General edge–isoperimetric inequalities

Part II. A local–global principle for lexicographical solutions

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1 Introduction

The lexicographical order $\mathcal{L}$ on a sequence space $\mathcal{X}^n = \{0, 1, \ldots, \alpha\}^n$, defined by $x^n \underset{\mathcal{L}}{\to<} y^n$ iff there exists a t such that $x_t < y_t$ and $x_s = y_s$ for $s < t$, is one of the most important and frequently encountered orders in combinatorial extremal theory. An early result in this area, Harper's solution of an edge–isoperimetric problem (EIP) in binary Hamming space ([13]) (generalized in [16] to nonbinary cases and rediscovered many times, e.g. [6], [9], [15]) says that first segments in $\mathcal{L}$ are optimal.

There are two kinds of EIP. They can be represented as extremal problems in graph theory. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. For any $A \subset \mathcal{V}$ define the set $\mathcal{B}(A)$ of all boundary edges, that is,

$$\mathcal{B}(A) = \Big\{\{x, y\} \in \mathcal{E} : |\{x, y\} \cap A| = 1\Big\} \tag{1.1}$$

and the set $\mathcal{L}(A)$ of all inner edges, that is,

$$\mathcal{I}(A) = \Big\{\{x, y\} \in \mathcal{E} : x, y \in A\Big\}. \tag{1.2}$$

1. Boundary–edge–isoperimetric problem (BEIP):

For given graph and positive integer m find a set $A \subset \mathcal{V}$ of cardinality m with minimal possible value of $|\mathcal{B}(A)|$.

2. Inner–edge–isoperimetric problem (IEIP):

For given graph and positive integer m find a set $A \subset \mathcal{V}$ with maximal possible value of $|\mathcal{I}(A)|$.

Notice that for regular graphs of degree d

$$|\mathcal{B}(A)| + 2|\mathcal{I}(A)| = d|A|$$

and that therefore the two problems are equivalent in the sense that a solution of one of these problems is at the some time a solution of the other.

We concentrate here on EIP's of the Cartesian sum graphs

$$G^n = G_1 \times G_2 \times \cdots \times G_n = (\mathcal{X}^n, \mathcal{E}^n) \tag{1.3}$$

of graphs $G_t = (\mathcal{X}_t, \mathcal{E}_t)$; $t = 1, 2, \ldots, n$; where $\mathcal{X}^n = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n$ and for $x^n = (x_1, \ldots, x_n), y^n = (y_1, \ldots, y_n) \in \mathcal{X}^n$ $\{x^n, y^n\} \in \mathcal{E}^n$ iff there exists a $t \in \{1, 2, \ldots, n\}$ such that for all $t' \neq t$ $x_{t'} = y_{t'}$ and $\{x_t, y_t\} \in \mathcal{E}_t$. Then EIP's in Hamming, Manhattan and Lee metrics can be understood as EIP's of Cartesian sum graphs of complete graphs, paths, and cycles, respectively. We speak of an (optimal) order for an EIP, if the initial segments of this order always achieve the extremal value. Then Harper's Theorem and its generalization show that $\mathcal{L}$ is an oder for the EIP in Cartesian sums of complete graphs.

Notice that the regularity implies that the BEIP and the IEIP become the same here. Of course $\mathcal{L}$ is not always optimal for these EIP's and for many of these problems there is no order at all. One can find such examples for Manhattan and Lee spaces in [1], [5], [8].

On the other hand one can ask

$$\text{“Is } \mathcal{L} \text{ optimal for an EIP in } G^n\text{?”} \tag{1.4}$$

Since there are $\prod_{t=1}^{n} |\mathcal{X}_t|$ orders on $\mathcal{X}^n$ one might expect the complexity of an algorithm deciding this question to be very high. Quite surprisingly, our main discovery, *a local–global principle*, shows that the problem is not NP–hard and not even P–hard. Actually, its complexity is independent of n!

For the convenience of the readers we limit ourselves here in (1.3) to the case of indentical factors, i.e. $G_t = G$ for $t = 1, 2, \dots, n$. We call here G^n the n–th power of G. It is not very hard to extend our main result to general Cartesian sum graphs. In another direction our work is more general. We introduce a fairly large family of set functions on G^n, including “boundary–edge” and “inner–edge” functions. Our local–global principle says that $\mathcal{L}$ is an optimal order for the extremal problems of the functions of this family in n–th power space exactly if it is optimal in the first and the second power spaces. This means that often the question (1.4) can be decided by a simple inspection!

In Section 2 we give the necessary definitions, state known facts from [2] and present a generalization of a lemma from [2]. Our main result (Theorem 1) is presented and proved in Section 3. Finally, as an example demonstrating the power of our local–global principle, we give an edge–isoperimetric theorem for the powers of complete bipartite graphs $C_{m,m}$ (Theorem 2) in Section 4.

