Number theoretic correlation inequalities for Dirichlet densities

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Abstract

Our main discovery is the inequality

$$\underline{\mathbf{D}}\,(A,B)\,\underline{\mathbf{D}}[A,B] \;\geq\; \underline{\mathbf{D}}\,A\;\underline{\mathbf{D}}\,B,$$

where A, B are arbitrary sets of positive integers, $(A,B) = \big\{(a,b) : a \in A, b \in B\big\}$ is the set of largest common divisors, $[A,B] = \big\{[a,b] : a \in A, b \in B\big\}$ is the set of least common multiples, and $\underline{\mathbf{D}}$ denotes the lower Dirichlet density. It is much more general than our recent inequality for multiples of sets, which in turn is sharper than Behrend's well–known inequality.

We also extend another recently discovered inequality, which does not seem to have number theoretic predecessors.

1 Introduction

The starting point of our investigations are two new inequalities of [3], which concern asymptotic densities of sets of multiples of certain sets of positive integers. They are readily stated.

For sets $A, B \subset \mathbb{N}$, the set of positive integers, consider the set of least common multiples $[A, B] = \big\{[a, b] : a \in A, b \in B\big\}$, the set of largest common divisors $(A, B) = \big\{(a, b) : a \in A, b \in B\big\}$, the set of products $A \times B = \{a \cdot b : a \in A, b \in B\}$, and the sets of their multiples $M(A) = A \times \mathbb{N}, M(B), M[A, B]$, $M(A, B)$, and $M(A \times B)$, resp.

We use the abbreviations $\langle 1, n\rangle = \{1, 2, \dots, n\}$ and for any set $C \subset \mathbb{N}$ $C_n = C \cap \langle 1, n\rangle$.

The asymptotic density $\mathbf{d}C$ is defined as $\mathbf{d}C = \lim\limits_{n\to\infty} C_n/n$, if the limit exists. The lower (resp. upper) asymptotic densities $\underline{\mathbf{d}}C$ and $\overline{\mathbf{d}}C$ are defined similarly with lim replaced by lim inf (resp. lim sup).

The discoveries in [3] are the inequalities

$$\mathbf{d}M(A, B)\mathbf{d}M[A, B] \geq \mathbf{d}M(A) \cdot \mathbf{d}M(B) \tag{1.1}$$

and

$$\mathbf{d}M(A) \cdot \mathbf{d}M(B) \geq \mathbf{d}M(A \times B), \tag{1.2}$$

where A and B are **finite**.

The first inequality is by the factor $\mathbf{d}M(A, B)$ sharper than Behrend's well–known inequality ([9], [25], [16]). This in turn is a generalisation of an earlier inequality of Rohrbach [27] and Heilbronn [18], which settled a conjecture of Hasse concerning an identity due to Dirichlet (see [27]). A simple proof was given in [28] via a probabilistic inequality, which actually is a special case of the earlier FKG–inequality [15].

The second inequality does not seem to have predecessors in number theory.

We mentioned already in [4] that structural similarities between Behrend's inequality and the AD–inequality (see Section 2) led us to conjecture the first inequality.

Now we go further in three directions. We mention first that these similarities are more than just analogies. In fact, we show in Section 2 that (1.1) is a *consequence* of the AD–inequality. Thus we have a unified view and the AD–inequality now not only includes several correlation inequalities from statistical physics ([15]) resp. probability theory ([17], [19]) and combinatorics ([22], [29], [12], [7]), but also well–known density inequalities ([27], [18], [9]) in number theory.

Secondly, this approach gives more than just another proof of (1.1), because it works for *arbitrary subsets* $C, D \subset \mathbb{N}$ and not just sets of multiples $M(A)$, $M(B)$ with A, B finite (Theorem 1 in Section 2).

