SETS OF INTEGERS AND QUASI–INTEGERS WITH PAIRWISE COMMON DIVISOR

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0. Introduction

Consider the set $\mathbb{N}_s(n) = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\} \cap \langle 1, n \rangle$ of positive integers between 1 and n, which are not divisible by the first $s-1$ primes $p_1, \dots, p_{s-1}$.

Erdös introduced in [4] (and also in [5], [6], [7], [9]) the quantity $f(n,k,s)$ as the largest integer ρ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \rho$, exists with no $k+1$ numbers being coprimes. Certainly the set

$$\mathbb{E}(n,k,s) = \big\{ u \in \mathbb{N}_s(n) : u = p_{s+i} v \ \text{ for some } \ i = 0,1,\dots,k-1 \big\} \tag{1}$$

does not have $k+1$ coprimes.

Conjecture 1 (Erdös [4]): $f(n,k,1) = |\mathbb{E}(n,k,1)|$ for all $n,k \in \mathbb{N}$ was disproved in [1].

This disproves of course also the General Conjecture (Erdös [7]): for all $n,k,s \in \mathbb{N}$

$$f(n,k,s) = |\mathbb{E}(n,k,s)|. \tag{2}$$

However, in [2] we proved (2) for every k,s and (relative to k,s) large n.

In the present paper we are concerned with the case $k=1$, which in [1] and [2] we called

Conjecture 2: $f(n,1,s) = |\mathbb{E}(n,1,s)|$ for all $n,s \in \mathbb{N}$.

Erdös mentioned in [7] that he did not even succeed in settling this special case of the General Conjecture.

Whereas in [1] we proved this by a completely different approach for $n \geq (p_{s+1} - p_s)^{-1} \prod_{i=1}^{s+1} p_i$, we establish it now for all n (Theorem 2).

We generalize and analyze Conjecture 2 first for quasi–primes in order to understand how the validity of Conjecture 2 depends on the distribution of the quasi–primes and primes. Our main result is a simply structured sufficient condition on this distribution (Theorem 1). Using sharp estimates on the prime number distribution by Rosser and Schoenfeld [14] we show that this condition holds for $\mathbb{Q} = \{p_s, p_{s+1}, \dots\}$, $s \geq 1$, as set of quasi–primes and thus Theorem 2 follows.

1. Basic definitions for natural numbers and quasi–numbers

Whenever possible we keep the notation of [2]. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{P} = \{p_1, p_2, \dots,\} = \{2, 3, 5, \dots\}$ denotes the set of all primes. $\mathbb{N}^*$ is the set of square free numbers.

For two numbers $u, v \in \mathbb{N}$ we write $u \mid v$ (resp. $u \nmid v$) iff u divides v (resp. u doesn't divide v), $[u, v]$ stands for the smallest common multiple of u and v , (u, v) is the largest common divisor of u and v , and we say that u and v have a common divisor, if $(u, v) > 1$. $\langle u, v \rangle$ denotes the interval $\{x \in \mathbb{N} : u \leq x \leq v\}$.

For any set $A \subset \mathbb{N}$ we introduce

$$A(n) = A \cap \langle 1, n \rangle \tag{1.1}$$

and $|A|$ as cardinality of A . The set of multiples of A is

$$M(A) = \{m \in \mathbb{N} : a \mid m \text{ for some } a \in A\}. \tag{1.2}$$

For set $\{a\}$ with one element we also write $M(a)$ instead of $M(\{a\})$. For $u \in \mathbb{N}$, $p^+(u)$ denotes the largest prime in its prime number representation

$$u = \prod_{i=1}^{\infty} p_i^{\alpha_i}, \quad \sum_{i=1}^{\infty} \alpha_i < \infty. \tag{1.3}$$

We also need the function π , where for $y \in \mathbb{N}$

$$\pi(y) = |\mathbb{P}(y)|, \tag{1.4}$$

and the set Φ , where

$$\Phi(u, y) = \big\{x \in \mathbb{N}(u) : (x, p) = 1 \text{ for all } p < y\big\}. \tag{1.5}$$

We note that $1 \in \Phi(u, y)$ for all $u \geq y$, $u \geq 1$.

Clearly, by (1.3) $u \in \mathbb{N}$ corresponds to a multiset $(\alpha_1, \alpha_2, \dots,)$. Therefore, instead of saying that $A \subset \mathbb{N}(z)$ has pairwise (nontrivial) common divisors, we adapt the following shorter multiset terminology.

Definition 1. $A \subset \mathbb{N}(z)$, $z \geq 1$, is said to be intersecting iff for all $a, b \in A$; $a = \prod\limits_{i=1}^{\infty} p_i^{\alpha_i}$, $b = \prod\limits_{i=1}^{\infty} p_i^{\beta_i}$; $\alpha_j \beta_j \neq 0$ for some j .

