HIGHER LEVEL EXTREMAL PROBLEMS

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1. Introduction

There seems to be an ever increasing number of sometimes fascinating combinatorial extremal problems. In particular Computer Science is a rich source of such problems. Our observation is that several of the known basic problems and many new ones can be formulated as parts of a transparent scheme of problems. It gives a certain orientation in a substantial part of known combinatorial results and at the same time it gives directions for new investigations (see section 9). Our present understanding of the subject grew with our recent work, especially, the papers [17], [9], [18], [20]. We focus here on the most important poset in combinatorics, the power set $\mathcal{P}(\Omega)$ of a finite set $\Omega = \{1, 2, \dots, n\}$, and we study problems involving the following binary relations between its elements:

$$\begin{aligned} &A \supset\subset B \quad (\text{comparable, that is, } A \subset B \text{ or } A \supset B), \\ &A \supset|\subset B \quad (\text{incomparable}), \quad A \cap B = \varnothing \quad (\text{disjoint}), \\ &A \cap B \neq \varnothing \quad (\text{intersecting}). \end{aligned} \tag{1.1}$$

The following problem, due to and solved by Sperner [12], has often been cited as a prototype of an extremal problem.

$\mathcal{A} \subset \mathcal{P}(\Omega)$ is an antichain, if its members are incomparable. What is the maximal cardinality of antichains? The problem is based on the relation "incomparable".

If this relation is replaced by "intersecting" and we also impose the constraint $\mathcal{A} \subset \mathcal{P}_k(\Omega) \triangleq \binom{\Omega}{k}$, then we arrive at the Erdös/Ko/Rado problem [13]. Analogous problems for the relations "comparable" or "disjoint" turn out to be trivial.

Furthermore, in the sequel we speak of the *unresricted* case, if $\mathcal{P}(\Omega)$ is the ground set, and of the *restricted* case, if $\mathcal{P}_k(\Omega)$ is the ground set.

A multitude of problems arises, if we go one step higher, that is, from sets to families of sets etc.

To fix ideas let us recall the notion of a cloud–antichain from [9].

$\{\mathcal{A}_i : 1 \leq i \leq N\}$ with disjoint non–empty $\mathcal{A}_i \subset \mathcal{P}(\Omega)$ is a cloud–antichain (CAC), if for all $i \neq j$ and all $A_i \in \mathcal{A}_i$, $A_j \in \mathcal{A}_j$

$$A_i \supset|\subset A_j. \tag{1.2}$$

Here not only the length N but also $|\mathcal{A}_i|$ $(1 \leq i \leq N)$ are parameters of interest. The case $\max\limits_i |\mathcal{A}_i| = 1$ is Sperner's case.

Notice that for every (i,j) we require (1.2) for *every* $A_i \in \mathcal{A}_i$ and *every* $A_j \in \mathcal{A}_j$. We therefore refer to this problem as being of *type* $(\forall, \forall)$.

A weaker requirement is that there *exists* an $A_i \in \mathcal{A}_i$ such that for *all* $A_j \in \mathcal{A}_j$ (1.2) holds. It is called of type $(\exists, \forall)$. An even weaker condition is that for *all* $A_i \in \mathcal{A}_i$ there *exists* an $A_j \in \mathcal{A}_j$ with the property (1.2). We speak of type $(\forall, \exists)$. Finally, the weakest condition is the type $(\exists, \exists)$, where again for every (i,j) there *exists* $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ with (1.2).

Analogously one can define families of clouds for the four types and for all the relations in (1.1). Also one may or may not require disjointness of the clouds. Among the extremal problems one might consider we emphasize two: For $N = 2$, what are the extremal pairs $(|\mathcal{A}_1|, |\mathcal{A}_2|)$?

Disregarding the values of the $|\mathcal{A}_i|$'s what is the maximal length N for the various situations above?

The first question is adressed in the papers [9], [28] and the second question is analysed in [29] for the relation "incomparable".

In [20] we studied the second question for the relations "disjoint" and "intersecting" in the restricted case for $k = 2$.

The main topic of this paper are exact and asymptotic results for the restricted and the unrestricted case for the relations "comparable", "disjoint" and "intersecting".

Key tools are results on related graph coloring problems (Theorem 1 in Section 2 and Theorem 2 in Section 3).

Our results on families of "comparable" clouds, of clouds with the "disjoint" relation and of clouds with the "intersecting" relation appear in Sections 4, 5, and 6.

The following chart is for the orientation of the reader about the present state of our knowledge about the various problems. Here we make the *Conventions*:

T = trivial, S = solved exactly, A = asymptotic solution.

