, Q_{s-1}) = 1$, for all $r \in \mathbb{N}_s$ and all $\ell \in \mathbb{N}$ one of the following integers $r + \ell\ Q_{s-1}, r + (\ell+1)Q_{s-1}, \dots, r + (\ell + p_s - 1)Q_{s-1}$ is divisible by p_s . Therefore by the definition (4.5)

$$|\mathbb{E}_j[1,n]| \geq \left\lfloor \frac{|[1,n] \cap G(r_j,s)|}{p_s} \right\rfloor \tag{4.13}$$

$$|\mathbb{E}(n,s)| = \sum_{j=1}^{R_{s-1}} |\mathbb{E}_j[1,n]| \geq \sum_{j=1}^{R_{s-1}} \left\lfloor \frac{[1,n] \cap G(r_j,s)}{p_s} \right\rfloor . \tag{4.14}$$

The result follows from (4.12) and (4.14).

Proof of Theorem: We try to show that for large n

$$|A_j[1,n]| \le |\mathbb{E}_j[1,n]| \quad \text{for} \quad j = 1, \dots, R_{s-1}. \tag{4.15}$$

The condition on n arises naturally this way. A is assumed to be optimal, that is, $|A| = f(n,1,s)$. We make here a space saving convention

$$A_j = A_j[1,n], \; E_j = \mathbb{E}_j[1,n]. \tag{4.16}$$

Two cases are distinguished.

Case: $A_j \cap E_j \neq \varnothing$.

Let r be any element of $A_j \cap E_j$. We partition A_j into the sets $A_j^1 = [1,r] \cap A_j$ and $A_j^2 = [r + p_s\, Q_{s-1}, n] \cap A_j$. Truly

$$A_j = A_j^1 \,\dot\cup\, A_j^2, \tag{4.17}$$

because $r + \ell \cdot Q_{s-1} \in A_j$ for $0 < \ell < p_s$ would imply that for some $p_{s'} (s' \ge s)$ $p_{s'} | r$ and $p_{s'} | r + \ell \cdot Q_{s-1}$, which is impossible since $p_{s'} \nmid \ell \cdot Q_{s-1}$.

The same argument applies to E_j . We can thus also write

$$E_j = E_j^1 \cup E_j^2, E_j^1 = [1,r] \cap E_j, E_j^2 = [r + p_s\, Q_{s-1}, n] \cap E_j. \tag{4.18}$$

Since $r \in E_j$, we have $p_s | r$ and $p_s | (r + p_s\, Q_{s-1})$.

Now by Lemma 2

$$|A_j^1| \le |E_j^1| \quad \text{and} \quad |A_j^2| \le |E_j^2|$$

and therefore $|A_j| = |A_j^1| + |A_j^2| \le |E_j^1| + |E_j^2| = |E_j|$.

Case: $A_j \cap E_j = \varnothing$.

This means that no member of A_j has p_s as factor. Write

$$A_j = \{r_j + \ell_1\, Q_{s-1}, r_j + \ell_2\, Q_{s-1}, \dots, r_j + \ell_{|A_j|} Q_{s-1}\}$$

with $0 \le \ell_1 < \ell_2 < \cdots < \ell_{|A_j|}$.

By the assumption on A_j in this case for some $s' \ge s+1$ $p_{s'} | r_j + \ell_k\, Q_{s-1}$ and $p_{s'} | r_j + \ell_{k+1}\, Q_{s-1}$ and hence $p_{s'} | (\ell_{k+1} - \ell_k)$. This implies

$$\ell_{k+1} - \ell_k \ge p_{s+1} \quad \text{for} \quad k = 1, \dots, |A_j| - 1 \tag{4.19}$$

and therefore

$$|A_j| \le \left\lceil \frac{|[1,n] \cap G(r_j,s)|}{p_{s+1}} \right\rceil . \tag{4.20}$$