In order to better understand, how properties depend on the multiset structure and how on the distribution of primes it is very useful to introduce quasi–(natural) numbers and quasi–primes. Results then also can be applied to a subset of the primes, if it is viewed as the set of quasi–primes.

A set $\mathbb{Q} = \{1 < r_1 < r_2 < \dots\}$ of positive real numbers, $\lim\limits_{i \to \infty} r_i = \infty$, is called a (complete) set of quasi–prime numbers, if every number in

$$\mathbb{X} = \left\{x \in \mathbb{R}^+ : x = \prod_{i=1}^{\infty} r_i^{\alpha_i}, \alpha_i \in \{0, 1, 2, \dots,\}, \sum_{i=1}^{\infty} \alpha_i < \infty\right\} \tag{1.6}$$

has a unique representation. (See also Remark 1 after Theorem 1.)

The set $\mathbb{X}$ is the set of quasi–numbers corresponding to the set of quasi–primes $\mathbb{Q}$.

We can now replace $\mathbb{P},\mathbb{N}$ *by* $\mathbb{Q},\mathbb{X}$ *in all concepts of this Section up to Definition 1* and thus for any $u,v\in\mathbb{X}$ $u\mid v$, $u\nmid v$, (u,v) , $[u,v]$, $\langle u,v\rangle$ $(=\{x\in\mathbb{X}: u\le x\le v\})$; for any $A\subset\mathbb{X}$ $A(z)$, $M(A)(=\{m\in\mathbb{X}: a\mid m$ for some $a\in A\})$; and "intersecting" are well defined. So are also the function π and the sets $\Phi(u,y)$ for $u\ge y, u\ge 1$.

We study $\mathcal{I}(z)$, the family of all intersecting $A\subset\mathbb{X}(z)$, and

$$f(z)=\max_{A\subset\mathcal{I}(z)}|A|,\ \ z\in\mathbb{X}. \tag{1.7}$$

The subfamily $\mathcal{O}(z)$ of $\mathcal{I}(z)$ consists of the optimal sets, that is,

$$\mathcal{O}(z)=\big\{A\in\mathcal{I}(z):|A|=f(z)\big\}. \tag{1.8}$$

A key role is played by the following configuration.

Definition 2. $A\subset\mathbb{X}(z)$ is called star, if

$$A=M(\{r\})\cap\mathbb{X}(z)\ \text{ for some }\ r\in\mathbb{Q}.$$

2. Auxiliary results concerning left compressed sets, "upsets" and "downsets"

There is not only one way to define "left pushing" of subsets of $\mathbb{X}$. Here the following is most convenient.

For any $i,j\in\mathbb{N}$, $j<i$, we define the operation "left pushing" $L_{i,j}$ on subsets of $\mathbb{X}$. For $A\subset\mathbb{X}$ let

$A_1=\big\{a\in A: a=a_1\cdot r_i^\alpha,\alpha\ge 1,(a_1,r_i\cdot r_j)=1,(a_1\cdot r_j^\alpha)\notin A\big\}$ and

$L_{i,j}(A)=(A\smallsetminus A_1)\cup A_1^*$, where

$A_1^*=\big\{a=a_1\cdot r_j^\alpha:(a_1,r_i\cdot r_j)=1$ and $a_1\cdot r_j^\alpha\in A_1\big\}$.

Clearly $|L_{i,j}(A)\cap\mathbb{X}(z)|\ge|A(z)|$ for every $z\in\mathbb{R}^+$.

It is easy to show, that the operation $L_{i,j}$ preserves the property "intersecting".

By finitely many (resp. countably many) "left pushing" operations $L_{i,j}$ one can transform every $A\subset\mathbb{X}(z)$, $z\in\mathbb{R}^+$, (resp. $A\subset\mathbb{X}$) into a "left compressed" set A' , where the concept of left compressedness is defined as follows:

Definition 3. $A\subset X$ is said to be left compressed if

$$L_{i,j}(A)=A\ \text{ for all }\ i,j\ \text{ with }\ i>j.$$

We note that there are left compressed sets A' and A'' , which are obtained by left pushing from the same set A .