Previously known results are referred to by articles in our references. Further bounds and conjectures are stated in Section 8.

Whereas here we consider only the canonical cases of *disjoint* clouds, in [29] also cases of *distinct* clouds were analysed.

	∀∀	∃∀	∀∃	∃∃	∀∀	∃∀	∀∃	∃∃
incom- parable	S [12]	A	S	S	T	T	T	T
compa- rable	T	?	A [31]	A	T	T	T	T
disjoint	T	?	A [8]	A	T	A(k=2)	A	A
inters.	S	S	S	S	S [13]	S(k=2) [20]	A S(k=2) [20]	A

unrestricted restricted

In Section 8 we state 10 problems and 9 conjectures about their (complete or partial) solutions.

In the last section of the paper a whole program of higher level extremal problems is sketched. Obvious extensions of our work arise for instance if the ground space is replaced by other lattices and other relations are incorporated.

As far as actual results go we draw attention to, what we call, excess problems. Their study was initiated in [27]. We report in Section 7 on exact results and also on asymptotic results which go considerably beyond the earlier work.

2. A graph coloring problem of type $(\exists,\exists)$

The family of sets $\mathcal{P}(\Omega)$ or $\mathcal{P}_k(\Omega)$ endowed with pairwise relations such as "intersect", "disjoint", "comparable" etc. can be viewed as a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with $\mathcal{V}=\mathcal{P}(\Omega)$ or $\mathcal{V}=\mathcal{P}_k(\Omega)$, where for instance in case of "disjoint" the relation $A\cap B=\varnothing$ is represented by an edge.

The study of cloud families of the $(\exists,\exists)$–type naturally leads to the following coloring concept. For any graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *a coloring of type* $(\exists,\exists)$ is a map

$$f:\mathcal{V}\to M_f=\{1,2,\dots,m_f\} \tag{2.1}$$

such that for any two colors, say, $i,j\in M_f$, an edge $(a,b)\in\mathcal{E}$ exists with $f(a)=i$ and $f(b)=j$.

We are interested in the quantity $m(\mathcal{G})=\max\{m_f: f \text{ is } (\exists,\exists)\text{–coloring of } \mathcal{G}\}$. Since obviously $\binom{m(\mathcal{G})}{2}\le|\mathcal{E}|$, we conclude that

$$m(\mathcal{G})\le\big(2|\mathcal{E}|+\frac{1}{4}\big)^{\frac12}+\frac12\le\sqrt{2}|\mathcal{E}|^{\frac12}+1. \tag{2.2}$$

We derive now a lower bound on $m(\mathcal{G})$ for any graph $G=(\mathcal{V},\mathcal{E})$ by the probabilistic method. We set

$$D=\max_{v\in\mathcal{V}}\deg(v)\quad\text{and}\quad N=2|\mathcal{E}|, \tag{2.3}$$

and we consider the set $\mathcal{E}^*=\big\{(a,b),(b,a):\{a,b\}\in\mathcal{E}\big\}=\{e_1,e_2,\dots,e_N\}$ of directed edges. We say that the (ordered) pair (a,b) is colored by the (ordered) pair (i,j), if $f(a)=i$ and $f(b)=j$.

Let now $F_1,\dots,F_M$, where $\{1,2,\dots,M\}=\mathcal{V}$, be independent, identically distributed random variables assuming values in $\{1,2,\dots,m\}$ with equal probabilities. Then $\mathbb{F}=(F_1,\dots,F_M)$ defines a random coloring of the vertices. To analyse its performance we consider the events

$$E_{ij}=\{\text{no edge in } \mathcal{E}^* \text{ is colored by } (i,j)\},$$

whose probability $\mathrm{prob}(E_{ij})$ is by symmetry independent of (i,j), and the random variable

$$S=\big|\big\{(i,j):\ \text{no edge in } \mathcal{E}^* \text{ is colored by } (i,j)\big\}\big|.$$

Clearly,

$$\mathbb{E}S = \sum_{(i,j)} Pr(E_{ij}) \le m^2 Pr(E_{ij}), \tag{2.4}$$

and if $\mathbb{E}S < 1$, then, with positive probability, every pair of colors occurs and thus $m(\mathcal{G}) \ge m$.

