ASYMPTOTICALLY DENSE NONBINARY CODES CORRECTING A CONSTANT NUMBER OF LOCALIZED ERRORS

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Abstract

The binary case was studied in [1], but the method used there doesn't give the tight answer for nonbinary cases and we presented in [2] another method for the corresponding result. Here we formulate the main theorem and prove the auxiliary statements used in [2].

During the transmission of q–ary words of length n over the channel at most t errors occur, and the encoder knows the set E of t positions, where these errors are possible. The decoder doesn't know anything about these positions. Let $\mathcal{E}_t = \big\{E \mid E \subseteq \{1,2,\ldots,n\}, |E| = t\big\}$ be the set of all subsets from $\{1,2,\ldots,n\}$ of size t and let $\mathcal{M}$ be a set of messages $(|\mathcal{M}| = M)$. A code word $x(m,E)$ depends not only on the message $m \in \mathcal{M}$ but also on the configuration of possible errors E. So there exists the natural correspondence between the message $m \in \mathcal{M}$ and the list of code words $\bigcup_{E\in\mathcal{E}_t}\big\{x(m,E)\big\}$, which we use for the transmission of this message. Thus the code X for the set of messages $\mathcal{M}$ represents a collection of M lists $\big\{\bigcup_{E\in\mathcal{E}_t}\{x(m,E)\}, m \in \mathcal{M}\big\}$. Sinde we can use the same word for different configurations, the size of a list can be essentially smaller than the size of the set $\mathcal{E}_t$ $\big(|\mathcal{E}_t| = \binom{n}{t}\big)$. Let us define the cylinder $C(a,A)$ with the base $a = (a,\ldots,a_n)$ and the support $A\big(A \subseteq \{1,2,\ldots,n\}\big)$ as the set of words $(y_1,\ldots,y_n)$ with $y_i = a_i$, if $i \notin A$. It is clear that the size of the cylinder $C(a,A)$ is equal to $q^{|A|}$ and the number of different cylinders with the same support A is equal to $q^{n-|A|}$.

As a result of the transmission of the codeword $x(m,E)$ every word of $C\big(x(m,E),E\big)$ can appear as output of the channel. The code X corrects t localized errors, if the decoder can correctly recover every message $m \in \mathcal{M}$. The following condition is necessary and sufficient for it:

$$C\big(x(m,E),E\big) \cap C\big(x(m',E'),E'\big) = \varnothing \text{ for all } E,E' \in \mathcal{E}_t, m,m' \in \mathcal{M}, m \neq m'. \quad (1)$$

The maximal number of messages, which we can transmit by a code correcting t localized errors, is denoted by $L_q(n,t)$.

Proposition 1:

$$L_q(n,t) \le \frac{q^n}{S_t}$$

where $S_t = \sum_{i=0}^{t}(q-1)^i C_n^i$ is the size of a sphere of radius t in the Hamming n–space.

A proof of this bound in the q–ary case can be given as for the binary case in [3] or [4]. The key inequality there has the following generalization.

Lemma 1. *Let $C(a_i,A_i),\dots,C(a_T,A_T)$ be cylinders with pairwise different supports $A_i \ne A_j, i \ne j$. Then for the size of the union of the cylinders*

$$\left|\bigcup_{i=1}^{T} C(a_i,A_i)\right| \ge \sum_{i=1}^{T}(q-1)^{|A_i|}.$$

Proof: We proceed by an induction on n. For $n=1$ the statement is obvious. Let $C(a_1,A_1),\dots,C(a_T,A_T)$ satisfy the condition of the Lemma. We consider now the new family $\mathbb{C}$ of cylinders:

a) if $n \notin A_i$, then $C(a_i,A_i) \in \mathbb{C}$

b) if $n \in A_i$, then $C(a_i^{(k)}, A_i \setminus n) \in \mathbb{C}$ for all $k (k=0,1,\dots,q-1)$, where $a_i^{(k)} = (a_{i1},\dots,a_{in-1},k)$.