Lemma 1. *For all* $z \in \mathbb{X}$

$$f(z) = \max_{A \in \mathcal{C}(z)} |A|.$$

Clearly, any $A \in \mathcal{O}(z)$ is an "upset":

$$A = M(A) \cap \mathbb{X}(z). \tag{2.1}$$

and it is also a "downset" in the following sense:

$$\text{for } a \in A, a = r_{i_1}^{\alpha_1} \dots r_{i_t}^{\alpha_t}, \alpha_i \geq 1 \text{ also } a' = r_{i_1} \dots r_{i_t} \in A. \tag{2.2}$$

For every $B \subset \mathbb{X}$ we introduce the unique primitive subset $P(B)$, which has the properties

$$b_1, b_2 \in P(B) \text{ implies } b_1 \nmid b_2 \text{ and } B \subset M\big(P(B)\big). \tag{2.3}$$

We know from (2.2) that for any $A \in \mathcal{O}(z)$ $P(A)$ consists only of squarefree quasi–numbers and that by (2.1)

$$A = M\big(P(A)\big) \cap \mathbb{X}(z). \tag{2.4}$$

From Lemma 1 we know that $\mathcal{O}(z) \cap \mathcal{C}(z) \neq \varnothing$.

Let now $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$ and $P(A) = \{a_1, \dots, a_m\}$, where the a_i's are written in lexicographic order. The set of multiples of $P(A)$ in $\mathbb{X}(z)$ can be written as a union of disjoint sets $B^i(z)$:

$$M\big(P(A)\big) \cap \mathbb{X}(z) = \dot\cup B^i(z), \tag{2.5}$$

$$B^i(z) = \big\{x \in M\big(P(A)\big) \cap X(z) : a_i \mid x, a_j \nmid x \text{ for } j = 1, \dots, i-1\big\}. \tag{2.6}$$

We can say more about $B^i(z)$, if we use the factorisation of the square free quasi–numbers a_i.

Lemma 2. *Let* $a_i = r_{j_1}, \dots, r_{j_\ell}; r_{j_1} < r_{j_2} \dots < r_{j_\ell}$ *, then*

$$B^i(z) = \left\{ x \in \mathbb{X}(z) : x = r_{j_1}^{\alpha_1} \dots r_{j_\ell}^{\alpha_\ell} T, \alpha_i \geq 1, \left(T, \prod_{r_i \leq r_{j_\ell}} r_i\right) = 1 \right\}.$$

Proof: This immediately follows from the facts that A is left compressed, "upset" and "downset".

Finally, a result for stars. Keep in mind that they contain a single prime and that Lemma 1 holds.

Lemma 3. *For any* $B \subset \mathcal{I}(z)$ *and* $B' \subset \mathbb{X}(z)$ *, which is left compressed and obtained from* B *by left pushing we have:* B *is a star exactly if* B' *is a star.*

3. The main result

Theorem 1. *Suppose the quasi–primes* $\mathbb{Q}$ *satisfy the following condition: for all* $u \in \mathbb{R}^+$ *and for all* $r_\ell, \ell \geq 2$

$$\text{(a)} \qquad 2|\Phi(u, r_\ell)| \leq |\Phi(u \cdot r_\ell, r_\ell)|.$$

Then, for all $z \in \mathbb{R}^+$, every optimal $A \in \mathcal{O}(z)$ is a "star". In particular

$$f(z) = |M(r_1) \cap X(z)| \quad \text{for all} \quad z \in \mathbb{X}.$$

Remarks:

1. This result and also Lemma 2 below immediately extend to the case where quasi–primes are defined without the requirement of the uniqueness of the representations in (1.6.), if multiplicities of representations are taken into consideration. $\mathbb{X}$ is thus just a free, discrete commutative semigroup in $\mathbb{R}^+_{\geq 1}$.
2. Without the uniqueness requirement we are led to a new problem by not counting multiplicities.
3. However, without the assumption $\lim\limits_{i \to \infty} r_i = \infty$ or without the assumption of discreteness the quasi–primes have a cluster point ρ and one can produce infinitely many infinite, intersecting sets in $\mathbb{X}(\rho^3 + \varepsilon)$, which are not stars.
4. In Section 5 we discuss the case of finitely many quasi–primes.

Proof: Let $A \in \mathcal{O}(z)$ and let $P(A) = \{a_1, \ldots, a_m\}$ be the primitive subset of A which generates A .

Under condition (a), the Theorem is equivalent to the statement:

for all $z \in \mathbb{X}$, $m = 1, a_1 = r_\ell$ for some quasi–prime r_ℓ .

Suppose, to the opposite, that for some $z \in \mathbb{X}$ there exists $A \in \mathcal{O}(z)$ which is not a star, i.e. if $P(A) = \{a_1, \ldots, a_m\}$ is the primitive, generating subset of A , then $m > 1$ and hence every element $a_i \in P(A)$ is a product of at least two different quasi–primes.