Now we write $[1,n] \cap G(r_j,s) = \{r_j, r_j + Q_{s-1}, \dots, r_j + (z-1)Q_{s-1}\}$ and conclude from (4.20) that

$$|A_j| \le \left\lceil \frac{z}{p_{s+1}} \right\rceil . \tag{4.21}$$

On the other hand by (4.13) we have

$$|E_j| \ge \left\lfloor \frac{z}{p_s} \right\rfloor .$$

The inequality $\left\lfloor \frac{z}{p_s} \right\rfloor \ge \left\lceil \frac{z}{p_{s+1}} \right\rceil$ would be insured if $\frac{z}{p_s}$ and $\frac{z}{p_{s+1}}$ are separated by an integer. Sufficient for this is

$$\frac{z}{p_s} - \frac{z}{p_{s+1}} \ge 1 \tag{4.22}$$

or (equivalently)

$$z \ge \frac{p_s p_{s+1}}{p_{s+1} - p_s}. \tag{4.23}$$

By the definition of z

$$(z-1)Q_{s-1} < n < z\, Q_{s-1} \tag{4.24}$$

and hence $z > \frac{n}{Q_{s-1}}$. Requiring $n \ge \frac{p_s p_{s+1}}{p_{s+1} - p_s} Q_{s-1}$ guarantees (4.23).

For these n $|E_j| \ge |A_j|$ in both cases and hence $|A| \le |\mathbb{E}(n,1,s)|$.

Finally we show uniqueness. For this we consider $[1,n] \cap G(r_j,s)$, which contains p_s . By (ii) in Lemma 2 one has $|E_j| = \left\lceil \frac{z}{p_s} \right\rceil$. Now, if $A_j \cap E_j = \varnothing$, then $|A_j| \le \left\lceil \frac{z}{p_{s+1}} \right\rceil$ and for $z \ge \frac{p_s\ p_s - 1}{p_{s+1} - p_s}$ one has $|E_j| > |A_j|$.
On the other hand, if $A_j \cap E_j \ne \varnothing$ and if $p_s \in A_j$, then all members of A must have p_s as a factor and so $A \subset \mathbb{E}(n,s)$. We are left with the case $p_s \notin A_j$ and for some $r \ne p_s$ $r \in A_j \cap E_j$. Here we consider partitions $A_j = A_j^1 \cup A_j^2$ and $E_j = E_j^1 \cup E_j^2$, which are described in (4.17) and (4.18). Now by (i) and (ii) in Lemma 2 one has

$$|A_j^1| \le |E_j^1| \text{ and } |A_j^2| \le |E_j^2|.$$

However, since $p_s \notin A_j$, by (iii) in Lemma 2 we have $|A_j^1| < |E_j^1|$. In any case an optimal A has to equal $\mathbb{E}(n,1,s)$.

Remark 3: Actually we proved a more general result. Replacing $[1, n]$ by $[n_1, n_2]$ the maximal cardinality of sets $A \subset \mathbb{N}_s \cap [n_1, n_2]$ without coprimes is assumed by $\mathbb{E}[n_1, n_2]$, if $n_2 - n_1$ is sufficiently large.

5. The examples

We present now the three examples mentioned in Section 2.

1.) We prove first Proposition 1.

The set proposed by Erdös is $\mathbb{E}(n, t+3, 1) = \left\{u \in \mathbb{N}_1(n) : \left(u, \prod_{i=1}^{t+3} p_i\right) > 1\right\}$. As competitor we suggest $A_n(t+3) = B \cup C$, where

$$B = \left\{u \in \mathbb{N}_1(n) : \left(u, \prod_{i=1}^{t-1} p_i\right) > 1\right\}$$

and

$$C = \{p_{t+i} \cdot p_{t+j} : 0 \leq i < j \leq 8\}. \tag{5.1}$$

Notice that by (H) for $n \in I_n$ $C \subset \mathbb{N}_1(n)$, that $B \cap C = \varnothing$, and that $|C| = \binom{9}{2} = 36$.