We have

$$\mathbb{C} = \bigcup_{k=0}^{q-1} \mathbb{C}^{(k)}$$

where $\mathbb{C}^{(k)}$ – all cylinders from $\mathbb{C}$ whose last coordinate is equal to $k (k=0,1,\dots,q-1)$. It is easy to show that

$$\left|\bigcup_{C\in\mathbb{C}} C\right| = \sum_{k=o}^{q-1}\left|\bigcup_{C\in\mathbb{C}^{(k)}} C\right|.$$

If follows from the condition of the Lemma that the support $A_i \setminus n$ of q cylinders $C(a_i^{(k)}, A_i \setminus n), k=0,1,\dots,q-1$ differs from the support of other cylinders at least for $q-1$ subfamilies $\mathbb{C}^k$, $k=0,1,\dots,q-1$. Thus one proves the Lemma using the induction step to estimate $|\bigcup_{C\in\mathbb{C}^{(k)}} C|$. It is easy to obtain Proposition 1 from Lemma 1. In fact for every union of cylinders $\bigcup_{E\in\mathcal{E}_t} C(\cdot,E)$ there exists some union of cylinders $\bigcup_{E\in\bigcup_{i=0}^t \mathcal{E}_i} C(\cdot,E)$ such that

$$\bigcup_{E\in\mathcal{E}_t} C(\cdot,E) = \bigcup_{E\in\bigcup_{i=1}^t \mathcal{E}_i} C(\cdot,E)$$

and therefore by Lemma 1

$$\left|\bigcup_{E\in\mathcal{E}_t} C(\cdot,E)\right| = \left|\bigcup_{E\in\bigcup_{i=1}^t \mathcal{E}_i} C(\cdot,E)\right| \geq S_t. \tag{2}$$

Now we have from the condition (1) that

$$\left(\bigcup_{E\in\mathcal{E}_t} C\big(x(m,E),E\big)\right) \cap \left(\bigcup_{E'\in\mathcal{E}_t} C\big(x(m',E'),E'\big)\right) = \varnothing \text{ for } m\neq m'.$$

From here and (2) the Proposition follows.

The following lower bound can be easily deduced by the standard greedy algorithm (maximal coding).

Proposition 2:

$$L_q(n,t) \geq \frac{q^n}{q^{2t}\binom{n}{t}}.$$

Proof: Let X be the code $X = \left\{\bigcup_{E\in\mathcal{E}_t}\{x(m,E)\},\quad m\in\mathcal{M}\right\}$ for M messages, correcting t localized errors. As

$$\left|\bigcup_{m\in\mathcal{M}}\bigcup_{E\in\mathcal{E}_t} C\big(x(m,E),E\big)\right| \leq M\cdot\binom{n}{t}\cdot q^t$$

and the number of different cylinders with the same support $E'(|E'|=t)$ is equal to q^{n-t}, for any support $E'\in\mathcal{E}_t$ there exists a cylinder $C(a,E')$ with

$$C(a,E')\cap C\big(x(m,E),E\big)=\varnothing \text{ for all } m\in\mathcal{M} \text{ and } E\in\mathcal{E}_t,$$

if

$$M\binom{n}{t}q^t < q^{n-t}. \tag{3}$$

Therefore, if the inequality (3) takes place, it is possible, according to the condition (1), to construct the code for $M+1$ messages, correcting t localized errors. Hence Proposition 2 follows.

Already Proposition 1 and 2 imply the asymptotic equivalence within a constant

$$L_q(n,t)\asymp\frac{q^n}{n^t},\text{ when } t \text{ is fixed and } n\to\infty.$$

We draw attention to the fact that this equivalence is known for nonbinary error–correcting codes, except $t=1$, only for $t=2$ and $q=3,4$ [5].

The following theorem gives the precise constant in the equivalence.

Theorem. *For every constant* t

$$L_q(n,t)=\frac{q^n}{S_t}\big(1+o(1)\big)=\frac{t!}{(q-1)^t}\cdot\frac{q^n}{n^t}\big(1+o(1)\big),$$

where $o(1)\to o$ *as* $n\to\infty$ *(* $o(1)$ *depends certainly on* t *and* q *).*

When proving this theorem [2] we refered to the following recurrence relation, having an independent interest.

Lemma 2. *If* $N \leq (q-1)T+1$ *, then*

$$L_q(N+T,1) \geq q^{N-1} L_q(T,1).$$

Proof: On the first N positions we always transmit a parity check $(\mod q)$ code of the size q^{N-1} . The last T positions we reserve for the code, which allows us to transmit $L_q(T,1)$ messages and to correct a single localized error. The method of transmission depends on the position of the localized error in the following way:

a) The error is on the last T positions. We use our code as a code, correcting a single localized error.

b) The error is in the first N positions. We use our code for the transmission of both, the message and the number of the position, where the error can occur. According to Proposition 1 and Lemma 2 at least $(q-1)T+1$ different words in the output of the channel of length T correspond to everyone of $L_q(T,1)$ messages and therefore we can make the successful transmission, if

$$N \leq (q-1)T+1.$$

The proof is complete.

References

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