According to Lemma 3 we can assume, that $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$, $P(A) = \{a_1, a_2, \ldots, a_m\}$; a_i's are written in lexicographic order, $m > 1$ and

$$p^+(a_m) = r_t, \; t \geq 2.$$

Write $P(A)$ in the form

$$P(A) = S_1 \cup S_2 \cup \cdots \cup S_t, \; t \geq 2, S_t \neq \varnothing,$$

where

$$S_i = \big\{ a \in P(A) : p^+(a) = r_i \big\}.$$

Since $A \in \mathcal{O}(z) \cap \mathcal{C}(z)$, we have

$$A = M\big(P(A)\big) \cap X(z) = \dot{\bigcup_{1 \le j \le t}} B(S_j), \quad \text{where}$$

$B(S_j) = \dot{\bigcup\limits_{a_i \in S_j}} B^i(z)$ and $B^i(z)$ are described in Lemma 2.

Now we consider $S_t = \{a_\ell, a_{\ell+1}, \ldots, a_m\}$ for some $\ell \le m$, and let $S_t = S_t^1 \dot\cup S_t^2$, where

$$S_t^1 = \{a_i \in S_t : r_{t-1} \mid a_i\}, S_t^2 = S_t \setminus S_t^1.$$

We have

$$B(S_t) = B(S_t^1) \dot\cup B(S_t^2), \quad \text{where} \tag{3.1}$$

$$B(S_t^j) = \bigcup_{a_i \in B_t^j} B^{(i)}(z); \; j = 1, 2.$$

Let $\widetilde{S}_t = \left\{\frac{a_\ell}{r_t}, \frac{a_{\ell+1}}{r_t}, \ldots, \frac{a_m}{r_t}\right\}$ and similarly $\widetilde{S}_t^j = \left\{\frac{a_i}{r_t} : a_i \in S_t^j\right\}$; $j = 1, 2$.
It is clear, that $\frac{a_i}{r_t} > 1$ for all $a_i \in S_t$.

Obviously $\widetilde{S}_t^1 \in \mathcal{I}(z)$, because all elements of $\widetilde{S}_t^1$ have common factor r_{t-1} . Let us show that $\widetilde{S}_t^2 \in \mathcal{I}(z)$ as well. Suppose, to the opposite, there exist $b_1, b_2 \in \widetilde{S}_t^2$ with $(b_1, b_2) = 1$.

We have $b_1 \cdot r_t, b_2 \cdot r_t \in S_t^2 \subset A$ and $(b_1 \cdot r_1, r_{t-1}) = 1, (b_2 \cdot r_t, r_{t-1}) = 1$.

Since $A \in \mathcal{C}(z)$ and $r_{t-1} \nmid b_1 {\cdot} b_2$ (see definition of S_t^2), we conclude that $r_{t-1} {\cdot} b_1 \in A$ as well. Hence the elements $r_{t-1} {\cdot} b_1, r_t {\cdot} b_2 \in A$ and at the same time $(r_{t-1} {\cdot} b_1, r_t {\cdot} b_2) = 1$, which is a contradiction. So, we have $\widetilde{S}_t^j \in \mathcal{I}(z)$; $j = 1, 2$; and hence

$$A_j = M\big((P(A) \setminus S_t) \cup \widetilde{S}_t^i\big) \cap \mathbb{X}(z) \in \mathcal{I}(z); \; j = 1, 2.$$

We are going to prove that at least one of the inequalities $|A_1| > |A|$, $|A_2| > |A|$ holds, and this will lead to a contradiction.

From (3.1) we know that

$$\max\big\{|B(S_t^1)|, |B(S_t^2)|\big\} \ge \frac{1}{2}|B(S_t)|.$$

Let us assume, say

$$|B(S_t^2)| \ge \frac{1}{2}|B(S_t)|, \tag{3.2}$$

and let us show that

$$|A_2| > |A| \tag{3.3}$$

(if $|B(S_t^1)| \ge \frac{1}{2}|B(S_t)|$ the situation is symmetrically the same).

Let $b \in \widetilde{S}_t^2$ and $b = r_{i_1} \cdot r_{i_2} \dots r_{i_s}$; $r_{i_1} < r_{i_2} < \cdots < r_{i_s} < r_t$. We know that

$$a_i = b \cdot r_t = r_{i_1} \dots r_{i_s} \cdot r_t \in S_t^2 \ \text{ for some } \ i \leq m,$$

and that (see Lemma 2), the contribution of $M(a_i)$ in $B(S_t)$ (and as well in A) are the elements in the form:

$$B^i(z) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot r_t^{\alpha_t} \cdot T; \ \text{ where } \ \alpha_i \geq 1 \ \text{ and } \ \left(T, \prod_{i \leq t} r_i \right) = 1 \right\}.$$

We write $B^i(z)$ in the following form:

$$B^i(z) = \dot{\bigcup_{(\alpha_1,\dots,\alpha_s),\alpha_i \geq 1}} D(\alpha_1, \dots, \alpha_s), \ \text{ where } \tag{3.4}$$

$$D(\alpha_1, \dots, \alpha_s) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot r_t \cdot T_1; \left(T_1, \prod_{i \leq t-1} r_i \right) = 1 \right\}. \tag{3.5}$$