The application of the AD–inequality is made to isomorphic images of A_n and B_n in finite lattices of multisets (that is, divisors of an integer). The transition to the (possibly infinite) sets A and B proceeds via Dirichlet series $\mathbf{D}(C,s) = \sum_{n \in C} n^{-s}$ and Dirichlet densities

$$\mathbf{D}C = \lim_{s \to 1^+} (s-1)\mathbf{D}(C,s), \tag{1.3}$$

if they exist. Otherwise we use the lower (resp. upper) Dirichlet densities $\underline{\mathbf{D}}C$ and $\overline{\mathbf{D}}C$, which are defined with lim replaced by lim inf (resp. lim sup).

Finally, we explore number theoretic analoga to other known correlation inequalities. Sets of multiples correspond to upsets in lattices.

Insofar our main finding is, that the inequality in (1.2) corresponds to and can also be derived via Dirichlet series from the van den Berg/Kesten inequality (see Section 3). It plays a role in (and was discovered in the context of) reliability theory (see [8], [24]) and also in percolation theory (see [10], [21]). This inequality holds only for upsets and as striking confirmation of our ideas we note that (1.2) does not extend to arbitrary sets! (Example 3 in Section 3.)

However our approach gives an extension of (1.2) to multiples of infinite sets in terms of Dirichlet densities (Theorem 2 in Section 3). Further perspectives of our ideas are discussed in Section 4. We conclude with relations between density concepts, which are used in the paper.

Dirichlet proved (see [25], page 96) that $\mathbf{D}A$ equals the so called logarithmic density

$$\boldsymbol{\delta} A = \lim_{n \to \infty} \frac{1}{\log n} \sum_{a \in A_n} \frac{1}{a}. \tag{1.4}$$

Inspection of the proof shows that the corresponding lower and upper densities are also equal:

$$\underline{\mathbf{D}}A = \underline{\boldsymbol{\delta}}A \quad \text{and} \quad \overline{\mathbf{D}}A = \overline{\boldsymbol{\delta}}A. \tag{1.5}$$

It is also well–known (see [25]) that

$$0 \leq \underline{\mathbf{d}}A \leq \underline{\mathbf{D}}A \leq \overline{\mathbf{D}}A \leq \overline{\mathbf{d}}A. \tag{1.6}$$

Consequently the existence of $\mathbf{d}A$ implies the existence of $\mathbf{D}A$. (For converse implications see [20], [23], [25]). A famous example of Besicovitch ([11], [16]) shows that infinite sets of multiples need not have an asymptotic density. However, Davenport and Erdös [13] proved that for every $A \subset \mathbb{N}$

$$\boldsymbol{\delta} M(A) = \underline{\mathbf{d}} M(A). \tag{1.7}$$

2 A number theoretic consequence of the AD–inequality

We state first the

AD–inequality.

Let $L \subset \{0,1\}^n$ be a sublattice of $\{0,1\}^n$ and let $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}_+$, then

$$\alpha(a)\beta(b) \leq \gamma(a \vee b)\delta(a \wedge b) \quad \textit{for all} \quad a, b \in L \tag{2.1}$$

implies

$$\alpha(A)\beta(B) \leq \gamma(A \vee B)\delta(A \wedge B) \quad \textit{for all} \quad A, B \in L, \tag{2.2}$$

where

$$\alpha(A) = \sum_{a \in A} \alpha(a) \quad \textit{etc. and} \quad A \vee B = \{a \vee b : a \in A, b \in B\} \quad \textit{etc.}$$

We state next our main result. It is an inequality, which is sharper and considerably more general than all its predecessors, inequalities by Rohrbach, Heilbronn, Behrend, and Ahlswede/Khachatrian (see [27], [18], [9], [4]). It holds for arbitrary sets and not just for sets of multiples of finite sets.

Presently we have no proof without transfinite methods. Whereas $A_n, B_n \subset < 1, n >$ we know in general only that $[A_n, B_n] \subset < 1, n^2 >$, which makes comparisons of densities difficult.

Theorem 1.

For arbitrary $A, B \subset \mathbb{N}$

$$\underline{\mathbf{D}}A \cdot \underline{\mathbf{D}}B \leq \underline{\mathbf{D}}[A, B] \cdot \underline{\mathbf{D}}(A, B).$$

Before we give a proof we present some immediate consequences.