2 Preliminaries

2.1 Definitions and known facts

We list all definitions and needed known facts in the first part of this section. The proofs of these facts are not very hard and can be found in [2]. For all $J \subset N \triangleq \{1, 2, \dots, n\}$, $x^n \in \mathcal{X}^n$ denote by x^J the subsequence of x^n obtained by deleting components x_t with $t \notin J$. $\mathcal{X}^J$ is defined analogously. Thus x^n and $\mathcal{X}^n$ can be rewritten as x^N and $\mathcal{X}^N$, respectively. Define for any $A \subset \mathcal{X}^N$ the general slices

$$A_J(x^{N \smallsetminus J}) \triangleq \{x^J \in \mathcal{X}^J : x^N \in A\} \;\text{ for }\; x^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J} \tag{2.1}$$

and the projections

$$A_J \triangleq \bigcup_{x^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J}} A_J(x^{N \smallsetminus J}). \tag{2.2}$$

For $J \subset N$ write the set of the lexicographically first m elements in $\mathcal{X}^J$ as $\mathcal{L}(\mathcal{X}^J, m)$. Then the general pushing down operations under $\mathcal{L}$ on $\mathcal{X}^J$ is defined by

$$D_J(\mathcal{L}, A) = \bigcup_{x^{n \smallsetminus J} \in A_{N \smallsetminus J}} \{y^N : y^{N \smallsetminus J} = x^{N \smallsetminus J} \text{ and } y^J \in \mathcal{L}(\mathcal{X}^J, |A_J(x^{N \smallsetminus J})|)\} \tag{2.3}$$

for all $A \subset \mathcal{X}^N$.

When $J = \{t\}$ we wirte also $D_J = D_t$. $A \subset \mathcal{X}^N$ is a downset, if $y^n \in A$ implies $x^n \in A$ in the case $x_t \leq y_t$ for all t. In other words a down set A of $\mathcal{X}^N$ is a set with $D_t(A) = A$ for all $t \in N$. For a given graph G $|\mathcal{B}(\cdot)|$ in (1.1) and $|\mathcal{I}(\cdot)|$ in (1.2) are functions on the subsets of the vertex set and we write them as $B(G, \cdot)$ and $I(G, \cdot)$. We shall state their essential properties and study them as abstract set functions.

For this goal we define the n–th power function φ^n of a set function φ on $2^{\mathcal{X}} \triangleq \{A : A \subset \mathcal{X}\}$ as a function on $2^{\mathcal{X}^n}$

$$\varphi^n(A) = \sum_{t=1}^{n} \sum_{\mathcal{X}^{N \smallsetminus \{t\}} \in x^{N \smallsetminus \{t\}}} \varphi\big(A_t(x^{N \smallsetminus \{t\}})\big), \tag{2.4}$$

where we abbreviate $A_{\{t\}}(\cdot)$ as $A_t\{\cdot\}$.

One readily verifies that for the n–th power G^n of a graph $G = (\mathcal{X}, \mathcal{E})$ in the sense of (1.3), $\varphi^n(\cdot) = -B(G^n, \cdot)$ (or $I(G^n, \cdot)$) if we let $\varphi(\cdot) = -B(G, \cdot)$ (or $I(G, \cdot)$). Thus EIP's are reduced to maximizing for given φ $\varphi^n(A)$ over all $A \subset \mathcal{X}^n$ with fixed $|A|$. The following properties were proved in [2]:

1) $\max_{|A|=m} \varphi^n(A)$ is assumed on a down set of $\mathcal{X}^n$, if the following conditions are satisfied

   I. (Nestedness): For all $k \in \mathcal{X} = \{0, 1, \ldots, \alpha\}$, $A \subset \mathcal{X}$ with $|A| = k + 1$,

$$\varphi(A) \leq \varphi([k]) \text{ where } [k] = \{0, 1, \ldots, k\}. \tag{2.5}$$

   II. (Submodularity): For $A, B \subset \mathcal{X}$

$$\varphi(A) + \varphi(B) \leq \varphi(A \cup B) + \varphi(A \cap B). \tag{2.6}$$

   III. $\varphi(\phi) = 0$.

   (We can always assume that III holds by replacing φ by φ', where $\varphi'(A) = \varphi(A) - \varphi(\phi)$. Further, obviously III holds for $\varphi(\cdot) = -B(G, \cdot)$ or $I(G, \cdot)$ for all G.)

2) For all graphs G, both $-B(G, \cdot)$ and $I(G, \cdot)$ satisfy II.

3) II implies that for any family $\{A_i\}_{i=1}^m$ of subsets of $\mathcal{X}$

$$\sum_{i=1}^{m} \varphi(A_i) \leq \sum_{k=1}^{m} \varphi\left(\bigcup_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \left(\bigcap_{j=1}^{k} A_{i_j} \right) \right). \tag{2.7}$$

4) Let

$$\triangle_\varphi(k) = \varphi([k]) - \varphi([k-1]), \tag{2.8}$$

(where we set $[-1]$ as empty set.) Then

$$\varphi^n(A) = \sum_{x^n \in A} \sum_{t=1}^{n} \triangle_\varphi(x_t), \tag{2.9}$$

if A is a down set.

Remarks:

1. Condition I (2.5) says exactly that $\mathcal{L}$ is an order for maximizing $\varphi^n(A)$ for fixed $|A|$, when $n = 1$. $-B(G, \cdot)$ $\big(I(G, \cdot)\big)$ satisfies I, if G has nested solutions for BEIP (IEIP) after labeling the vertices properly.