Now we look at the contribution of $M(b)$ in $A_2 = M\big((P(A) \setminus S_t) \cup \widetilde{S}_t^2\big) \cap \mathbb{X}(z)$, namely we look only at the elements in A_2 (denoted by $B(b)$), which are divisible by b , but not divisible by any element from $\big(P(A) \setminus S_t\big) \cup \big(\widetilde{S}_t^2 \setminus b\big)$.
Since $A \subset C(z)$ and r_t is the largest quasi–prime in $P(A)$, we conclude that

$$B(b) \supseteq B^*(b) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot \widetilde{T}, \alpha_i \geq 1, \ \text{ where } \ \left(\widetilde{T}, \prod_{i \leq t-1} r_i \right) = 1 \right\},$$

and we can write

$$B^*(b) = \dot{\bigcup_{(\alpha_1,\dots,\alpha_s),\alpha_i \geq 1}} \widetilde{D}(\alpha_1, \dots, \alpha_s), \tag{3.6}$$

where

$$\widetilde{D}(\alpha_1, \dots, \alpha_s) = \left\{ x \in \mathbb{X}(z) : x = r_{i_1}^{\alpha_1} \dots r_{i_s}^{\alpha_s} \cdot \widetilde{T}, \alpha_i \geq 1, \left(\widetilde{T}, \prod_{i \leq t-1} r_i \right) = 1 \right\}. \tag{3.7}$$

Hence

$$|B(b)| \geq |B^*(b)| = \sum_{(\alpha_1,\dots,\alpha_s),\alpha_i \geq 1} \widetilde{D}(\alpha_1, \dots, \alpha_s). \tag{3.8}$$

At first we prove that $|A_2| \geq |A|$. In the light of (3.2), (3.4–3.8), for this it is sufficient to show that

$$|\widetilde{D}(\alpha_1, \dots, \alpha_s)| \geq 2|D(\alpha_1, \dots, \alpha_s)|, \tag{3.9}$$

for all $(\alpha_1, \ldots, \alpha_s), \alpha_i \geq 1$.

However, this is exactly the condition (a) in the Theorem for $u = \frac{z}{r_{i_1}^{\alpha_1} \ldots r_{i_s}^{\alpha_s} \cdot r_t}$ and $\ell = t$. Hence $|A_2| \geq |A|$.

To prove (3.3), that is $|A_2| > |A|$, it is sufficient to show the existence of $(\alpha_1, \ldots, \alpha_s)$, $\alpha_i \geq 1$, for which in (3.9), strict inequality holds. For this we take $\beta \in \mathbb{N}$ and $(\alpha_1, \ldots, \alpha_s) = (\beta, 1, 1, \ldots, 1)$ such that

$$\frac{z}{r_t} < r_{i_1}^{\beta} \cdot r_{i_2} \cdot r_{i_s} \leq z.$$

This is always possible, because

$$r_{i_1} \cdot r_{i_2} \ldots r_{i_s} \cdot r_t \leq z \text{ implies } r_{i_1} \ldots r_{i_s} \leq \frac{z}{r_t} \text{ and } r_{i_1} < \cdots < r_s < r_t.$$

We have $|\widetilde{D}(\beta, 1, \ldots, 1)| = 1$ and $|D(\beta, 1, \ldots, 1)| = 0$. Hence $|A_2| > |A|$, which is a contradiction, since $A_2 \in \mathcal{I}(z)$. This completes the proof.

Lemma 4. *Sufficient for condition (a) in Theorem 1 to hold is the condition*

(b) $$2\pi(v) \leq \pi(r_2 \cdot v) \quad \textit{for all} \quad v \in \mathbb{R}^+.$$

Proof: Under condition (b) it is sufficient to prove for every $u \in \mathbb{R}^+, r_\ell, (\ell \geq 2)$ that $|\Phi(u, r_\ell)| \leq |\Phi_1(u \cdot r_\ell, r_\ell)|$ where $\Phi_1(u \cdot r_\ell, r_\ell) = \{x \in \Phi(u \cdot r_\ell, r_\ell) : u < x \leq u \cdot r_\ell\}$.

We avoid the trivial cases $u < 1$, for which $\Phi(u, r_\ell) = \varnothing$, and $1 \leq u < r_\ell$, for which $\Phi(u, r_\ell) = \{1\}$ and $r_\ell \in \Phi_1(u \cdot r_\ell, r_\ell)$. Hence, we assume $u \geq r_\ell$.