Therefore we have

$$|A_n(t+3)| = |B| + 36. \tag{5.2}$$

Furthermore, no $k+1 = t+4$ numbers of $A_n(t+3)$ are coprimes, because we can take in B at most $t-1$ and in C at most 4 pairwise relatively prime integers.

For comparison we write $\mathbb{E}(n, t+3, 1)$ in the form $\mathbb{E}(n, t+3, 1) = B \,\dot\cup\, D$, where

$$D = \{p_t, p_{t+1}, p_{t+2}, p_{t+3}\} \cup \{p_t^2, p_{t+1}^2, p_{t+2}^2, p_{t+3}^2\} \cup \{p_{t+i} \cdot p_{t+j} : 0 \leq i \leq 3, 1 \leq j \leq 8, i < j\}.$$

Notice here that by (H) for $n \in I_n$ p_t^3 (and a fortiori $p_{t+1}^3 \dots$) exceeds n and so does $p_t \cdot p_{t+9}$ (and a fortiori $p_{t+1} p_{t+9} \dots$).

Since $|D| = 4 + 4 + 8 + 7 + 6 + 5 = 34$ we conclude with (5.2) that
$|A_n(t+3)| - |\mathbb{E}(n, t+3, 1)| = |B| + 36 - (|B| + 34) = 2 > 0.$

The hypothesis (H) remains to be verified. It is perhaps interesting to know that among the prime numbers less than 5000 there is only one t, which satisfies (H), namely $t = 209$. The relevant primes $p_t, \dots, p_{t+9}$ are

P_{209}	P_{210}	P_{211}	P_{212}	P_{213}	P_{214}	P_{215}	P_{216}	P_{217}	P_{218}
1289	1291	1297	1301	1303	1307	1319	1321	1327	1361

We calculate (in our heads of course) that

$P_{209} \cdot P_{218} = 1289 \cdot 1361 = 1754329 > P_{216} \cdot P_{217} = 1321 \cdot 1327 = 1752967$

and that $P_{209}^2 = 1289^2 > 1361 = P_{218}$.

Hence for $k = 212$ and for all n with $P_{209} \cdot P_{218} = 1754329 > n \geq 1752967 = P_{216} \cdot P_{217}$ one has $f(n,k,1) \geq |\mathbb{E}(n,k,1)| + 2$. Curiously, $P_{209} \cdot P_{218} - P_{216} \cdot P_{217} = 1362 = P_{218} + 1$. Also, if P_{209} where smaller by 2 these 4 primes would not suffice for the construction.

2.) Notice that in the previous notation by (5.1) $C \cap \mathbb{N}^* = C$ and that $|D \cap \mathbb{N}^*| = |D| - 4$. Since $|C| - |D| = 2$, we conclude that

$|A_n(t+3) \cap \mathbb{N}^*| - |\mathbb{E}^*(n, t+3, 1)| = |(B \cap \mathbb{N}^*) \,\dot\cup\, C| - |(B \cap \mathbb{N}^*) \,\dot\cup\, (D \cap \mathbb{N}^*)| = 6 > 0$.

3.) Choose $s = 25$ and consider $p_{25} = 101$, $p_{26} = 103$, $p_{27} = 107$, $p_{28} = 109$, $p_{29} = 113$, $p_{30} = 127$. Verify that $109 \cdot 113 < 101 \cdot 127$ and choose $n \in [109 \cdot 113, 101 \cdot 127)$.

For these parameters

$\mathbb{E}^*(n, 2, 25) = \{101 \cdot m : m \in \mathbb{N}\} \cup \{103 \cdot m : m \in \mathbb{N}\} \cap \left\{ u \in \mathbb{N}_1^*(n) : \left(u, \prod_{i=1}^{24} p_i\right) = 1 \right\}$

$= \{101; 101 \cdot 103, 101 \cdot 107, 101 \cdot 109, 101 \cdot 113\} \cup \{103; 103 \cdot 107, 103 \cdot 109, 103 \cdot 113\}$

and $|\mathbb{E}^*(n, 25)| = 9$.