We derive now conditions under which $\mathbb{E}S < 1$ holds. For this we define first

$$E_{ij}^{(t)} = \big\{e_t \text{ is not colored by } (i,j)\big\}, E_{ij}^t = \big\{e_1, e_2, \dots, e_t \text{ are not colored by } (i,j)\big\} = \bigcup_{s=1}^t E_{ij}^{(s)},$$

and write

$$Pr(E_{ij}) = Pr(E_{ij}^{(1)}) Pr(E_{ij}^{(2)}|E_{ij}^1) \dots Pr(E_{ij}^{(n)}|E_{ij}^{n-1}) \dots Pr(E_{ij}^{(N)}|E_{ij}^{N-1}). \tag{2.5}$$

We now estimate $Pr(E_{ij}^{(n)}|E_{ij}^{n-1})$ from above.

With $\overline{E}_{ij}^{(n)} = \{e_n$ is colored by $(i,j)\}$ we write

$$Pr(E_{ij}^{(n)}|E_{ij}^{n-1}) = 1 - Pr(\overline{E}_{ij}^{(n)}|E_{ij}^{n-1}) = 1 - \frac{Pr(\overline{E}_{ij}^{(n)} \cap E_{ij}^{n-1})}{Pr(E_{ij}^{n-1})}.$$

First we estimate $Pr(\overline{E}_{ij}^{(n)} \cap E_{ij}^{n-1})$ from below.

Write $e_t = (a_t, b_t)$, $1 \le t \le N$, and set

$$\mathcal{V}_n' = \{a_t : 1 \le t < n, b_t = b_n\}, \mathcal{V}_n'' = \{b_t : 1 \le t < n, a_t = a_n\}.$$

In the event $\overline{E}_{ij}^{(n)} \cap E_{ij}^{n-1}$ we must have $F_a \ne i$, $F_b \ne j$ for all $a \in \mathcal{V}_n'$ and all $b \in \mathcal{V}_n''$. Consider the edges

$$\mathcal{E}_n^* = \big\{e_t : 1 \le t \le n-1, \{a_t, b_t\} \cap \big(\mathcal{V}_n' \cup \mathcal{V}_n'' \cup \{a_n, b_n\}\big) = \varnothing\big\}$$

and the events

$$E_{ij}^* = \big\{\text{no } e_t \text{ in } \mathcal{E}_n^* \text{ is colored by } (i,j)\big\}.$$

Then we have

$$Pr\big(\overline{E}_{ij}^{(n)} \cap E_{ij}^{n-1}\big) = Pr\big(E_{ij}^{n-1}|\overline{E}_{ij}^{(n)}\big) Pr\big(\overline{E}_{ij}^{(n)}\big) = Pr\big(E_{ij}^{n-1}|\overline{E}_{ij}^{(n)}\big)\frac{1}{m^2} \tag{2.6}$$

Now E_{ij}^{n-1} clearly occurs, if all vertices in $\mathcal{V}_n' \cup \mathcal{V}_n''$ are not colored by either i or j *and* if E_{ij}^* occurs. The last two events are, also conditionally on $\overline{E}_{ij}^{(n-1)}$, independent and have probabilities $\big(1 - \frac{2}{m}\big)^{|\mathcal{V}_n' \cup \mathcal{V}_n''|}$ and $Pr\big(E_{ij}^*\big)$, resp. Therefore we conclude from (2.6)

$$Pr\big(\overline{E}_{ij}^{n-1} \cap E_{ij}^{n-1}\big) \ge \frac{1}{m^2}\big(1 - \frac{2}{m}\big)^{|\mathcal{V}_n' \cup \mathcal{V}_n''|} \cdot Pr\big(E_{ij}^*\big). \tag{2.7}$$

Secondly we have obviously

$$Pr\big(E_{ij}^{n-1}\big) \le Pr\big(E_{ij}^{*}\big). \tag{2.8}$$

The two inequalities imply $Pr\big(E_{ij}^{(n)}|E_{ij}^{n-1}\big) \le 1 - \frac{1}{m^2}\big(1-\frac{2}{m}\big)^{|\mathcal{V}'_n \cup \mathcal{V}''_n|} \le 1 - \frac{1}{m^2}\big(1 - \frac{2}{m}\big)^{2D}$ and thus by (2.5)

$$Pr(E_{ij}) \le \big(1 - \frac{1}{m^2}(1-\frac{2}{m})^{2D}\big)^N.$$

Together with (2.4) this implies

$$\mathbb{E}S \le m^2\big(1 - \frac{1}{m^2}(1-\frac{2}{m})^{2D}\big)^N.$$

We rewrite the right hand side expression and get

$$\mathbb{E}S \le m^2 \exp\big\{N \log\big(1 - \frac{1}{m^2}(1-\frac{2}{m})^{2D}\big)\big\}.$$

Since $\log(1-x) \le -x$ for $0 \le x < 1$ we continue with

$$\mathbb{E}S \le m^2 \exp\big\{-\frac{N}{m^2}(1-\frac{2}{m})^{2D}\big\}. \tag{2.9}$$

There are general choices of the parameters N, D, m for which (2.9) implies the desired $\mathbb{E}S < 1$.