Let $F(u, r_\ell) = \{a \in \Phi(u, r_\ell), a \neq 1 : a \cdot p^+(a) \leq u\} \cup \{1\}$. It is clear that for any $b \in \Phi(u, r_\ell), b \neq 1$, we have $\frac{b}{p^+(b)} \in F(u, r_\ell)$ and that

(3.10) $$|\Phi(u, r_\ell)| = 1 + \sum_{a \in F(u, r_\ell)} |\tau(a)|,$$

where $\tau(a) = \{r \in Q : r_\ell \leq p^+(a) \leq r \leq \frac{u}{a}\}$ and integer 1 in (3.10) stands to account for the element $1 \in \Phi(u, r_\ell)$.

On the other hand we have

(3.11) $$|\Phi_1(u \cdot r_\ell, r_\ell)| \geq \sum_{a \in F(u, r_\ell)} |\tau_1(a)|, \quad \text{where}$$

$$\tau_1(a) = \left\{r \in Q : \frac{u}{a} < r \leq \frac{u \cdot r_\ell}{a}\right\}.$$

We have

$$|\tau(a)| \leq \pi\left(\frac{u}{a}\right) - \ell + 1 \leq \pi\left(\frac{u}{a}\right) - 1 \ (\ell \geq 2) \text{ and by condition (b)}$$

(3.12) $$|\tau_1(a)| = \pi\left(\frac{u \cdot r_\ell}{a}\right) - \pi\left(\frac{u}{a}\right) \geq \pi\left(\frac{u}{a}\right).$$

Hence $|\tau_1(a)| > |\tau(a)|$ for all $a \in F(u, r_\ell)$ and, since $F(u, r_\ell) \neq \varnothing \quad (u \geq r_\ell)$, from (3.10),(3.11),(3.12) we get

$$|\Phi_1(u \cdot r_\ell, r_\ell)| \geq |\Phi(u, r_\ell)|.$$

4. Proof of Erdös' "Conjecture 2"

For a positive integer s let $\mathbb{N}_s = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\}$ and let $\mathbb{N}_s(n) = \mathbb{N}_s \cap \langle 1, n \rangle$.

Erdös introduced in [4] (and also in [5], [6], [7], [9]) the quantity $f(n,k,s)$ as the largest integer ρ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \rho$, exists with no $k+1$ numbers being coprimes.

Certainly the set

$$\mathbb{E}(n,k,s) = \left\{ u \in \mathbb{N}_s(n) : u = p_{s+i} v \ \text{ for some } \ i = 0, 1, \dots, k-1 \right\} \tag{4.1}$$

does not have $k+1$ coprimes.

The case $s = 1$, in which $\mathbb{N}_1(n) = \langle 1, n \rangle$, is of particular interest.

Conjecture 1 (Erdös [4]):

$$f(n,k,1) = |\mathbb{E}(n,k,1)| \ \text{ for all } \ n, k \in \mathbb{N} \tag{4.2}$$

was disproved in [1].

This disproves of course also the **General Conjecture** (Erdös [7]): for all $n, k, s \in \mathbb{N}$

$$f(n,k,s) = |\mathbb{E}(n,k,s)|. \tag{4.3}$$

However, in [2] we proved (4.3) for every k, s and (relative to k, s) large n . For further related work we refer to [8–10].

Erdös mentions in [7] that he did not succeed in settling even the case $k = 1$. This special case of the General Conjecture was called in [1] and [2]

Conjecture 2: $f(n,1,s) = |\mathbb{E}(n,1,s)|$ for all $n, s \in \mathbb{N}$.

Notice that

$$\mathbb{E}(n,1,s) = \left\{ u \in \mathbb{N}_s(n) : p_s \mid u \right\}, \ \text{ i.e. } \ \mathbb{E}(n,1,s) \ \text{ is a star.} \tag{4.4}$$

In the language of quasi–primes we can define

$$\mathbb{Q} = \{ r_1, r_2, \dots, r_\ell \dots \} = \{ p_s, p_{s+1}, \dots, p_{s+\ell-1}, \dots \} \tag{4.5}$$

and the corresponding quasi–integers $\mathbb{X}$.

Now, Conjecture 2 is equivalent to

$$f(n,1,s) = |M(p_s) \cap \mathbb{X}(n)| \ \text{ for all } \ n, s \in \mathbb{N}. \tag{4.6}$$

Notice that $\mathbb{X}(n)$ is the set of those natural numbers not larger than n , which are entirely composed from the primes not smaller than p_s . Clearly, condition (1.6) for quasi–prime is satisfied.

Theorem 2.