As competitor we choose

$$A_n^*(2, 25) = \{P_{25+i} \cdot P_{25+j} : 0 \leq i < j \leq 4\}.$$

Its largest element $109 \cdot 113$ does not exceed n and since only 5 primes are involved as factors, no 3 products with 2 factors can be relatively prime. However,

$$|A_n^*(2, 25)| = \binom{5}{2} = 10 > 9.$$

6. Proof of Theorem 3

Let us define now

$$Q_{\mathbb{P}'} = \prod_{p \in \mathbb{P}'} p \tag{6.1}$$

and replace Q_{s-1} by $Q_{\mathbb{P}'}$ in the earlier definitions. Thus we replace

$G(r,s)$ by $G(r,\mathbb{P}') = \{u \in \mathbb{N} : u \equiv r \mod Q_{\mathbb{P}'}\} \cap \mathbb{N}_{\mathbb{P}'}$ in Section 4 and establish the generalizations of Lemmas 1, 2 and also of Theorem 2.

Just keep in mind that $\mathbb{P}'$ takes the role of $\{p_1, \ldots, p_{s-1}\}$, q_1 takes the role of p_s , and q_2 takes the role of p_{s+1} .

Thus the sufficient condition $n \geq \frac{p_s \; p_{s+1}}{p_{s+1} - p_s} Q_{s-1}$ is to be replaced by

$$n \geq \frac{q_1 q_2}{q_2 - q_1} Q_{\mathbb{P}'}. \tag{6.2}$$

References

[1] P. Erdös, "Remarks in number theory, IV (in Hungarian)", Mat. Lapok 13, 228–255, 1962.

[2] P. Erdös, A. Sárközy and E. Szemerédi, "On some extremal properties of sequences of integers", Annales Univ. Sci. Budapest. Eötvös 12, 131–135, 1969.

[3] P. Erdös, A. Sárközy and E. Szemerédi, "On some extremal properties of sequences of integers, II", Publications Math. 27, 117–125, 1980.

[4] P. Erdös, "Problems and results on combinatorial number theory", Chapt. 12 in "A Survey of Combinatorial Theory", J.N. Srivastava et al, eds., North–Holland Publishing Company, 1973.

[5] P. Erdös, "A survey of problems in combinatorial number theory", Annals of Discrete Mathematics 6, 89–115, 1980.

[6] C. Szabo and G. Toth, "Maximal sequences not containing 4 pairwise coprimes integers (in Hungarian)", Mat. Lapok 32, 253–257, 1985.

[7] P. Erdös and A. Sárközy, "On sets of coprime integers in intervals", Preprint No. 9/1992 of the Mathematical Institute of the Hungarian Academy of Sciences.

[8] J. Marica and J. Schönheim, "Differences of sets and a problem of Graham", Canad. Math. Bull. vol. 12, no. 5, 635–637, 1969.

[9] R. Ahlswede and D.E. Daykin, "An inequality for the weights of two families of sets, their unions and intersections", Z. Wahrscheinlichkeitstheorie u. verw. Geb. 43, 183–185, 1978.

[10] R. Ahlswede and D.E. Daykin, "The number of values of combinatorial functions", Bull. London Math. Soc. 11, 49–51, 1979.

[11] R. Ahlswede and D.E. Daykin, "Inequalities for a pair of maps $S \times S \to S$ with S a finite set", Math. Zeitschrift 165, 267–289, 1979.

[12] B. Bollobás, "Combinatorics", Cambridge University Press, 1986.

[13] R.A. Rankin, "The difference between consecutive prime numbers", J. London Math. Soc. 13, 242–247, 1938.

[14] P. Erdös, "On the differences of consecutive primes", Quart. J. Oxford 6, 124–128, 1935.