A NEW DIRECTION IN EXTREMAL THEORY

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1. Introduction

The notion of an antichain in a partially ordered set was generalized in [1] and [2] to the seemingly natural and useful notion of a "cloud–antichain" $(\mathcal{A}_i)_{i=1}^N$.

Whereas in antichains elements of a partially ordered set are compared in cloud–antichains sets of elements take their role. Elements in different sets $\mathcal{A}_i$, called clouds, are required to be incomparable. More formally, for every two clouds $\mathcal{A}_i$ and $\mathcal{A}_j$ we have

$$A_i \not> A_j \ \text{ for } \underline{\text{all}} \ A_i \in \mathcal{A}_i \ \text{ and } \underline{\text{all}} \ A_j \in \mathcal{A}_j \ . \tag{1.1}$$

This concept is a degree more sophisticated than those usually studied. Its logical structure suggests to call it of type $(\forall, \forall)$. Clearly, this makes us also curious about definitions of the types $(\exists, \forall), (\forall, \exists)$, and $(\exists, \exists)$. In order to test whether there is any substance to these speculations about concepts, we study them here in connection with the simple notion of adjacency of edges in a graph.

Amazingly, this leads to several new extremal problems. We hope that readers find some of the solutions as beautiful as they appear to us. Extensions to hypergraphs (in fact already to k–uniform hypergraphs) constitute a formidable program.

A very special case of the Erdös–Ko–Rado Theorem is the statement that a graph $G_{n,N}$ with n vertices and N edges, which have pairwise a common vertex (that is, are pairwise adjacent), satisfies

$$N \leq n-1, \text{ if } \ n > 3 \ . \tag{1.2}$$

This fact is of course obvious and so is the fact that the optimal value of N is assumed for stars (in the terminology of [10]).

We introduce now our new problems. It is often convenient to view edges as two–element sets, that is as elements of $\binom{\mathcal{V}_n}{2}$, where $\mathcal{V}_n = \{1, 2, \dots, n\}$.

Thus edges are adjacent if the sets intersect. In the sequel $(\mathcal{A}_i)_{i=1}^N$ is always a family of disjoint subsets of $\binom{\mathcal{V}_n}{2}$. It is said to be of type $(\forall, \forall, I)$, if for all $i \neq j$

$$A_i \cap A_j \neq \varnothing \ \text{ for } \underline{\text{all}} \ A_i \in \mathcal{A}_i \ \text{ and } \underline{\text{all}} \ A_j \in \mathcal{A}_j \ . \tag{1.3}$$

Similarly, $(\mathcal{A}_i)_{i=1}^N$ is of type $(\exists, \forall, I)$, if for all $i \neq j$

$$\text{there } \underline{\text{exists}} \text{ an } \ A_i \in \mathcal{A}_i \ \text{ such that for } \underline{\text{all}} \ A_j \in \mathcal{A}_j A_i \cap A_j \neq 0, \tag{1.4}$$

it is of type $(\forall, \exists, I)$, if for all $i \neq j$

$$\text{for } \underline{\text{all}} \ A_i \in \mathcal{A}_i \ \text{ there exists an } \ A_j \in \mathcal{A}_j \ \text{ with } \ A_i \cap A_j \neq \varnothing, \tag{1.5}$$

and it is of type $(\exists, \exists, I)$, if for all $i \neq j$

$$\text{there } \underline{\text{exists}} \text{ an } A_i \in \mathcal{A}_i \text{ and there } \underline{\text{exists}} \text{ an } A_j \in \mathcal{A}_j \text{ with } A_i \cap A_j \neq \varnothing \ . \tag{1.6}$$

We also speak of $(\forall, \forall)$–intersecting systems, etc. The maximal cardinalities N of such systems are denoted by $I_n(\forall, \forall)$, $I_n(\exists, \forall)$, $I_n(\forall, \exists)$ and $I_n(\exists, \exists)$, resp. Here the first quantity is readily seen to equal the maximal cardinality I_n of the usual intersecting system.

The other three quantities are investigated in Section 3. The first two of them are determined exactly (Theorem 1, 2). The third grows like $n^{3/2}$ (Theorem 3). A seemingly small change in our definitions, namely, replacement of non–disjointness conditions $A_j \cap A_j \neq \varnothing$ by disjointness conditions $A_i \cap A_j = \varnothing$ in the definitions above leads to new notions of disjoint systems, whose maximal cardinalities are denoted by $D_n(\exists, \forall)$, $D_n(\forall, \exists)$, and $D_n(\exists, \exists)$. We have determined the growth of these extremal values in Section 4 (Theorems 4, 5, and 6).