Corollary 1.

(i) *For* **finite** $A, B \subset \mathbb{B}$ $\mathbf{d}M(A)\mathbf{d}M(B) \leq \mathbf{d}M[A, B]\mathbf{d}M(A, B)$ ([4]).

(ii) *For* **arbitrary** $A, B \subset \mathbb{N}$ $\mathbf{D}M(A)\mathbf{D}M(B) \leq \mathbf{D}M[A, B]\mathbf{D}M[A, B]$.

Furthermore, this inequality can be given in equivalent forms by writing $\boldsymbol{\delta}$ *or* $\underline{\mathbf{d}}$ *instead of* $\mathbf{D}$.

Proof: First observe that

$$M[A, B] = \big[M(A), M(B)\big] \text{ and } M(A, B) = \big(M(A), M(B)\big).$$

These identities and application of Theorem 1 to the sets $M(A)$ and $M(B)$ give

$$\underline{\mathbf{D}}M(A)\cdot\underline{\mathbf{D}}M(B)\le\underline{\mathbf{D}}M[A,B]\cdot\underline{\mathbf{D}}M(A,B).$$

Now (i) follows, because multiples of finite sets have asymptotic density and by (1.6) we can replace $\underline{\mathbf{D}}$ by $\mathbf{d}$.

Finally (ii) follows, because we know from (1.7) and (1.5) that for any $C\subset\mathbb{N}$ $\underline{\mathbf{d}}M(C)=\boldsymbol{\delta}M(C)=\mathbf{D}M(C)$.

Proof of Theorem 1:

We consider the Dirichlet series D associated with A, B, $[A,B]$, and (A,B) in the domain $\{s:s\in\mathbb{R},s>1\}$, where they all converge.

It is also clear that for $A_n=A\cap<1,n>$ and $B_n=B\cap<1,n>$

$$\begin{aligned}&\lim_{n\to\infty}D(A_n,s)=D(A,s),\ \lim_{n\to\infty}D(B_n,s)=D(B,s),\\&\lim_{n\to\infty}D\big([A_n,B_n],s\big)=D\big([A,B],s\big),\quad\text{and}\quad\lim_{n\to\infty}D\big((A_n,B_n),s\big)=D\big((A,B),s\big).\end{aligned}\tag{2.3}$$

Using these facts we derive below from the AD–inequality the inequality

$$D(A,s)\cdot D(B,t)\le D\big([A,B],s\big)D\big((A,B),t\big)\quad\text{for}\quad 1<s<t.\tag{2.4}$$

This immediately yields

$$\liminf_{s\to1^+}D(A,s)\cdot D(B,t)\le\liminf_{s\to1^+}D\big([A,B],s\big)\cdot D\big((A,B),t\big)$$

and thus $\underline{\mathbf{D}}A\cdot\underline{\mathbf{D}}B\le\underline{\mathbf{D}}[A,B]\cdot\underline{\mathbf{D}}(A,B)$.

We prove now (2.4).

Let $\{p_1,p_2,\dots,p_m\}$ be the set of all primes in $<1,n>$.

Clearly $A_n\cup B_n\cup[A_n,B_n]\cup(A_n,B_n)\subset L'=\left\{\prod_{i=1}^m p_i^{\pi_i}:0\le\pi_i\le n\right\}$ and $c'\in L'$ has a unique representation

$$c'=\prod_{i=1}^m p_i^{\pi_i(c')}.\tag{2.5}$$

$f:L'\to\{0,1\}^{m\cdot n}$, defined by $f(c')=c=(c_{11},\dots,c_{n1},c_{12},\dots,c_{n2},\dots,c_{1m},\dots,c_{nm})$ with

$$c_{ji}=\begin{cases}1 & \text{if}\quad \pi_i(c')\ge j\\ 0 & \text{if}\quad \pi_i(c')<j,\end{cases}\tag{2.6}$$

embedds the lattice L' isomorphically into the lattice $\big(\{0,1\}^{m\cdot n},\vee,\wedge\big)$. Denote the image by L.