2. Condition II (2.6) is a key issue for pushing down to work. (2.7) is an extension of condition II to more than two sets.

3. By known facts 1), 2) and 4) the EIP's of power graph G^n have been reduced to maximizing $\sum_{x^n \in A} f(x^n)$ over all down sets A of $\mathcal{X}^n$ for a sum–type function $f(x^n) \triangleq \sum_{t=1}^n \triangle_\varphi(x_t)$ (if G has nested solutions for the corresponding EIP's). The importance of the extremal values of $\sum_{x^n \in A} f(x^n)$ over the down sets of $\mathcal{X}^n$ was known to Ahlswede and Katona [4] nearly 20 years ago and the problems for $\mathcal{X} = \{0, 1\}$ were well studied there. In the sequel, we always assume $|\mathcal{X}| \geq 3$. Actually comparing Theorem 1 in Section 3 of the present paper with the solutions of the corresponding problems in the binary case in [4], one may immediately notice that the local–global principle does not extend to binary alphabet. In the sequel we consider maximizing $\varphi^n(A)$ (or $\varphi(A)$) for A of given size and speak of an order for φ^n (or φ).

2.2 A lemma concerning general pushing down operations

In this subsection we generalize Lemma 2 of [2] to general pushing down operations. $\mathcal{L}$ in the following lemma can be replaced by any order. For any $J \subset N = \{1, 2, \ldots, n\}$, $A \in \mathcal{X}^J$, and φ we write

$$\varphi^J(A) = \sum_{j \in J} \sum_{x^{J \smallsetminus \{j\}} \in \mathcal{X}^{J \smallsetminus \{j\}}} \varphi\big(A_j(x^{J \smallsetminus \{j\}})\big), \tag{2.10}$$

where $A_j(\cdot) = A_{\{j\}}(\cdot)$ is defined by (2.1).

Lemma 1.

if φ satisfies II and if $\mathcal{L}$ is an order for φ^J $(J \subset N)$ then for all $A \subset \mathcal{X}^N$

$$|D_J(\mathcal{L}, A)| = |A| \tag{2.11}$$

and

$$\varphi^N(A) \le \varphi^N\big(D_J(\mathcal{L}, A)\big), \tag{2.12}$$

where $D_J(\mathcal{L}, \cdot)$ is defined by (2.3) and φ^n is written as φ^N.

Proof: (2.11) follows from the definition of $D_J(\mathcal{L}, \cdot)$. Let $B = D_J(\mathcal{L}, A)$.

Then by (2.1), (2.4), and (2.10) for all $C \subset \mathcal{X}^N$, (in particular for $C = A$ or B)

$$\begin{aligned}\varphi^N(C) &= \sum_{t \in J} \sum_{x^{N \smallsetminus \{t\}} \in \mathcal{X}^{N \smallsetminus \{t\}}} \varphi\big(C_t(x^{N \smallsetminus \{t\}})\big) + \sum_{t \in N \smallsetminus J} \sum_{x^{N \smallsetminus \{t\}} \in \mathcal{X}^{N \smallsetminus \{t\}}} \varphi\big(C_t(x^{N \smallsetminus \{t\}})\big) \\ &= \sum_{x^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J}} \varphi^J\big(C_J(x^{N \smallsetminus J})\big) + \sum_{x^J \in \mathcal{X}^J} \varphi^{N \smallsetminus J}\ \big(C_{N \smallsetminus J}(x^J)\big).\end{aligned} \tag{2.13}$$

By the definition of $D_J(\mathcal{L}, \cdot)$ in (2.3) and $B = D_J(\mathcal{L}, A)$, we have $B_J(x^{N \smallsetminus J}) = \mathcal{L}\big(\mathcal{X}^J, |A_J(x^{N \smallsetminus J})|\big)$ for all $x^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J}$. Thus, $\mathcal{L}$ being an (optimal) order for φ^J implies for all $x^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J}$

$$\varphi^J\big(A_J(x^{N \smallsetminus J})\big) \le \varphi^J\big(B_J(x^{N \smallsetminus J})\big). \tag{2.14}$$

Denote $A_{N \smallsetminus J}(x^J)$ $\big(B_{N \smallsetminus J}(x^J)\big)$ by A_k (B_k) if x^J is the lexicographically k–th sequence in $\mathcal{X}^J$. We have for all $y^{N \smallsetminus J} \in \mathcal{X}^{N \smallsetminus J}$ $y^{N \smallsetminus J} \in B_k$ exactly if there are at least k A_ℓ's containing $y^{N \smallsetminus J}$, or in other words

$$B_k = \bigcup_{1 \le i_1 < i_2 < \dots < i_k \le |\mathcal{X}|^{|J|}} \left(\bigcap_{j=1}^{k} A_{i_j} \right). \tag{2.15}$$