In the course of our investigations further notions arose. We speak of the type $(\exists, \forall, I')$, if (1.4) holds only <u>one–sided</u>, that is for at least one of the pairs (i,j) and (j,i) . By analogy the types $(\forall, \exists, I')$ and $(\exists, \forall, D')$, $(\forall, \exists, D')$ are defined. The extremal cardinalities are denoted by $I'_n(\exists, \forall)$, $I'_n(\forall, \exists)$, $D'_n(\exists, \forall)$, and $D'_n(\forall, \exists)$. In Section 5 we comment on these functions. As a Corollary to Theorems 5, 6 we obtain the asymptotic growth of $D'_n(\forall, \exists)$. We have no idea about $D'_n(\exists, \forall)$ which goes beyond the inequalities $D_n(\exists, \forall) \leq D'_n(\exists, \forall) \leq D'_n(\forall, \exists)$. Concerning $I'_n(\exists, \forall)$ and $I'_n(\forall, \exists)$ we present two constructions, which we believe to be optimal.

Finally, we emphasize that there are strong connections of our concepts to those which have been developped in the context of so called "Intersection Theorems" (cf. [11], [12], [10]), the successor of [7]. There are even connections to known theorems in special cases, however, generally our concepts seem to go in new directions.

2. Auxiliary Results

In this paper we essentially start from first principles. With the notion of clouds we continue the terminology of [1] and [2]. This and other concepts used are such that they directly can be generalized to hypergraphs, in particular to the uniform hypergraphs defined by the k–element subsets of an n–set . We use the well–known fact that for every prime power p^m there is a projective plane of order p^m , which has $n = p^{2m} + p^m + 1$ points and lines.

Frequently we use the knowledge of the edge chromatic number of the complete graphs which is based on the matchings $P_i = \{(x,y) : x + y \equiv i \mod n\}$. Even though this is a rather simple case of Baranyai's Theorem, <u>we always refer to that Theorem</u>, because we want to suggest, what we believe to be, the right intuition for an understanding of the presently open extensions of our results to hypergraphs.

General form of Baranyai's Theorem

Let $n_1, \dots, n_t$ be natural numbers such that $\sum_{i=1}^t n_i = \binom{n}{k}$, then for $\mathcal{V}_n = \{1, 2, \dots, n\}$ $\binom{\mathcal{V}_n}{k}$ can be partitioned into disjoint sets $P_1, \dots, P_t$ such that $|P_i| = n_i$ and each $\ell \in \mathcal{A}$ is contained in exactly $\lceil \frac{n_i \cdot k}{n} \rceil$ or $\lfloor \frac{n_i \cdot k}{n} \rfloor$ members of P_i .

Finally we use a result of Erdös and Hanani [5]. If $A(n, 2k-2, k)$ is the maximal cardinality of a family of k–element subsets of $\mathcal{V}_n$, which pairwise intersect in at most one point (or, equivalently, have symmetric difference of size at least $2k-2$), then

$$\lim_{n \to \infty} A(n, 2k-2, k) n^{-2} = \frac{1}{k(k-1)} \ .$$

(Improvements of this result are presented in the language of codes on page 529 of [8]).

3. Intersecting Systems

Theorem 1:

$$I_n(\exists, \forall) = \begin{cases} n-1 & \text{for } n \in \mathbb{N} - \{3, 5\} \\ n & \text{for } n = 3, 5 \ . \end{cases}$$

Proof: Let us consider a cloud system $\mathcal{A}_1, \dots, \mathcal{A}_N$ of type $(\exists, \forall, I)$. If $|\mathcal{A}_i| = 1$ for $i = 1, \dots, N$, then $N \le n-1$ except for $n = 3$, where the bound is 3. Suppose now that $|\mathcal{A}_1| \ge 2$ and that for $A_{1i} \in \mathcal{A}_i$ $(i = 2, \dots, N)$

$$A_{1i} \cap X \ne \varnothing \ \forall X \in \mathcal{A}_1 \ . \tag{3.1}$$

We distinguish between two cases

$\underline{\exists B, C \in \mathcal{A}_1 \text{ with } B \cap C \ne \varnothing :}$ Now at most $1 + n - 3$ edges are adjacent with B <u>and</u> C . Thus we have $N \le 1 + 1 + n - 3 \le n - 1$.

$\underline{\exists B, C \in \mathcal{A}_1 \text{ with } B \cap C = \varnothing :}$ Now only 4 edges are adjacent with B and C , and we have $N \le 5$.