The quadruple $(\alpha, \beta, \alpha, \beta)$, where

$$\alpha(c) = \frac{1}{(c')^s} = \prod_{i=1}^m p_i^{-\pi_i(c')s}, \beta(c) = \frac{1}{(c')^t} = \prod_{i=1}^m p_i^{-\pi_i(c')t} \tag{2.7}$$

for $c \in L$ and equals 0 otherwise, satisfies (2.1), because $(a, b) \leq a, b \leq [a, b]$, $(a, b) \cdot [a, b] = a \cdot b$ and thus for $1 < s < t$

$$\frac{1}{a^s} \frac{1}{b^t} \leq \frac{1}{[a, b]^s} \cdot \frac{1}{(a, b)^t}. \tag{2.8}$$

Since

$$[a, b] = \prod_{i=1}^m p_i^{\max(\pi_i(a), \pi_i(b))}, (a, b) = \prod_{i=1}^m p_i^{\min(\pi_i(a), \pi_i(b))} \tag{2.9}$$

the AD–inequality implies now (2.2), that is,

$$\sum_{a \in A_n} \frac{1}{a^s} \cdot \sum_{b \in B_n} \frac{1}{b^t} \leq \sum_{c \in [A_n, B_n]} \frac{1}{c^s} \cdot \sum_{d \in (A_n, B_n)} \frac{1}{d^t}, \tag{2.10}$$

or that

$$D(A_n, s) D(B_n, t) \leq D\big([A_n, B_n], s\big) D\big((A_n, B_n), t\big). \tag{2.11}$$

This and (2.3) finally imply (2.4).

Obviously, from (2.4) we can derive also other inequalities for arbitrary $A, B \subset \mathbb{N}$, namely $\underline{\mathbf{D}}A \cdot \overline{\mathbf{D}}B \leq \underline{\mathbf{D}}[A, B] \cdot \overline{\mathbf{D}}(A, B)$, $\overline{\mathbf{D}}A \cdot \overline{\mathbf{D}}B \leq \overline{\mathbf{D}}[A, B] \cdot \overline{\mathbf{D}}(A, B)$, and $\overline{\mathbf{D}}A \cdot \underline{\mathbf{D}}B \leq \overline{\mathbf{D}}[A, B] \cdot \underline{\mathbf{D}}(A, B)$.

It is a wide field of research to investigate for which sets the various densities exist.

Now we deduce from Theorem 1 an inequality for non–multiples $N(C) = \mathbb{N} \setminus M(C)$.

Corollary 2.

(i) *For* **finite** $A, B \subset \mathbb{N}$ $\mathbf{d}N(A) \cdot \mathbf{d}N(B) \leq \mathbf{d}\big[N(A), N(B)\big] \mathbf{d}N(A \cup B)$.

(ii) *For* **arbitrary** $A, B \subset \mathbb{N}$ $\mathbf{D}N(A) \mathbf{D}N(B) \leq \mathbf{D}\big[N(A), N(B)\big] \mathbf{D}N(A \cup B)$.
Here $\mathbf{D}$ *can be replaced by* $\boldsymbol{\delta}$ *or* $\underline{\mathbf{d}}$.

Proof:

First observe that $\big(N(A), N(B)\big) = N(A \cup B)$. This identity and application of Theorem 1 to the sets $N(A)$ and $N(B)$ give

$$\mathbf{D}N(A) \cdot \mathbf{D}N(B) \leq \underline{\mathbf{D}}\big[N(A), N(B)\big] \cdot \mathbf{D}N(A \cup B).$$

Here $\underline{\mathbf{D}}$ can be replaced by $\mathbf{D}$, because $\big[N(A), N(B)\big] = N(A) \cup N(B)$ and $\mathbf{D}N(A) + \mathbf{D}N(B) - \mathbf{D}N(A \cup B) = \mathbf{D}\big(N(A) \cup N(B)\big)$.