(i) *Conjecture 2 is true.*

(ii) *For all* $s, n \in \mathbb{N}$, *every optimal configuration is a "star".*

(iii) *The optimal configuration is unique if and only if*

$$|M(p_s) \cap \mathbb{N}_s(n)| > |M(p_{s+1}) \cap \mathbb{N}_s(n)|,$$

which is equivalent to the inequality

$$|\Phi\left(\frac{n}{p_s}, p_s\right)| > |\Phi\left(\frac{n}{p_{s+1}}, p_s\right).|$$

Remark 5: We believe, that also for $k = 2, 3$

$$f(n, k, s) = |\mathbb{E}(n, k, s)| \text{ for all } n, s \in \mathbb{N}.$$

For $k = 4$ our counterexample in [1] applies. Moreover, we believe that every optimal configuration in the case $k = 2$ is a union of two stars. In the case $k = 3$ it is not always true, which shows the following

Example: Let $s \in \mathbb{N}$ be such that $p_s \cdot p_{s+7} > p_{s+5} \cdot p_{s+6}$ (as such primes we can take for instance the primes from the mentioned counterexample) and let $p_{s+5} \cdot p_{s+6} \leq n \leq p_s \cdot p_{s+7}$. We verify that

$$|\mathbb{E}(n, 2, s)| = |M(p_s, p_{s+1}, p_{s+2}) \cap \mathbb{N}(n)| = 21.$$

On the other hand the set

$$A = \{p_i \cdot p_j,\ s \leq i < j \leq s + 6\}$$

has no 4 coprime elements and is not a union of the stars, but again

$$|A| = 21.$$

Proof of Theorem 2: We prove (ii). Since $M(p_s) \cap \mathbb{N}_s(n)$ is not smaller than any competing star, this implies (i) and (iii). In the light of Theorem 1 and Lemma 4, it is sufficient to show that

$$2\pi(v) \leq \pi(p_{s+1} \cdot v) \text{ for all } v \in \mathbb{R}^+. \tag{4.7}$$

Since for $v < P_s$, $\pi(v) = 0$, we can assume $v \geq P_s$.

(4.7) is equivalent to

$$2\big(\Pi(v) - s + 1\big) \leq \Pi(p_{s+1} \cdot v) - s + 1 \tag{4.8}$$

where $\Pi(\cdot)$ is usual the counting function of primes. To show (4.8) it is sufficient to prove for all $v \in \mathbb{R}^+$

$$2\Pi(v) \leq \Pi(3v). \tag{4.9}$$

For this it suffices to show (4.9) only for $v \in \mathbb{P}$.

We use the very sharp estimates on the distribution of primes due to Rosser and Schoenfeld [14]:

$$\frac{v}{\log v - \frac{1}{2}} < \Pi(v) < \frac{v}{\log v - \frac{3}{2}} \quad \text{for every } \; v \geq 67. \tag{4.10}$$

From (4.10) we get

$$2\Pi(v) < \Pi(3v) \quad \text{for all } \; v > 298.$$

The cases $v < 298$, $v \in \mathbb{P}$ are verified by inspection. We just mention that for $v \in \{3,5,7,13,19\}$ one has even the equality $2\Pi(v) = \Pi(3v)$.

5. Examples of sets of quasi–primes for which almost all optimal intersecting sets of quasi–numbers are not stars

Suppose we are given only a *finite number* of quasi–primes:

$$1 < r_1 < r_2 < \cdots < r_m, \; m \geq 3,$$

satisfying (1.6).

The sets $\mathbb{X}, \mathbb{X}^*, \mathbb{X}(z), \mathcal{I}(z), \mathcal{O}(z)$ are defined as in Section 1. Here $\mathbb{X}^*$ has exactly 2^m elements. We are again interested in the quantity

$$f(z) = \max_{A \in \mathcal{I}(z)} |A|, \; z \in \mathbb{X}.$$

For all $y \in \mathbb{X}^*$, $y = r_1^{\alpha_1} \dots, r_m^{\alpha_m}$; $\alpha_i \in \{0,1\}$, let $w(y) = \sum\limits_{i=1}^{m} \alpha_i$ and let

$$T(y) = \{x \in \mathbb{X}, x = r_1^{\beta_1} \dots, r_m^{\beta_m} : \beta_i \geq 1 \; \text{ iff } \; \alpha_i = 1\}.$$

We distinguish two cases.

Case I: $m = 2m_1 + 1$

Define $\mathbb{X}_1^* = \left\{ x \in \mathbb{X}^* : w(x) \geq m_1 + 1 \right\}$.