In all cases $I_n(\exists, \forall) \le n - 1$ for $n \ge 6$ and the reverse inequality is true, because $n - 1 \le I_n(\forall, \forall) \le I_n(\exists, \forall)$. The cases $n = 1, 2, 3$ and 4 are settled by inspection. It remains to be seen that $I_5(\exists, \forall) \ge 5$. Application of Baranyai's Theorem with the parameters $n = 5, k = 2, n_i = 2$ for $i = 1, \dots, 5$ gives disjoint clouds $\mathcal{A}_i = P_i (i = 1, 2, \dots, 5)$ with non–adjacent members. The system $(\mathcal{A}_i)_{i=1}^5$ is readily verified to be of type $(\exists, \forall, I)$.

The next result is deeper.

Theorem 2:

$$I_n(\forall,\exists) = \begin{cases} n & \text{for } \ n \in \mathbb{N} - \{1,2,4\} \\ n-1 & \text{for } \ n = 1,2,4 \ . \end{cases}$$

Proof: We establish first the inequality

$$I_n(\forall,\exists) \leq n \ . \tag{3.2}$$

Let $(\mathcal{C}_i)_{i=1}^N$ be an $(\forall,\exists,I)$–system in $\mathcal{V}_n = \{1,2,\dots,n\}$. To every cloud $\mathcal{C}_i$ we associate the vertex set C_i covered by all edges, that is,

$$C_i = \bigcup_{C \in \mathcal{C}_i} C \ . \tag{3.3}$$

We also define for $x \in \mathcal{V}_n$

$$J(x) = \{i : 1 \leq i \leq N, x \in C_i\}\,, \tag{3.4}$$

$$\mathcal{L}(x) = \{\mathcal{C}_i : i \in J(x)\} \ . \tag{3.5}$$

The key observation is

$$|\mathcal{L}(x)| \le \left| \bigcap_{i\in\overline{J}(x)} C_i \right| = n - \left| \bigcup_{i\in\overline{J}(x)} \overline{C}_i \right| \ . \tag{3.6}$$

Here the bar stands for complementation of a set in its ground set.

To verify (3.6) notice first that for $j \in J(x)$ there is a y_j with $\{x, y_j\} \in \mathcal{C}_j$ and that by disjointness of the clouds these y_j–'s are distinct. We have therefore

$$|\mathcal{L}_x| = |\{y_j : j \in J(x)\}| \ .$$

Next, by the $(\forall, \exists, I)$–property for every $j \in J(x)$ and every $i \in \overline{J}(x)$ there is a $\{u, v\} \in \mathcal{C}_i$ with $\{x, y_j\} \cap \{u, v\} \neq \varnothing$ and since $x \notin \{u, v\}$ necessarily $y_j \in \{u, v\}$. This yields $\{y_j : j \in J(x)\} \subset C_i$ for $i \in \overline{J}(x)$ and thus (3.6).

Using the abbreviation $\mu_x \triangleq \left|\bigcap_{i\in\overline{J}(x)} \overline{C}_i\right|$ we can write (3.6) in the form

$$N - \left|\overline{J}(x)\right| \le n - \mu_x \ . \tag{3.7}$$

If $\mu_x = 0$, then $|J(x)| = N$ and all clouds contain an edge with vertex x . In this case therefore $N \le n - 1$ and (3.2) holds. If $\mu_x \neq 0$ for all $x \in \mathcal{V}_n$, then we derive from (3.7)

$$\sum_x \frac{1}{\mu_x} \left(N - |\overline{J}(x)|\right) \le \sum_x \frac{1}{\mu_x} n - n \ .$$

Since $\sum_x \frac{1}{\mu_x} \left|\overline{J}(x)\right| = \sum_{i=1}^N \sum_{x\in\overline{C}_i} \frac{1}{\mu_x} \le \sum_{i=1}^N \sum_{x\in\overline{C}_i} \left(\overline{C}_i\right)^{-1} = N$ by definition of μ_x , we conclude that

$$\sum_x \frac{1}{\mu_x} N - N \le \sum_x \frac{1}{\mu_x} n - n \ . \tag{3.8}$$

Finally, it follows from $\mu_x \le n$ for all $x \in \mathcal{V}_n$ that $\sum_x \frac{1}{\mu(x)} \ge 1$.

Moreover, $\sum_x \frac{1}{\mu(x)} - 1 > 0$ and thus (3.8) implies (3.2) unless $|\mathcal{L}_x| = 0$ for all $x \in \mathcal{V}_n$. In the latter case there cannot be any cloud.