This gives (ii) and, since for a finite set C

$\mathbf{D}N(C) = 1 - \mathbf{D}M(C) = 1 - \mathbf{d}M(C) = \mathbf{d}N(C)$, also (i).

Notice that (i) is by the factor $\mathbf{d}\big[N(A), N(B)\big]$ better than Behrend's inequality.

In [4] we have shown that (i) in Corollary 1 is equivalent to

$$\mathbf{d}N(A)\mathbf{d}N(B) \leq \mathbf{d}N(A \cup B) - \mathbf{d}N(A, B)\big(1 - \mathbf{d}N[A, B]\big), \tag{2.12}$$

which is by the term $\mathbf{d}N(A, B)\big(1 - \mathbf{d}N[A, B]\big)$ better than Behrend's inequality!

Quite surprisingly the two inequalities are different and none implies the other!

Example 1:

$A = \{3, 4\}$, $B = \{6\}$, $\mathbf{d}N(A) = 1 - \left(\frac{1}{3} + \frac{1}{4} - \frac{1}{12}\right) = \frac{1}{2}$, $\mathbf{d}N(B) = \frac{5}{6}$, $\mathbf{d}N(A \cup B) = \mathbf{d}N(A) = \frac{1}{2}$, $\mathbf{d}\big[N(A), N(B)\big] = \frac{11}{12}$, $(A, B) = \{2, 3\}$, $\mathbf{d}N(A, B) = \frac{1}{3}$, $[A, B] = \{6, 12\}$, $\mathbf{d}N[A, B] = \mathbf{d}N(\{6\}) = \mathbf{d}N(B) = \frac{5}{6}$.

We have therefore

$\mathbf{d}N(A \cup B) - \mathbf{d}N(A, B)\big(1 - \mathbf{d}N[A, B]\big) = \frac{1}{2} - \frac{1}{3}\left(1 - \frac{5}{6}\right) = \frac{4}{9} < \frac{11}{24} = \frac{1}{2} \cdot \frac{11}{12} = \mathbf{d}N(A \cup B) \cdot \mathbf{d}\big[N(A), N(B)\big]$.

Example 2:

$A = \{2\}$, $B = \{2, 3\}$, $\mathbf{d}N(A) = \frac{1}{2}$, $\mathbf{d}N(B) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1}{3}$, $\mathbf{d}N(A \cup B) = \mathbf{d}N(B) = \frac{1}{3}$, by R_{10} $\big[N(A), N(B)\big] = N(A) \cup N(B) = N(A)$ and thus $\mathbf{d}\big[N(A), N(B)\big] = \frac{1}{2}$, $\mathbf{d}N(A \cup B) = \mathbf{d}N(B) = \frac{1}{3}$, $\mathbf{d}N(A, B) = 0$.

We have therefore

$\mathbf{d}N(A \cup B) - \mathbf{d}N(A, B)\big(1 - \mathbf{d}N[A, B]\big) = \frac{1}{3} > \frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2} = \mathbf{d}N(A \cup B)\mathbf{d}\big[N(A), N(B)\big]$.

We mention that in the case $A = \{4\}$, $B = \{6\}$, the order is as in example 1, but with *equality* in (2.12).

3 From "new better than used" (NBU) to the inequality $\mathbf{D}M(A \times B) \leq \mathbf{D}M(A) \cdot \mathbf{D}M(B)$

The role played by the AD–inequality in the forgoing section is now played by the BK–inequality, which we now introduce.

For $a = (a_1, \ldots, a_n)$ and $b = \{b_1, \ldots, b_n) \in \mathbb{R}^n$, $a \geq b$ means $a_t \geq b_t$ for $t = 1, 2, \ldots, n$. A set $A \subset \mathbb{R}^n$ is called increasing, if $a \in A$ and $b \geq a$ implies $b \in A$.