Proposition 1. *Let* $m = 2m_1 + 1$ *be odd. There exists a constant* $z_0 = z(r_1, \dots, r_m)$ *such that for all* $z > z_0$, $|\mathcal{O}(z)| = 1$ *and* $A \in \mathcal{O}(z)$ *has the form*

$$A = M(\mathbb{X}_1^*) \cap X(z) = \bigcup_{y \in \mathbb{X}_1^*} T(y) \cap \mathbb{X}(z).$$

Proof: Suppose $B \in \mathcal{O}(z)$. Since by optimality B is "downset" and "upset", we have

$$B = \bigcup_{y \in Y} T(y) \cap X(z) \quad \text{for some } \; Y \subset \mathbb{X}^*.$$

It is clear, that $|Y| \le 2^{m-1}$, because by the intersecting property $y \in Y$ implies $\overline{y} = \frac{r_1 \dots r_m}{y} \notin Y$.

Write $Y = Y_1 \dot\cup Y_2$, where

$$Y_1 = \{y \in Y : w(y) \le m_1\} \quad \text{and} \quad Y_2 = \{y \in Y : w(y) \ge m_1 + 1\}.$$

Our aim is to prove, that for large enough z one always has $Y_1 = \varnothing$, from where the Proposition follows. Since $\mathbb{X}^*$ is finite, for this it is sufficient to show that for all $y \in \mathbb{X}^*$ with $w(y) \le m_1$

$$|T(y) \cap \mathbb{X}(z)| < |T(\overline{y}) \cap \mathbb{X}(z)|, \quad \text{if } z > z(y). \tag{5.1}$$

Let $y = r_1^{\alpha_1} \dots r_m^{\alpha_m}$, $\alpha_i \in \{0,1\}$, and let $\mathcal{I}(y) \subset \{1, 2, \dots, m\}$, $|\mathcal{I}(y)| = w(y)$, be the positions with $\alpha_i = 1$.

We introduce

$$c_i = \log r_i \quad \text{for } i = 1, \dots, m. \tag{5.2}$$

Then it is easy to see that $|T(y) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(y)} c_i \gamma_i \le \log z \quad \text{in} \quad \gamma_i \in \mathbb{N}$$

and $|T(\overline{y}) \cap \mathbb{X}(z)|$ is the number of solutions of

$$\sum_{i \in \mathcal{I}(\overline{y})} c_i \delta_i \le \log z \quad \text{in} \quad \delta_i \in \mathbb{N}.$$

We verify that

$$|T(y) \cap \mathbb{X}(z)| \sim c_* (\log z)^{w(y)}, \quad \text{where} \quad c_* = \frac{1}{\prod\limits_{i \in \mathcal{I}(y)} c_i \cdot \big(w(y)\big)!}$$

and

$$|T(\overline{y}) \cap \mathbb{X}(z)| \sim c_{**} (\log z)^{m - w(y)}, \quad \text{where} \quad c_{**} = \frac{1}{\prod\limits_{i \in \mathcal{I}(\overline{y})} c_i \cdot \big(m - w(y)\big)!}$$

Since $w(y) \le m_1$, $m - w(y) \ge m_1 + 1$, then there exists a $z(y)$ for which (5.1) is satisfied.

Case II: $m = 2m_1$

Let $\mathbb{X}_1^* = \{x \in \mathbb{X}^* : w(x) \ge m_1 + 1\}$ and $\mathbb{X}_0^* = \{x \in \mathbb{X}^* : w(x) = m_1\}$.

For every $y \in \mathbb{X}_0^*$ let

$$g(y) = \prod_{i \in \mathcal{I}(y)} c_i \quad \text{with } c_i \text{ defined as in (5.2).}$$

Finally, define $\widetilde{\mathbb{X}}_0^* = \{y \in \mathbb{X}_1^* : g(y) \le g(\overline{y})\}$. If $g(y) = g(\overline{y})$ we take as an element of $\widetilde{\mathbb{X}}_0^*$ one of them, so $|\mathbb{X}_0^*| = \frac{\binom{2m_1}{m_1}}{2}$.

Using the same approach as in the proof of Proposition 1 we get

Proposition 2. *Let* $m = 2m_1$ *be even. There exists a constant* $z_0 = z(r_1, \ldots, r_m)$ *such that for all* $z > z_0$ *an optimal set* $A \in \mathcal{O}(z)$ *is*

$$A = M(\mathbb{X}_1^* \cup \widetilde{\mathbb{X}}_0^*) \cap \mathbb{X}(z) = \bigcup_{x \in \mathbb{X}_1^* \cup \widetilde{\mathbb{X}}_0^*} T(y) \cap \mathbb{X}(z)$$

and, if $g(y) \neq g(\overline{y})$ *for all* $y \in X_0^*$ *, then the optimal set is unique.*

From these Propositions follows that for *finite* sets Q of quasi–primes, for all sufficiently large z , the optimal intersecting sets are *not stars.*

By choosing Q's of infinitely many quasi–primes, which are sufficiently far from each other, say $r_{i+1} > \exp(r_i)$, (details are omitted), one can make sure, that again for all sufficiently large z , the optimal intersecting sets are *never stars.*

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