We construct now $(\forall, \exists, I)$–systems achieving the bounds claimed.

<u>Case $n = 2\ell + 1$:</u>

We construct clouds with the help of Baranyai's Theorem (Section 2) for the parameters $n_i = \ell$ for $i = 1, \dots, t$ with $t = \binom{2\ell+1}{2}\ell^{-1}$, $k = 2$ as follows

$$\mathcal{A}_i = P_i,\ 1 \le i \le t\ . \tag{3.9}$$

Since there is exactly one point not covered by $\mathcal{A}_i$ the system $(\mathcal{A}_i)_{i=1}^t$ is of type $(\forall, \exists, I)$. We achieve the desired bound, because $t = \frac{(2\ell+1)2\ell}{2\cdot\ell} = n$.

<u>Case $n = 2\ell$:</u>

Here the previous approach gives only $t = 2\ell - 1 = n - 1$. This is optimal only for $n = 2$ and, as inspection shows, for $n = 4$. For the other cases there is a new construction based on a concatenation argument. To every $(\forall, \exists, I)$–system in $\{1, 2, \ldots, m\}$ we can associate an $(\forall, \exists, I)$–system in $\{1, 2, \ldots, 2m\}$ which has twice as many clouds. They are obtained from the original clouds $\mathcal{A}_1, \ldots, \mathcal{A}_N$ as follows:

$$\{x, y\} \to \begin{array}{l} \{\{2x-1, 2y-1\}, \{2x, 2y\}\}, \\ \{\{2x-1, 2y\}, \{2x, 2y-1\}\}, \end{array} \tag{3.10}$$

that is an edge $\{x, y\}$ occuring in a cloud is replaced by two edges $\{2x-1, 2y-1\}, \{2x, 2y\}$. Thus a cloud $\mathcal{A}_i$ is transformed into a new cloud $\mathcal{A}'_{2i-1}$. The clouds $\mathcal{A}'_{2i}$ are obtained by using the second replacement in (3.10). The replacement rules ensure that $\left(\mathcal{A}'_j\right)_{i=1}^{2N}$ is an $(\forall, \exists, I)$–system .

We know now that $I_m(\forall, \exists) = m$ implies $I_{2m}(\forall, \exists) = 2m$. This and the result of the previous case settle the problem for all numbers divisible by an odd prime.

The remaining cases $n = 2^\ell, \ell \ge 3$, were settled if we know that $I_8(\forall, \exists) = 8$. Here we have the following construction. There are 8 clouds all having 3 edges. The 4 edges not used partition $\mathcal{V}_8$. Here we choose for that partition $\{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}\}$.

The clouds are

$$\begin{array}{ll}
\mathcal{C}_1 = \{\{1,3\}, \{2,5\}, \{4,6\}\} & \mathcal{C}_5 = \{\{1,6\}, \{2,8\}, \{5,7\}\} \\
\mathcal{C}_2 = \{\{1,5\}, \{2,4\}, \{3,6\}\} & \mathcal{C}_6 = \{\{1,7\}, \{2,6\}, \{5,8\}\} \\
\mathcal{C}_3 = \{\{1,4\}, \{2,7\}, \{3,8\}\} & \mathcal{C}_7 = \{\{3,5\}, \{4,8\}, \{6,7\}\} \\
\mathcal{C}_4 = \{\{1,8\}, \{2,3\}, \{4,7\}\} & \mathcal{C}_8 = \{\{3,7\}, \{4,5\}, \{6,8\}\}
\end{array}$$

and their associated vertex sets are

$$\begin{array}{l}
C_1 = C_2 = \mathcal{V}_8 \smallsetminus \{7,8\},\ C_3 = C_4 = \mathcal{V}_8 \smallsetminus \{5,6\} \\
C_5 = C_6 = \mathcal{V}_8 \smallsetminus \{3,4\},\ C_7 = C_8 = \mathcal{V}_8 \smallsetminus \{1,2\}\ .
\end{array}$$

Remarks

1.) The above arguments also give a closely related result of independent interest. Let M_n be the maximal number N for which there exist graphs with n vertices admitting a proper edge–coloring with N colors such that one cannot change the color of a single edge without destroying the coloring, then clearly $M_n \le I_n(\forall, \exists)$.

Moreover, since our constructions in the proof of Theorem 2 use only clouds with disjoint edges, also the opposite inequality holds.