Furthermore, we observe that by (2.10) the submodularity II of φ implies the submodularity of $\varphi^{J'}$, and therefore φ and $\mathcal{X}$ in (2.7) can be replaced by $\varphi^{J'}$ and $\mathcal{X}^{J'}$ for all $J' \subset N$. Applying the resulting inequality and (2.15) to $J' = N \smallsetminus J$, $m = |\mathcal{X}|^{|J|}$, $\sum_{x^J \in \mathcal{X}^J} \varphi^{N \smallsetminus J}\big(A_{N \smallsetminus J}(x^J)\big)$, which equals $\sum_{i=1}^m \varphi^{N \smallsetminus J}(A_k)$, and $\sum_{x^J \in \mathcal{X}^J} \varphi^{N \smallsetminus J}\big(B_{N \smallsetminus J}(x^J)\big)$, which equals $\sum_{k=1}^m \varphi^{N \smallsetminus J}(B_k)$, we obtain

$$\sum_{x^J \in \mathcal{X}^J} \varphi^{N \smallsetminus J}\big(A_{N \smallsetminus J}(x^J)\big) \le \sum_{x^J \in \mathcal{X}^J} \varphi^{N \smallsetminus J}\big(B_{N \smallsetminus J}(x^J)\big), \tag{2.16}$$

which with (2.13) and (2.14) together implies (2.12).

3 A necessary and sufficient condition for the lexicographical order to be optimal for edge–isoperimetric problems satisfying I – III

Before we state and prove here our main result (Theorem 1) we first derive two auxiliary results (Lemma 2 and Lemma 3).

Lemma 2.

Assume that $\mathcal{X} = \{0, 1, \dots, \alpha\}, \alpha \geq 2$ and $n > 2$. Suppose now that $A \subset \mathcal{X}^N$ and that for all $J \subset N = \{1, 2, \dots, n\}$, $J \neq N$,

$$D_J(\mathcal{L}, A) = A, \tag{3.1}$$

then for $a^N \in A$ the following sequences belong to A:

(i) *all $x^N \in \mathcal{X}^N$ with $x_1 = a_1$ and $x^{N \smallsetminus \{1\}} <_{\mathcal{L}} a^{N \smallsetminus \{1\}}$*

(ii) *$b^N = (a_1 - 1, \alpha, \dots, \alpha, a_n)$, if $a_1 \geq 1$, and all x^N with $x_1 = a_1 - 1$ and $x^{N \smallsetminus \{1\}} <_{\mathcal{L}} b^{N \smallsetminus \{1\}}$.*

(iii) *in case $a_1 \geq 2$ all x^N with $x_1 \leq a_1 - 2$.*

Proof: Since $D_{N \smallsetminus \{1\}}(\mathcal{L}, A) = A$ (i) holds. Thus by (i) for (ii) and (iii) it is sufficient to show that

$$b^N \in A, \quad \text{if} \ \ a_1 - 1 \geq 0 \tag{3.2}$$

and

$$c^N = (a_1 - 2, \alpha, \dots, \alpha) \in A, \quad \text{if} \ \ a_1 - 2 \geq 0. \tag{3.3}$$

Now (3.2) follows from $a^N \in A$, $D_{N \smallsetminus \{n\}}(A) = A$ and $(a_1 - 1, \alpha, \dots, \alpha)\big(\in \mathcal{X}^{N \smallsetminus \{n\}}\big) <_{\mathcal{L}} a^{N \smallsetminus \{n\}}$.

Finally, (3.3) follows from (3.2), $D_{N \smallsetminus \{2\}}(A) = A$ and for $(a_1 - 2, \alpha, \dots, \alpha), (a_1 - 1, \alpha, \dots, \alpha, a_n) \in \mathcal{X}^{N \smallsetminus \{2\}}$ $(a_1 - 2, \alpha, \dots, \alpha) <_{\mathcal{L}} (a_1 - 1, \alpha, \dots, \alpha, a_n)$ (second components are deleted from both b^N and c^N, and this is possible because $n > 2$).

Lemma 3.

Assume $|\mathcal{X}| \geq 3$ and let φ satisfy I – III in Section 2.

(i) *If $\mathcal{L}$ is the optimal order for φ^2, then*

$$\triangle_\varphi(1) \leq \triangle_\varphi(2) \tag{3.4}$$

and for $a, b, i \in \mathcal{X} = \{0, 1, \dots, \alpha\}$ with $a < b \le b + i \le \alpha$ and either $a = 0$ or $b + i = \alpha$

$$\sum_{j=0}^{i} \triangle_\varphi(a+j) \le \sum_{j=0}^{i} \triangle_\varphi(b+j). \tag{3.5}$$

(ii) *If $\mathcal{L}$ is the order for φ^m, then for $a \in \mathcal{X}$, $a \ge 1$, and $x_i \in \mathcal{X}$ $(i = 2, \dots, m)$ not all equal to α*

$$\sum_{i=2}^{m} \triangle_\varphi(x_i) + \triangle_\varphi(a) \le (m-1)\triangle_\varphi(\alpha) + \triangle_\varphi(a-1). \tag{3.6}$$

Proof: (i) If $\mathcal{L}$ is the order for φ^2, then in particular

$$\varphi^2\big(\{00, 01, 10\}\big) \le \varphi^2\big(\{00, 01, 02\}\big) \tag{3.7}$$

and therefore (2.9) in property 4) of Section 2 gives (3.4).

While showing (3.5) we can assume that

$$a + i < b, \tag{3.8}$$

because otherwise we can delete the common terms on both sides of (3.5).