In reliability theory (see [8] for a systematic account) a non–negative random variable X is called "new better than used" or in short NBU, if its probability distribution on $\mathbb{R}_+$ satisfies for all $x_1, x_2 \geq 0$

$$P\{X > x_1 + x_2 | X > x_1\} \leq P\{X > x_2\}, \tag{3.1}$$

or equivalently,

$$P\{X > x_1 + x_2\} \leq P\{X > x_1\} P\{X > x_2\}. \tag{3.2}$$

Motivated by the study of critical probabilities in percolation theory (see [21], [10]) van der Berg and Kesten introduced and analysed the following concept.

A random vector $X = (X_1, \ldots, X_n)$ is *strongly new better than used* (SNBU), if for all increasing Borel sets $A, B \subset \mathbb{R}^n$

$$P\{X \in A + B\} \leq P\{X \in A\} P\{X \in B\}. \tag{3.3}$$

They found the

BK–inequality.

If $X_1, X_2, \ldots, X_n$ are NBU and independent, then

$$X = (X_1, \ldots, X_n) \quad \textit{is SNBU.} \tag{3.4}$$

Since $\mathbb{N} \cup \{o\} \subset \mathbb{R}_+$ this inequality applies also for products of chains.

We state and prove now our second inequality.

Theorem 2.

For **arbitrary** $A, B \subset \mathbb{N}$ $\mathbf{D}M(A) \cdot \mathbf{D}M(B) \geq \mathbf{D}M(A \times B)$.

Here $\mathbf{D}$ *can be replaced by* $\underline{\mathbf{d}}$ *and also by* $\mathbf{d}$ *if A and B are finite.*

Proof: We use the product of chains $L(\ell,m)=\{0,1,\dots,\ell\}^m$ and the probability distribution $\nu^{(m)}=\prod\limits_{i=1}^{m}\nu_i$, where

$$\nu_i(\ell_i)=p_i^{-s\ell_i}\cdot\left(\sum_{\lambda=0}^{\ell}p_i^{-s\lambda}\right)^{-1} \tag{3.5}$$

and thus

$$\nu^{(m)}(\ell_1,\dots,\ell_m)=\prod_{i=1}^{m}p_i^{-s\ell_i}\cdot\prod_{i=1}^{m}\left(\sum_{\lambda=0}^{\ell}p_i^{-s\lambda}\right)^{-1}. \tag{3.6}$$

Notice that for any $A,B\subset\mathbb{N}$

$$M(A)=A\times\mathbb{N},M(A\times B)=A\times B\times\mathbb{N}. \tag{3.7}$$

Define $\mathbb{N}(\ell,m)=\left\{\prod\limits_{i=1}^{m}p_i^{\ell_i}:(\ell_1,\dots,\ell_m)\in L(\ell,m)\right\}$ and notice that

$$M(A)\cap\mathbb{N}(\ell,m)=\Big(\big((A\cap\mathbb{N}(\ell,m)\big)\times\mathbb{N}(\ell,m)\Big)\cap\mathbb{N}(\ell,m), \tag{3.8}$$

$$M(A\times B)\cap\mathbb{N}(\ell,m)=\Big(\big((A\cap\mathbb{N}(\ell,m)\big)\times\big(B\cap\mathbb{N}(\ell,m)\big)\times\mathbb{N}(\ell,m)\Big)\cap\mathbb{N}(\ell,m) \tag{3.9}$$

and that

$$\lim_{\ell,m\to\infty}M(A)\cap\mathbb{N}(\ell,m)=M(A). \tag{3.10}$$

Since all summands in our Dirichlet series are non–negative we have therefore also for $s>1$

$$\lim_{\ell,m\to\infty}D\big(M(A)\cap\mathbb{N}(\ell,m),s\big)=D\big(M(A),s\big) \tag{3.11}$$

and

$$\lim_{\ell,m\to\infty}D\big(M(A\times B)\cap\mathbb{N}(\ell,m),s\big)=D\big(M(A\times B),s\big). \tag{3.12}$$

We show next that the ν_i's are NBU:

$$\nu_i\big(\{\lambda_1,\dots,\ell\}\big)\cdot\nu_i\big(\{\lambda_2,\dots,\ell\}\big)\geq\nu_i\big(\{\lambda_1+\lambda_2,\dots,\ell\}\big).$$

For $\lambda_1+\lambda_2>\ell$ this inequality obviously holds.