Theorem 2':

$$M_n = \begin{cases} n & \text{for } \ n \in \mathbb{N} - \{1,2,4\} \\ n-1 & \text{for } \ n = 1,2,4 \ . \end{cases}$$

2.) If we allow clouds to consist of edges <u>and vertices</u> and consider again systems with the $(\forall, \exists, I)$–property , then we can study their maximal cardinalities $I_n^*(\forall, \exists)$. We have

$$I_n^*(\forall, \exists) = n \ \text{ for all } \ n \in \mathbb{N} \ . \tag{3.11}$$

Proof: For every n the value n is assumed by the system $\{\{1\}\}, \{\{1,2\}\}, \ldots, \{\{1,n\}\}$.

Clearly, if no cloud uses a vertex, then Theorem 1 applies and gives the bound n , and if a vertex, say x , is in some cloud, then all other clouds contain an edge $\{x,y\}$ and again the number of clouds cannot exceed n .

Whereas the quantities considered until now grow linearly in n next we obtain an $n^{3/2}$–law .

Theorem 3: $\lim_{n\to\infty} I_n(\exists, \exists) n^{-\frac{3}{2}} = 1$.

Proof: Let $(\mathcal{C}_i)_{i=1}^N$ be an $(\exists, \exists, I)$–system in $\mathcal{V}_n$. Since an edge is adjacent with $2(n-2)$ edges we have the inequality

$$N \le |\mathcal{C}_i| \, 2(n-2) \ \text{ for all } \ i = 1,2,\ldots,N \ . \tag{3.12}$$

Furthermore, since the total number of edges is $\binom{n}{2}$, we also have

$$\min_i |\mathcal{C}_i| \le \binom{n}{2} N^{-1} \ . \tag{3.13}$$

From (3.12) and (3.13) we deduce $N \le \binom{n}{2} N^{-1} 2(n-2)$ and hence

$$N \le [n(n-1)(n-2)]^{1/2} \le n^{3/2} \ . \tag{3.14}$$

We provide now the construction, which asymptotically achieves this bound. It uses the existence of projective planes of prime power order and again Baranyai's Theorem.

Let $k = p^m (p \ne 2)$, $n = k^2 + k + 1$, and let $L_1, \ldots, L_n$ be the lines in the projective plane with n points. We use the facts

$$|L_i| = k+1 \ \text{ for } \ i = 1,2,\ldots,n \tag{3.15}$$

$$|L_i \cap L_j| = 1 \ \text{ for } \ i \ne j \ . \tag{3.16}$$

Since $k+1$ is even, by Baranyai's Theorem we can partition the set of all $\binom{k+1}{2}$ edges in L_i into k clouds of $\frac{k+1}{2}$ edges each. These clouds have among each other the desired intersection property. Since all clouds live on exactly all $k+1$ points of a line, by (3.16) also two clouds living on different lines have the intersection property. Now the number of clouds N satisfies

$$N = k \cdot n \tag{3.17}$$

and since $k = \sqrt{n - \frac{3}{4}} - \frac{1}{2}$, we have $N = n\left(\sqrt{n - \frac{3}{4}} - \frac{1}{2}\right)$.

Now for arbitrary, but sufficiently large n , we use a familiar argument based on the following density property of primes:

For all sufficiently large m there is a prime $p(m)$ between $m - \frac{1}{10}m^{2/3}$ and m . This and sharper results are stated and quoted on page XX of [9]. Choose now m such that

$$(m+1)^2 + (m+1) + 1 \geq n \geq m^2 + m + 1 \ .$$

Since $n \geq p(m)^2 + p(m) + 1$, since $p(m) \geq m - \frac{1}{10}m^{2/3} \gtrsim \sqrt{n}$ and since $I_n(\exists, \exists)$ is monotonically increasing in n we conclude that

$$I_n(\exists, \exists) \geq p(m)(p(m)^2 + p(m) + 1) \stackrel{\sim}{>} n^{3/2} \ .$$

This and (3.14) imply the result.

4. Disjoint Systems

We state first the main results of this Section.

Theorem 4: $\lim_{n\to\infty} D_n(\exists, \forall)n^{-2} = \frac{1}{6}$.

Theorem 5: $\lim_{n\to\infty} D_n(\forall, \exists)n^{-2} = \frac{1}{4}$.

Theorem 6: $\lim_{n\to\infty} D_n(\exists, \exists)n^{-2} = \frac{1}{4}$.

A. The upper bounds

Lemma 1: $D_n(\exists,\exists) \le \frac{n^2}{4}$.