Define now the interval

$$\langle x^\ell, y^\ell \rangle = \{z^\ell \in \mathcal{X}^\ell : x^\ell \le_{\mathcal{L}} z^\ell \le_{\mathcal{L}} y^\ell\}, \langle y^\ell \rangle = \langle 0^\ell, y^\ell \rangle. \tag{3.9}$$

Case $a = 0$: By (3.8)
$\big(\langle (1, b+i)\rangle \smallsetminus \langle (1, b), (1, b+i)\rangle\big) \cup \langle (2, 0), (2, i)\rangle = \big\{(0, 0), (0, 1), \dots, (0, \alpha), (1, 0), \dots,$
$(1, b-1), (2, 0), \dots, (2, i)\big\} = B$, say, is a downset and $|B| = |\langle (1, b+i)\rangle|$.
Since $\mathcal{L}$ is the order for φ^2 we conclude with (2.9) that

$$\begin{aligned}
0 &\le \varphi\big(\langle (1, b+i)\rangle\big) - \varphi(B) \\
&= \sum_{x^2 \in \langle (1,b+i)\rangle \smallsetminus B} \big(\triangle_\varphi(x_1) + \triangle_\varphi(x_2)\big) - \sum_{x^2 \in B \smallsetminus \langle (1,b+i)\rangle} \big(\triangle_\varphi(y_1) + \triangle_\varphi(y_2)\big) \\
&= \left((i+1)\triangle_\varphi(1) + \sum_{j=0}^{i} \triangle_\varphi(b+j)\right) - \left((i+1)\triangle_\varphi(2) + \sum_{j=0}^{i} \triangle_\varphi(j)\right)
\end{aligned}$$

or equivalently

$$\sum_{j=0}^{i} \triangle_\varphi(b+j) - \sum_{j=0}^{i} \triangle_\varphi(j) \ge (i+1)\big(\triangle_\varphi(2) - \triangle_\varphi(1)\big). \tag{3.10}$$

This and (3.4) imply (3.5) in this case.

Case $a \neq 0$ and $b+i=\alpha$:

Instead of $\langle(1,b+i)\rangle$ and B we consider now $\langle(2,a-1)\rangle$ and

$$\begin{aligned} B' &= \big(\langle(2,a-1)\rangle \smallsetminus \langle(1,b),(1,\alpha)\rangle\big) \cup \langle(2,a),(2,a+i)\rangle \\ &= \big\{(0,0),\dots,(0,\alpha),(1,0),\dots,(1,b-1),(2,0),\dots,(2,a-1),\dots,(2,a+i)\big\} \end{aligned}$$

and by the previous argument we obtain (3.5) also in this case.

(ii) For $x^\ell, y^\ell \in \mathcal{X}^\ell$ define now the half–open interval

$$\langle x^\ell, y^\ell) = \langle x^\ell, y^\ell\rangle \smallsetminus \{y^\ell\}. \tag{3.11}$$

Then for $x^m=(a,x_2,\dots,x_m)$, $y^m=(a-1,\alpha,\dots,\alpha)$

$$\varphi\big(\langle 0^m, x^m)\big) \geq \varphi\big(\langle x^m\rangle \smallsetminus \{y^m\}\big). \tag{3.12}$$

Since both arguments in (3.12) are downsets we can use (2.9) to estimate them. Then we delete the common terms on both sides and obtain (3.6).

Quite surprisingly we found the following result.

Theorem 1.

If $|\mathcal{X}| \geq 3$, then for any set function $\varphi: 2^{\mathcal{X}} \to \mathbb{R}$ satisfying I – III and any integer $n \geq 2$ $\mathcal{L}$ on $\mathcal{X}^n$ is optimal for φ^n iff $\mathcal{L}$ on $\mathcal{X}^2$ is optimal for φ^2. (Condition I says that $\mathcal{L}$ is an optimal order for φ.)

Proof: Assume to the contrary that $\mathcal{L}$ is optimal for φ^2 but not for φ^n and that $n \geq 3$ is smallest with this property.

By Lemma 1 in Section 2 we can assume that for all $J \neq N$

$$D_J(\mathcal{L}, A) = A \tag{3.13}$$

for an optimal set A, and therefore A is a downset.

Moreover, among all optimal sets satisfying (3.13) we choose as A one which achieves the minimal value of $|A \,\triangle \langle\theta(|A|)\rangle|$, where $\theta(|A|)$ is the $|A|$–th smallest element in $\mathcal{X}^n$ in lexicographical order (i.e. $\theta(A) = \mathcal{L}(\mathcal{X}^n, |A|)$). Since by assumption $\mathcal{L}$ is not the order of φ^n, we have

$$|A \,\triangle\langle\theta(|A|)\rangle| > 0 \quad \text{or} \quad A \neq \langle\theta(|A|)\rangle. \tag{3.14}$$

Let now a^n be the lexicographically last element of A. Then obviously

$$A \subsetneq \langle a^n\rangle. \tag{3.15}$$

Since $\mathcal{L}$ is optimal for φ^{n-1}, $a_1 \neq 0$.

