Note

Counterexample to the Frankl/Pach conjecture for uniform, dense families

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$\mathbb{N}$ denotes the set of positive integers and for $\ell, n \in \mathbb{N}$, $\ell \leq n$ we set

$$2^{[n]} = \{F : F \subset [1,n]\}, \binom{[n]}{\ell} = \{F \in 2^{[n]} : |F| = \ell].$$

A family $\mathcal{F} \subset 2^{[n]}$ is called ℓ–dense , $\ell \in \mathbb{N}$, if there exists an ℓ–element subset $D \in \binom{[n]}{\ell}$ with $\mathcal{F}(D) = \{F \cap D : F \in \mathcal{F}\}$ satisfying

$$|\mathcal{F}(D)| = 2^\ell. \tag{1}$$

A well–known result of Sauer [1], Shelah/Perles [2], and Vapnik/Ĉervonenkis [3] says that any $\mathcal{F} \subset 2^{[n]}$ is ℓ–dense , if

$$|\mathcal{F}| > \sum_{i<\ell} \binom{n}{i}. \tag{2}$$

Frankl/Pach [4] proved that any ℓ–uniform $\mathcal{F}$, that is $\mathcal{F} \subset \binom{[n]}{\ell}$, is ℓ–dense , if

$$|\mathcal{F}| > \binom{n}{\ell-1}, \tag{3}$$

and they *conjectured* that for every ℓ–uniform , but not ℓ–dense , $\mathcal{F}$ with $n > 2\ell$ necessarily

$$|\mathcal{F}| \leq \binom{n-1}{\ell-1}. \tag{4}$$

It was pointed out in [4], [5], and also by Erdös [6] that the truth of this conjecture would mean a sharpening of the Erdös/Ko/Rado Theorem [7]. While (4) holds for $\ell = 2$, it is unfortunately false for $\ell \geq 3$ and $n \geq 2\ell$.

Example: Let $\mathcal{F} = \mathcal{F}_1 \dot\cup \mathcal{F}_2 \dot\cup \mathcal{F}_3 \dot\cup \mathcal{F}_4 \subset \binom{[n]}{\ell}$, where

$$\mathcal{F}_1 = \left\{F \in \binom{[n]}{\ell} : 1 \in F, 2 \notin F\right\}, \mathcal{F}_2 = \left\{F \in \binom{[n]}{\ell} : 1,2,3 \in F\right\},$$

$$\mathcal{F}_3 = \left\{F \in \binom{[n]}{\ell} : 1,2,4 \in F, 3 \notin F\right\}, \mathcal{F}_4 = \left\{F \in \binom{[n]}{\ell} : 1 \notin F, 2,3, \in F\right\}.$$

It remains to be seen that for no $D \in \binom{[n]}{\ell}$ (1) holds.

Obviously, a candidate $D = \{d_1, \dots, d_\ell\}$ must satisfy $D \in \mathcal{F}$.

It is convenient to use also the sequence representation $(f_1, \ldots, f_\ell)$ for $D \cap F$, where

$$f_t = \begin{cases} 1 & \text{if } d_t \in D \cap F \\ 0 & \text{if } d_t \notin D \cap F. \end{cases}$$

Now let $\mathcal{F}_1 = \mathcal{F}_1^1 \dot\cup \mathcal{F}_1^2$, where

$$\mathcal{F}_1^1 = \{F \in \mathcal{F}_1 : 3 \in F\} \text{ and } \mathcal{F}_1^2 = \mathcal{F}_1 \smallsetminus \mathcal{F}_1^1.$$

Notice that $D \notin \mathcal{F}_1^1$, because $(0, 0, \ldots, 0) \notin \mathcal{F}(D)$.

Also, $D \notin \mathcal{F}_1^2$, because $(0, 1, 1, \ldots, 1) \notin \mathcal{F}(D)$. Hence $D \notin \mathcal{F}_1$.

Furthermore, $D \notin \mathcal{F}_2 \cup \mathcal{F}_3$, because $(0, 0, \ldots, 0) \notin \mathcal{F}(D)$.

Finally, let $\mathcal{F}_4 = \mathcal{F}_4^1 \dot\cup \mathcal{F}_4^2$, where $\mathcal{F}_4^1 = \{F \in \mathcal{F}_4 : 4 \in F\}$ and $\mathcal{F}_4^2 = \mathcal{F}_4 \smallsetminus \mathcal{F}_4^1$.

Now we have $D \notin \mathcal{F}_4^1$, because $(1, 0, 0, \ldots, 0) \notin \mathcal{F}(D)$, and $D \notin \mathcal{F}_4^2$, because $(1, 0, 1, 1, \ldots, 1) \notin \mathcal{F}(D)$. Thus $\mathcal{F}$ is not ℓ–dense, however,

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| = \tbinom{n-2}{\ell-1} + \tbinom{n-3}{\ell-3} + \tbinom{n-4}{\ell-3} + \tbinom{n-3}{\ell-2} = \tbinom{n-1}{\ell-1} + \tbinom{n-4}{\ell-3} > \tbinom{n-1}{\ell-1}.$$

Remark: Frankl/Watanabe [5] even conjectured that for every k–uniform, but not ℓ–dense, $\mathcal{F}$ for $k > \ell > 2$ necessarily

$$|\mathcal{F}| \leq \binom{n-k+\ell-1}{\ell-1} \quad \text{for } n > n_0(k). \tag{5}$$

Of course, our example can be used to disprove this for every k, ℓ, n; $n > k > \ell > 2$.

References

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