Proof: In an $(\exists,\exists,D)$–system $\mathcal{C}_1,\dots,\mathcal{C}_N$ there are at most $\lfloor \frac{n}{2} \rfloor$ clouds with one edge only. Therefore

$$\left\lfloor \frac{n}{2} \right\rfloor + 2\left(N - \left\lfloor \frac{n}{2} \right\rfloor\right) \le \binom{n}{2}$$

and hence $N \le \frac{n^2}{4} - \frac{n}{4} + \frac{1}{2}\lfloor \frac{n}{2} \rfloor \le \frac{n^2}{4}$.

Lemma 2: $D_n(\exists,\forall) \le \frac{n^2}{6}$.

Proof: For an $(\exists,\forall,D)$–system $(\mathcal{C}_i)_{i=1}^N$ let the labelling be chosen such that the first N_1 clouds have exactly one edge, the next N_2 clouds have exactly two edges, etc.

We know that

$$N_1 \le \left\lfloor \frac{n}{2} \right\rfloor \ . \tag{4.1}$$

Next we consider $(\mathcal{C}_i^*)_{i=N_1+1}^{N_1+N_2}$, where

$$\mathcal{C}_i^* = \binom{C_i}{2},\ C_i = \bigcup_{X\in\mathcal{C}_i} X \ , \tag{4.2}$$

and notice that these clouds form again an $(\exists,\forall,D)$–system .

It uses $\sum_{i=N_1+1}^{N_1+N_2} \left|\binom{C_i}{2}\right| \ge 3\ N_2$ edges.

Furthermore, one readily verifies that

$$\binom{C_i}{2} \cap \mathcal{C}_j = \varnothing \ \text{ for } \ i \in \{N_1+1,\dots,N_1+N_2\} \ \text{ and}$$
$$j \notin \{N_1+1,\dots,N_1+N_2\} \ . \tag{4.3}$$

We conclude that $N_1 + 3\ N_2 + \sum_{j\ge 3} j\ N_j \le \binom{n}{2}$ and by (4.1) $3\sum_{j\ge 1} N_j \le \binom{n}{2} + n$. Thus $N \le \frac{n^2}{6}$.

B. Constructions yielding the lower bounds

From the Theorem by Erdös and Hanani we know that

$$\lim_{n\to\infty} A(n,4,3)n^{-2} = \frac{1}{6} \ . \tag{4.4}$$

For any family $\mathcal{A}$ with parameters $n, 4$ and 3 it follows from $|A \triangle A'| \geq 4$ for $A, A' \in \mathcal{A}$ that the system of clouds $\left\{\mathcal{C} = \binom{C}{2} : C \in \mathcal{A}\right\}$ associated with $\mathcal{A}$ is of $(\exists, \forall, D)$–type .

Therefore

$$D_n(\exists, \forall) \geq A(n,4,3) \tag{4.5}$$

and Theorem 4 follows from (4.4), (4.5) and Lemma 2.

The proofs of Theorem 5 and 6 will be complete when we have shown that

$$\underline{\lim}_{n\to\infty} D_n(\forall, \exists)n^{-2} \geq \frac{1}{4} \ . \tag{4.6}$$

We construct now an $(\forall, \exists, D)$–system yielding this result. It suffices to consider values for n of the form

$$n = k \cdot u^2, \ k = 4t \tag{4.7}$$

with k and u tending to infinity. Partition now $\mathcal{V}_n$ into sets $W_1, \dots, W_k$ such that all have a cardinality u^2 . Let $\mathcal{W}_{k,u} = \{W_i : 1 \leq i \leq k\}$ be the vertex set of a complete graph. By Baranyai's Theorem its set of edges can be partitioned into sets $E_1, \dots, E_{k-1}$ of disjoint edges such that

$$|E_s| = \frac{k}{2} \ \text{ for } \ s = 1, 2, \dots, k-1 \ . \tag{4.8}$$

Since k is divisible by 4, we can define a partition F_s of E_s into non–adjacent pairs of edges.

Thus the members of F_s are of the form $\{(W_i, W_j), (W_\ell, W_m)\}$ with all four indices being different and $i < j, \ell < m$. Clearly, the elements of $W_i \times W_j$ etc. are edges in $\mathcal{V}_n$.

Below we match every edge in $W_i \times W_j$ with an edge in $W_\ell \times W_m$ and let two matched edges form a cloud. This will be done for all members of F_s and all s .