It immediately follows from Lemma 2 that all elements in $\langle a^n \rangle \setminus A$ are of the form

$$(a_1 - 1, \alpha, \dots, \alpha, x), \;\; x > a_n. \tag{3.16}$$

Moreover, by (3.13) none of the a_i's equals α. Thus by (3.16),

$$\langle a^n \rangle \setminus A = \big\{(a_1 - 1), \alpha, \dots, \alpha, x) : \beta \le x \le \alpha\big\} \tag{3.17}$$

for some $\alpha \ge \beta > a_n$.

Since A is a downset,

$$\big\{(a_1, \dots, a_{n-1}, y) : 0 \le y \le a_n\big\} \subset A. \tag{3.18}$$

Considering $a_n < \beta$, one can choose a, b, and i such that

$$a = 0, a + i = a_n, b = \beta \;\text{ and }\; \beta + i \le \alpha \tag{3.19}$$

or

$$a > 0, a + i = a_n, b = \beta \;\text{ and }\; \beta + i = \alpha. \tag{3.20}$$

In both cases we have $a < b$. When $a_n \le \alpha - \beta$ we choose (3.19) and otherwise (3.20).

Now we remove the "top part"

$$\big\{(a_1, a_2, \dots, a_{n-1}, y) : y = a, a+1, \dots, a+i\big\} \subset A$$

from A and add

$$\big\{(a_1 - 1, \alpha, \dots, \alpha, x) : x = b, b+1, \dots, b+i\big\}, \text{ which is disjoint from } A,$$

to A and obtain a set A' with $|A'| = |A|$ and $|A' \triangle \theta(|A|)| < |A \triangle \theta(|A|)|$.

From the structure of A, as described in Lemma 2, it is not hard to see that A' also satisfies (3.13).

On the other hand, by (2.9) and Lemma 3

$$\begin{aligned}
\varphi^n(A) - \varphi^n(A') &= \sum_{j=0}^{i} \triangle_\varphi(a+j) + (i+1)\left(\sum_{j=2}^{n-1} \triangle_\varphi(a_j) + \triangle_\varphi(a_1)\right) \\
&- \sum_{j=0}^{i} \triangle_\varphi(b+j) + ((n-2)\triangle_\varphi(\alpha) + \triangle_\varphi(a_1 - 1))\,(i+1) \\
&= \left(\sum_{j=0}^{i} \triangle_\varphi(a+j) - \sum_{j=0}^{i} \triangle_\varphi(b+j)\right) \\
&+ (i+1)\left(\left(\sum_{j=2}^{n-1} \triangle_\varphi(a_j) + \triangle_\varphi(a_1)\right) - \big((n-2)\triangle_\varphi(\alpha) + \triangle_\varphi(a_1 - 1)\big)\right) \le 0
\end{aligned}$$

in contradiction to the definition of A.

4 An application of Theorem 1 to complete bipartite graphs

This last section demonstrates how useful the *local–global principle* of Theorem 1 is. We begin with the simple C_4 and then consider all complete bipartite graphs.

A. *The lexicographical order $\mathcal{L}$ is optimal for $I(C_4^n, \cdot)$ or, by regularity, equivalently for $B(C_4^n, \cdot)$.*

   We know from Theorem 1 that it suffices to show the property for $I(C_4^2, \cdot)$. This in turn is readily done by the following simple observation:

   For all $A \subset \{0,1,2,3\}^2$, $|A| \le 8 = \frac{1}{2}|\{0,1,2,3\}^2|$

$$I(C_4^2, A) \le \begin{cases} |A| - 1 & \text{if } \ |A| < 4 \\ 2|A| - 4 & \text{if } \ 4 \mid |A| \\ 2|A| - 5 & \text{otherwise.} \end{cases}$$

B. *The complete bipartite graph $C_{m,m}$.*

   Let $\mathcal{X} = \{0, 1, \dots, 2m-1\}$ be the vertex set with edges exactly between even and odd numbers. This defines $C_{m,m}$, which is a regular graph, so that boundary – and inner isoperimetric problems are equivalent.

   We set $\varphi(\cdot) = I(C_{m,m}, \cdot)$ and readily verify that I – III hold and that

$$\triangle_\varphi(\ell) = \left\lceil \frac{\ell}{2} \right\rceil \quad \text{for} \ \ \ell = 0, 1, \dots, 2m-1. \tag{4.1}$$

Theorem 2.

The first segments in lexicographical order $\mathcal{L}$ give optimal sets for the (inner and boundary) edge–isoperimetric problems for $C_{m,m}^2$, and therefore for $C_{m,m}^n$.

Proof: Let us consider $C_{m.m}^2$.

In order to proceed by induction on m we introduce the notation $\Psi_m = \triangle_\varphi$.

Case $m = 2$**:**

Notice that $C_4 = C_{2,2}$ and the result was established above.

Case $m \to m+1$**:**

For any $A \subset [2m+1]^2$ we have to show that replacement of A by the first $|A|$–segment of $[2m+1]^2$ under $\mathcal{L}$ cannot decrease $\varphi^2(A)$. We shall make use of the fact that the folowing 3 properties are equivalent:

(α) $A \subset [2m+1]^2 = \mathcal{X}^2$ is optimal among the subsets of cardinality $|A|$.

(β) $A^c = \mathcal{X}^2 \setminus A$ is optimal among the subsets of cardinality $|A^c| = |\mathcal{X}^2| - |A|$.