Lemma.

For any $\lambda_1, \lambda_2 \in \{0, 1, \ldots, \ell\}, \lambda_1 + \lambda_2 \leq \ell$

$$\sum_{\lambda \geq \lambda_1}^{\ell} \frac{1}{p^\lambda} \cdot \sum_{\lambda \geq \lambda_2}^{\ell} \frac{1}{p^\lambda} \geq \sum_{\lambda \geq \lambda_1 + \lambda_2}^{\ell} \frac{1}{p^\lambda} \cdot \sum_{\lambda = 0}^{\ell} \frac{1}{p^\lambda}.$$

Proof: The claimed statement is equivalent with

$$(1 + p + \cdots + p^{\ell - \lambda_1})(1 + p + \cdots + p^{\ell - \lambda_2}) \geq (1 + p + \cdots + p^{\ell - \lambda_1 - \lambda_2})(1 + p + \cdots + p^{\ell})$$

and consequently with

$$(p^{\ell - \lambda_1 + 1} - 1)(p^{\ell - \lambda_2 + 1} - 1) \geq (p^{\ell - \lambda_1 - \lambda_2 + 1} - 1)(p^{\ell + 1} - 1).$$

This in turn is equivalent with

$$p^{\ell - \lambda_1 - \lambda_2 + 1} + p^{\ell + 1} \geq p^{\ell - \lambda_1 + 1} + p^{\ell - \lambda_2 + 1}$$

and with

$$1 + p^{\lambda_1 + \lambda_2} \geq p^{\lambda_2} + p^{\lambda_1},$$

which is true.

Since $U = M(A) \cap \mathbb{N}(\ell, m)$ and $V = M(B) \cap \mathbb{N}(\ell, m)$ are increasing sets, we can apply the BK–inequality and get with $W = M(A \times B) \cap \mathbb{N}(\ell, m)$

$$\sum_{u \in U} \frac{1}{u^s} \left(\sum_{n \in \mathbb{N}(\ell, m)} \frac{1}{n^s} \right)^{-1} \cdot \sum_{v \in V} \frac{1}{v^s} \left(\sum_{n \in \mathbb{N}(\ell, m)} \frac{1}{n^s} \right)^{-1} \geq \sum_{w \in W} \frac{1}{w^s} \left(\sum_{n \in \mathbb{N}(\ell, m)} \frac{1}{n^s} \right)^{-1}$$

and consequently

$$\sum_{u \in U} \frac{1}{u^s} \cdot \sum_{v \in V} \frac{1}{v^s} \geq \sum_{w \in W} \frac{1}{w^s} \sum_{n \in \mathbb{N}(\ell, m)} \frac{1}{n^s}. \tag{3.13}$$

Now, together with (3.11) and (3.12) this implies

$$D\big(M(A), s\big) \cdot D\big(M(B), s\big) \geq D\big(M(A \times B), s\big) \cdot \zeta(s), \tag{3.14}$$

if ζ is Riemann's Zetafunction. We know that the logarithmic and also the Dirichlet densities exist for sets of multiples. Since also $\lim_{s \to 1^+} (s - 1)\zeta(s) = 1$, we complete the proof with (1.3).

Example 3:

In the case $A = \{1\}$, $B = \mathbb{N}$ we have $1 = \mathbf{d}(A \times B) \not\leq \mathbf{d}A \cdot \mathbf{d}B = 0$. This shows that for the present inequality it is essential to work with sets of multiples.

4 Concluding remarks

Supplementary material can be found in the preprint [6]. Besides investigations of the existence of the various densities it concerns in particular firstly a discussion of possible number theoretic implications of the work [3] and secondly inequalities of an elementary nature. We give here a brief sketch.

1. We dare to say that we did not just discover new density inequalities, but that we discovered a method to produce density inequalities from combinatorial correlation inequalities.

   Since the AD–inequality is much more general and also sharper than its predecessors ([17], [22], [15], [29], [19]) it received strong attention. However, it went almost unnoticed in the subsequent literature that this inequality is a very, very special case of the much more general inequalities of [3].