Certain omissions are then necessary to convert this system of clouds into an $(\forall, \exists, D)$–system . These matchings are constructed via the following basic bijection. Let $\varphi, \psi : \{1, 2, \dots, u^2\} \times \{1, 2, \dots, u^2\} \to \{1, 2, \dots, u^2\}$ be two maps defined by

$$\left.\begin{array}{l}\varphi(a,b)=pu+q+1\\ \psi(a,b)=(x-1)u+y\end{array}\right\}\ \text{if}\ \left\{\begin{array}{l}a=pu+x,1\le x\le u\\ b=pu+y,1\le y\le u\end{array}\right. . \tag{4.9}$$

Label now the elements in $\mathcal{V}_n$ such that

$$W_i=\{(i,s):1\le s\le u^2\}\ ;\ i=1,2,\dots,k\ ; \tag{4.10}$$

and define for all quadruples of indices occuring $f:W_i\times W_j\to W_\ell\times W_m$ by

$$f\left((i,s),(j,t)\right)=\left((\ell,\varphi(s,t)),(m,\psi(s,t))\right) \tag{4.11}$$

The $(\forall,\exists,D)$–property must now hold for all pairs of clouds.

Three situations arise. In the case both clouds have an edge from $W_i\times W_j$ and an edge from $W_\ell\times W_m$ the property obviously holds. The same is the case if the clouds are based on two different members of the same F_s . Finally we have to investigate the case where one cloud is based on $\{\{W_i,W_j\},\{W_\ell,W_m\}\}\in F_s$ and the other on $\{\{W_i',W_j'\},\{W_\ell',W_m'\}\}\in F_{s'}$ $(s\ne s')$.

The $(\forall,\exists,D)$–property can be violated only if

$$\begin{array}{c}\{i',j'\}\subset\{i,j,\ell,m\}\ \text{ or }\ \{\ell',m'\}\subset\{i,j,\ell,m\}\ \text{ or}\\ \{i,j\}\subset\{i',j',\ell',m'\}\ \text{ or }\ \{\ell,m\}\subset\{i',j',\ell',m'\}\ .\end{array}$$

It suffices to consider a case like

$$\begin{array}{l}\{(1,a),(2,b)\},\{(3,\varphi(a,b)),(4,\psi(a,b))\},\\ \{(1,a'),(3,b')\},\{(\cdot,\cdot)\}\end{array} \tag{4.12}$$

with $a=a'$ and $b'=\varphi(a,b)$.

For fixed a , how many pairs (a,b') have to be excluded in $\{1,\dots,u^2\}\times\{1,\dots,u^2\}$?

Clearly, the conflict in (4.11) is resolved, if for every $a\in\{1,\dots,u^2\}$ we omit the set

$$G_\varphi(a)=\left\{(a,b')=(a,\varphi(a,b)):b\in\{1,\dots,u^2\}\right)\ .$$

If we write $a=pu+x$, $b=qu+y$, $\varphi(a,b)=pu+q+1$, then we see that

$$|G_\varphi(a)|=u\ . \tag{4.13}$$

From the discussion above we know that there are 4 cases like the one in (4.12). In total we have to exclude at most $4\cdot u^2\cdot u$ elements from the u^4 elements of $\{1,2,\dots,u^2\}\times\{1,2,\dots,u^2\}$.

The total number of clouds left is therefore
$(u^4 - 4u^3)\frac{k}{4}(k-1) = \frac{n^2}{4} - \frac{u^4 \cdot k}{4} - u^3 k(k-1) \geq \frac{n^2}{4}\left(1 - \frac{1}{k} - \frac{4}{u}\right)$, which implies (4.6).

Remark. A result of [12], Corollary 8.7 in [13], implies (4.6). This can readily be verified by choosing the A in [13] to consist of two non–adjacent edges. For k–uniform hypergraphs $(k > 2)$ there does not seem to be such a simple connection. Also, our proof is constructive and the proof in [12], being based on random selection, obviously is not.

5. Remarks about non–symmetric relations

The inequalities

$$D_n(\forall,\exists) \leq D'_n(\forall,\exists) \leq D_n(\exists,\exists) \tag{5.1}$$

and Theorem 5 and 6 imply

Corollary 7: $\lim_{n\to\infty} D'_n(\forall,\exists)n^{-2} = 1/4$.

We turn to the 1–sided intersection property

Conjecture 8:

$$I'_n(\exists,\forall) = \begin{cases} n-1 & \text{for } \ n=1,2 \\ n & \text{for } \ n=3,4 \\ n+2 & \text{for } \ n=5 \\ n+1 & \text{for } \ n\geq 6 \ . \end{cases}$$

Constructions achieving lower bounds:

The cases $n=1,2,3,4$ being obvious we desribe the constructions for $n=5$ and for $n>5$.