(γ) The set $\widetilde{A} = \big\{(2m+1-x, 2m+1-y) : (x,y) \in A^c\big\}$ is optimal among the subsets of cardinality $|\mathcal{X}^2| - |A|$.

Notice that A is downset iff $\widetilde{A}$ is downset.

Case 1:

There is *no* $(x,y) \in A$ with $x \geq 2m$. Therefore we are able to rule out the last two columns $\big\{(x,y) : x = 2m, 2m+1\big\}$, which are useless.

Let "top" and "bottom" of the part remaining be

$$T = \big\{(x,y) : 0 \leq x \leq 2m-1, 2 \leq y \leq 2m+1\big\} \tag{4.2}$$

and

$$B = \big\{(x,y) : 0 \leq x \leq 2m-1, 0 \leq y \leq 2m-1\big\} \tag{4.3}$$

respectively.

Define O_T (resp. O_B) as the operator keeping the $A \setminus T$ (resp. $A \setminus B$) part unchanged and changing $A_T \triangleq A \cap T$ (resp. $A_B \triangleq A \cap B$) to the first $|A_T|$ (resp. $|A_B|$) elements of T (resp. B) in order $\mathcal{L}$.

Obviously both, $O_T(A)$ and $O_B(A)$, are downsets whenever A is downset. Furthermore, by induction hypothesis, for $B \subset \{0, 1, \ldots, 2m-1\}^2$,

$$\varphi^2\big(O_B(A)\big) \geq \varphi^2(A). \tag{4.4}$$

In fact, the induction hypothesis also implies $\varphi^2\big(O_T(A)\big) \geq \varphi^2(A)$, since by (4.1) for $2 \leq y \leq 2m+1$ $\Psi_{m+1}(y) = 1 + \Psi_m(y-2)$ holds, so that we can shift 2 units down and transform T to $\big\{(x,y) : 0 \leq x \leq 2m-1, 0 \leq y \leq 2m-1\big\}$ and apply the induction hypothesis. However $O_T(A) \neq A$ (resp. $O_B(A) \neq A$), implies that the operator O_T (resp. O_B) decreases strictly the maximal element in the order $\mathcal{L}$ of A_T (resp. A_B). Thus one can use O_T and O_B repeatedly finitely often, and obtain as resulting set $A^{'}$ a downset with

$$\varphi^2(A^{'}) \geq \varphi^2(A), O_T(A^{'}) = A^{'} \text{ and } O_B(A^{'}) = A^{'}. \tag{4.5}$$

Thus, $A^{'}_T$ and $A^{'}_B$ must have the following forms:

$$A^{'}_T = \big\{(x,y) : 0 \leq x \leq a-1, 2 \leq y \leq 2m+1\big\} \cup \big\{(a,y) : 2 \leq y \leq u\big\}$$
$$\text{for some } 0 \leq t \leq 2m-1 \text{ and } 2 \leq u \leq 2m+1,$$

and

$$A^{'}_B = \big\{(x,y) : 0 \leq x \leq b-1, 0 \leq y \leq 2m-1\big\} \cup \big\{(x,b) : 0 \leq y \leq v\big\}$$
$$\text{for some } t \leq b \leq 2m-1 \text{ and } 0 \leq v \leq \min(u, 2m-1).$$

Now, considering that $A^{'}$ is a downset and $A^{'} = A^{'}_T \cup A^{'}_B$, a simple calculation shows that in case $A^{'}$ is not the first segment of $\mathcal{X}^2 = [2m+1]^2$, we have $a = b-1$, $u \in \{2m-1, 2m\}$, $v \in \{0,1\}$, and $A^{'}$ must be one of the following four subsets described in (a) – (d). Write

$$A^* = \big\{(x,y) : 0 \leq x \leq a-1, 0 \leq y \leq 2m+1\big\}, \quad \text{then}$$

(a) $A^{'} = A^* \cup \big\{(a,y)|0 \leq y \leq 2m-1\big\} \cup \big\{(a+1,y) : y = 0,1\big\}$,

(b) $A^{'} = A^* \cup \big\{(a,y)|0 \leq y \leq 2m\big\} \cup \big\{(a+1,y) : y = 0,1\big\}$,

(c) $A^{'} = A^* \cup \big\{(a,y)|0 \leq y \leq 2m-1\big\} \cup \big\{(a+1,0)\big\}$,

(d) $A^{'} = A^* \cup \big\{(a,y)|0 \leq y \leq 2m\big\} \cup \big\{(a+1,0)\big\}$.

Let now $K_1 = (a, 2m+1)$, $K_2 = (a, 2m)$, $K_3 = (a+1, 1)$ and $K_4 = (a+1, 0)$. We make the following operations, which change $A^{'}$ to the first segment in $\mathcal{L}$.

case (a): Remove K_3 and K_4 from $A^{'}$ and add K_1 and K_2 to it,

case (b): Remove K_3 from $A^{'}$ and add K_1 to it,

case (c): Remove K_4 from $A^{'}$ and add K_2 to it,

case (d): Remove K_4 from $A^{'}$ and add K_1 to it.