   More importantly, the discovery of [3] was that the basis of such correlation inequalities are not lattice properties (as was believed earlier), but cartesian product properties of the operations used.

   As the AD–inequality passes from a "local" property (2.1) to a "global" property (2.2), those more general inequalities also constitute local–global principles.

   It should be explored, which of these inequalities lead to number theoretic inequalities ("twins"). Conversely, now number theoretic questions may give hints for the search after combinatorial or probabilistic correlation inequalities.

   Recently, a beautiful generalization of AD has been given in (to our knowledge) independent papers [1] and [26]. We state without proof its number theoretic "twins" obtained by our approach for these sets.

   For arbitrary $A, B, C \subset \mathbb{N}$

   $$\underline{\mathbf{D}}A \cdot \underline{\mathbf{D}}B \cdot \underline{\mathbf{D}}C \leq \underline{\mathbf{D}}[A,B,C]\underline{\mathbf{D}}\big[(A,B),(A,C),(B,C)\big]\underline{\mathbf{D}}(A,B,C). \tag{4.1}$$

   From Theorem 1 we also get the bound

   $$\underline{\mathbf{D}}A \cdot \underline{\mathbf{D}}B \cdot \underline{\mathbf{D}}C \leq \big(\underline{\mathbf{D}}[A,B]\underline{\mathbf{D}}(A,B) \cdot \underline{\mathbf{D}}[A,C] \cdot \mathbf{D}(A,C) \cdot \mathbf{D}[B,C]\mathbf{D}(B,C)\big)^2. \tag{4.2}$$

   We conjecture that the upper bound in (4.1) is always at least as good as the upper bound in (4.2). If true, this gives an inequality for these bounds.

2. We consider here only finite sets $A, B \subset \mathbb{N}$.

   Some observations were made by asking just out of curiosity whether in (1.1) we can replace the operation M by the operation N, if we simultaneously reverse the inequality sign. Quite luckily, this is the case, but the inequality is very elementary:

   $$\mathbf{d}N[A,B]\mathbf{d}N(A \cup B) \leq \mathbf{d}N(A)\mathbf{d}N(B). \tag{4.3}$$

Equality holds exactly if $N(A) \supset N(B)$ or $N(B) \supset N(A)$.

Since $N(A,B) \subset N(A \cup B)$, we also have

$$\mathbf{d}N[A,B]\mathbf{d}N(A,B) \leq \mathbf{d}N(A)\mathbf{d}N(B). \tag{4.4}$$

Slightly more sophisticated is

$$\mathbf{d}\big(M(A) \cap M(B)\big)\mathbf{d}\big(M(A) \cup M(B)\big) \leq \mathbf{d}M(A)\mathbf{d}M(B), \tag{4.5}$$

where equality holds exactly if

$$\mathbf{d}M(A) \supset \mathbf{d}M(B) \ \text{ or } \ \mathbf{d}M(B) \supset \mathbf{d}M(B).$$

Since $M(A) \cup M(B) = M(A \cup B)$ and $M(A \cap B) \subset M(A) \cap M(B)$ this implies

$$\mathbf{d}M(A \cap B)\mathbf{d}M(A \cup B) \leq \mathbf{d}M(A)\mathbf{d}M(B), \tag{4.6}$$

where equality holds exactly if

$$B \subset M(A \cap B) \ \text{ or } \ A \subset M(A \cap B).$$

Combining (4.5) and Theorem 2, that is, by taking the maximum of the left–hand sides in the inequalities, we get an inequality truly better than any one of them.

We conclude with a combinatorial "twin" of (4.6):

For downsets $U, V \subset L$

$$(|L| - |U \vee V|)(|L| - |U \wedge V|) \leq (|L| - |U|)(L - |V|), \tag{4.7}$$

where equality holds exactly if $U \subset V$ or $V \subset U$.

Consistent with our observation about Theorem 2 is, that (4.7) does not extend to the non–monotone case.

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