$\underline{n=5:}$ $\mathcal{C}_i = \{5,i\}$ for $i=1,2,3,4$ and $\mathcal{C}_5,\mathcal{C}_6,\mathcal{C}_7$ are the sets of edges defined by Baranyai–type partitions of $\{1,2,3,4\}$ into disjoint edges. For larger values of n these clouds based on partitions no longer have mutually the $(\exists,\forall,I')$ property. That is the reason why the number five plays a special role.

$\underline{n>6:}$ Choose $\mathcal{C}_i = \{n,i\}$ for $i=1,2,\dots,n-1$ and
$\mathcal{C}_n = \{\{1,i\}:1,2,\dots,j\}\cup\{\{n-1,i\}:i=j+1,\dots,n-2\}$ for $1<j<n-2$.

Finally set $\mathcal{C}_{n+1} = \binom{\mathcal{V}_2}{2} - \bigcup_{i=1}^n \mathcal{C}_i$.

Actually we can prove optimality of these configurations, too.

However, a formal proof takes an amount of writing which seems unproportional to the significance of this result. Having not written a formal proof we cannot state more than a conjecture.
Finally for $n\geq 3$ we have the

Conjecture 9:

$$I'_n(\forall,\exists) = \begin{cases} 2n-3, & \text{if } \ n \ \text{ is odd} \\ 2n-4, & \text{if } \ n \ \text{ is even .} \end{cases}$$

Constructions achieving these values are as follows.

$\underline{n = 2m+1:}$ Let $P_1, \dots, P_{2m-1}$ be partitions of $\{1, 2, \dots, 2m\}$ into disjoint edges according to Baranyai. Define $\mathcal{C}_i = P_i$ for $i = 1, \dots, 2m-1$ and $\mathcal{C}_{2m-1+j} = \{\{n, j\}\}$ for $j = 1, \dots, n-1$.

Then $(\mathcal{C}_i)_{i=1}^{2n-3}$ is an $(\forall, \exists, I')$–system .

$\underline{n = 2m:}$ Let $P_1, \dots, P_{2m-3}$ be now the usual partitions of $\{1, 2, \dots, 2m-2\}$ in this case. Let now $\mathcal{C}_i = P_i$ for $i = 1, \dots, 2m-3$ and $\mathcal{C}_{2m-3+j} = \{\{n, j\}\}$ for $j = 1, \dots, 2m-2$. Finally set $\mathcal{C}_{2n-4} = \{\{n-1, \ell\} : \ell \neq n-1\}$. Now again $(\mathcal{C}_i)_{i=1}^{2n-4}$ is an $(\forall, \exists, I')$–system .

References

[1] R. Ahlswede and Z. Zhang, "An identity in combinatorial extremal theory", Advances in Mathematics, 80, 2, 137–151, 1990.

[2] R. Ahlswede and Z. Zhang, "On cloud–antichains and related configurations", Discrete Mathematics 85, 225–245, 1990.

[3] Z. Baranyai, "On the factorization of the complete uniform hypergraph", Colloquia Mathematica Societatis, János Bolyai 10. Infinite and finite sets, Keszthely, Hungary, 91–108, (eds.: A. Hajnal, R. Rado, V.T. Sós), 1973.

[4] M.K. Fort jr., and G.A. Hedlund, "Minimal coverings of pairs by triples", Pacific J. Math. 8, 709–719, 1958.

[5] P. Erdös and H. Hanani, "On a limit theorem in combinatorial analysis", Publ. Math. Debrecen 10, 10–13, 1963.

[6] J. Schönheim, "On maximal systems of k–tuples ", Stud. Sci. Hung. 1, 363–368, 1966.

[7] P. Erdös, Chao Ko and R. Rado, "Intersection theorems for systems of finite sets", Quart. J. Math. Oxford (2), 12, 313–320, 1961.

[8] F.J. Mac Williams and N.J.A. Sloane, "The Theory of Error–Corecting Codes", North–Holland, 1977.

[9] B. Bollobás, "Extremal Graph Theory", Academic Press, 1978.

[10] R. Ahlswede, G. Katona, "Graphs with maximal number of adjacent pairs of edges", Acta Math. Acad. Sc. Hung. 32, 1–2, 97–120, 1978.

[11] M. Simonovits and V.T. Sós, "Intersection theorems on structures", Annals of Discrete Math. 6, 301–313, 1980.

[12] P. Frankl and Z. Füredi, "Colored packing of sets", Annals of Discrete Math. 34, 165–178, 1987.

[13] Z. Füredi, "Matching and covers in hypergraphs", Graphs and Combinatorics 4, 115–206, 1988.