Now the value of $\Psi_{m+1}(= \triangle_\varphi)$ at $K_i (i = 1,2,3,4)$ is by (4.1)

$$K_1 : \Psi_{m+1}(a) + \Psi_{2m+1}(2m+1) = \left\lceil \frac{a}{2} \right\rceil + m + 1$$

$$K_2 : \Psi_{2m+1}(a) + \Psi_{2m+1}(2m) = \left\lceil \frac{a}{2} \right\rceil + m$$

$$K_3 : \Psi_{2m+1}(a+1) + \Psi_{2m+1}(1) = \left\lceil \frac{a+1}{2} \right\rceil + 1$$

$$K_4 : \Psi_{2m+1}(a+1) + \Psi_{2m+1}(0) = \left\lceil \frac{a+1}{2} \right\rceil.$$

Remember that we assumed $m \geq 2$. The operations in (a) – (d) cannot make things worse. This concludes the proof in the first case.

Moreover, by symmetry and the equivalences (α) – (γ), the following cases can be excluded too:

$$\big\{(x,y) : 0 \leq x \leq 2m+1, y \geq 2m\big\} \cap A = \phi, \big\{(x,y) : x \leq 1, 0 \leq y \leq 2m+1\big\} \subset A,$$
$$\text{and } \big\{(x,y) : 0 \leq x \leq 2m+1, y \leq 1\big\} \subset A..6 \tag{4}$$

Thus we can always assume

$$(0, 2m), (2m, 0) \in A \quad \text{and} \quad (1, 2m+1), (2m+1, 1) \notin A. \tag{4.7}$$

Case 2:

(4.7) holds, and at least one of the elements $(0, 2m + 1)$ and $(2m + 1, 0)$ is not in A. Assume $(2m + 1, 0) \notin A$ and consequently that $A \cap \{(x, y) : x = 2m + 1\} = \phi$. Let $C\ell_{2m} = \{(x, y) \in A : x = 2m\}$ and $R_{2m} = \{(x, y) \in A : y = 2m\}$.

Then (4.7) implies $C\ell_{2m} \neq \phi$, $R_{2m} \neq \phi$.

W.l.o.g. we can assume that

$$|C\ell_{2m}| \triangleq c \leq |R_{2m}|, \tag{4.8}$$

because otherwise we can exchange the roles of rows and columns and, if necessary, move the $2m + 1$ st column to $2m + 1$ st row.

When $(0, 2m+1) \notin A$, i.e. there is no element of A in the last row at all, we remove $C\ell_{2m} = \{(2m, 0), (2m, 1) \dots (2m, c-1)\}$ from A and add $\{(0, 2m+1), (1, 2m+1) \dots (c-1, 2m+1)\}$ to A.

By (4.8) the resulting set is a downset and (4.1) shows that it has a larger value of φ^2. If $(0, 2m + 1) \in A$ and $|C\ell_{2m}| < |R_{2m}|$, one can also remove $C\ell_{2m}$ from A, but add $\{(1, 2m+1) \dots (c, 2m+1)\}$ to A. When $(0, 2m+1) \in A$ and $c = |C\ell_{2m}| = |R_{2m}| \neq 1$, we remove $\{(2m, 1), (2m, 2) \dots (2m, c-1)\}$ from A and add $\{(1, 2m+1) \dots (c-1, 1)\}$ to it.

For all the 3 subcases above we change our A to a downset in case 1 or a set of type (4.6) for case 1. Finally we assume

$$(0, 2m + 1) \in A, C\ell_{2m} = \{(2m, 0)\}, R_{2m} = \{(0, 2m)\}. \tag{4.9}$$

Thus $(0, 2m + 1) \in A$ implies $(2m + 1, 0) \notin \widetilde{A}$, where $\widetilde{A}$ is defined in (γ).

Now $(2m, 1) \notin A$ and A is a downset. This together implies $(2m+1, 1) \notin A$ and therefore $(1, 2m), (0, 2m) \in \widetilde{A}$. Similarly, since $(1, 2m) \notin A$, $(2m, 1), (2m, 0) \in \widetilde{A}$.

Therefore, both, the $2m$–th row and the $2m$–th column of $\widetilde{A}$ have at least 2 elements. By (α) – (γ), instead of A we can consider $\widetilde{A}$, which has been settled in the previous subcases of this case.

Case 3:

$(0, 2m + 1), (2m + 1, 0) \in A$. Here $(2m + 1, 0) \notin \widetilde{A}$ and $(0, 2m + 1) \notin \widetilde{A}$. By (γ) we can consider $\widetilde{A}$ and reduce our problem to the case 2.

Finally the Theorem follows now from Theorem 1.

Remark 4:

The result does not extend to general $C_{m,m'}$. Already, for $C_{2m,2m-1}$ $\mathcal{L}$ is not optimal for the inner isoperimetric problem. The first $4m - 4$ elements of $[2m - 1] \times [2m - 2]$ in the order $\mathcal{L}$ are

$$\{(0, 0), (0, 1), \dots, (0, 2m - 1), (1, 0), \dots, (1, 2m - 3)\}.$$

However, $\{(0, 0), \dots, (0, 2m - 2), (1, 0), \dots, (1, 2m - 2)\}$ has more inner edges.

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