For our purposes we can choose always $m \ge D$. Then by definition of the exponential function

$$\mathbb{E}S \le m^2 \exp\big\{-\frac{N}{m^2}e^{-4}\big\}$$

and we get the sufficient condition

$$2\log m < \frac{N}{m^2}e^{-4}. \tag{2.10}$$

If now $m = \big(\frac{N}{e^4 \log N}\big)^{\frac{1}{2}}$, then $2\log m = \log\frac{N}{e^4\log N} < \log N = \frac{N}{m^2}e^{-4}$ and the condition (2.10) holds.

We summarize our findings.

Theorem 1. *For any graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *we have with* $N = 2|\mathcal{E}|$

(a) $m(\mathcal{G}) \le N^{\frac{1}{2}} + 1$.

Moreover, if

$$D = \max_{x\in\mathcal{V}} \deg(x) \le \left(\frac{N}{e^4 \log N}\right)^{\frac{1}{2}}, \tag{2.11}$$

then

(b) $m(\mathcal{G}) \ge \left(\frac{N}{e^4\log N}\right)^{\frac{1}{2}}$.

3. A graph coloring problem of type $(\forall, \exists)$

The study of cloud families of the $(\forall, \exists)$–type leads to the following coloring concept for any graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

A coloring of type $(\forall, \exists)$ is a map

$$g : \mathcal{V} \to M_g = \{1, 2, \dots, m_g\}$$

such that for any two colors, say, $i, j \in M_g$ and for any $a \in \mathcal{V}$ with $g(a) = i$ there is an edge $\{a, b\} \in \mathcal{E}$ with $g(b) = j$.

We are interested in

$$m^*(\mathcal{G}) = \max\{m_g : g \text{ is } (\forall, \exists)\text{–coloring of } \mathcal{G}\}.$$

Theorem 2. *For any graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *we have*

(a) $\big(\log |\mathcal{V}|\big)^{-1}(d+1) \leq m^*(\mathcal{G}) \leq d+1$ *, where* $d = \min_{x \in \mathcal{V}} \deg(x)$.

(b) $m^{*'}(\mathcal{G}) \triangleq \max\big|\big\{m^*(\mathcal{G}') : \mathcal{G}' \text{ is subgraph of } \mathcal{G}\big\}\big| \leq D+1$.

In [18] we used a coloring concept for hypergraphs $(\mathcal{V}, \mathcal{F})$. It is said to carry m colors, if there is a vertex coloring with m colors such that all colors occur in every edge $F \in \mathcal{F}$. Let $m(\mathcal{V}, \mathcal{F})$ be the maximal number of colors carried by $(\mathcal{V}, \mathcal{F})$.

Coloring Lemma. *AZ([18]) For any hypergraph* $(\mathcal{V}, \mathcal{F})$

$$m(\mathcal{V}, \mathcal{F}) \geq \left\lfloor (\log |\mathcal{F}|)^{-1} \min_{F \in \mathcal{F}} |F| \right\rfloor .$$

Proof of Theorem 2:

Associate with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the hypergraph $\mathcal{H}(\mathcal{G}) = (\mathcal{V}, \mathcal{F})$, where $\mathcal{F} = \big\{\mathcal{N}(x) : x \in \mathcal{V}\big\}$ and $\mathcal{N}(x) = \big\{y \in \mathcal{V} : \{x, y\} \in \mathcal{E}\big\} \cup \{x\}$.

Clearly,

$$m(\mathcal{V}, \mathcal{F}) = m^*(\mathcal{G}) \quad \text{and} \quad \min_{F \in \mathcal{F}} |F| = d+1.$$

Since obviously $m^*(\mathcal{G}) \leq d+1$ and a fortiori (b) holds, Theorem 2 follows with Coloring Lemma AZ.

Remark:

Finally we mention a notion corresponding to $(\exists, \forall)$–type families.

A coloring of type $(\exists, \forall)$ is a map

$$h : \mathcal{V} \to M_h = \{1, 2, \dots, m_h\}$$

such that for any two colors $i, j \in M_h$ an $a \in \mathcal{V}$ exists with $h(a) = i$ and $h(b) = j$ for all $b \in \mathcal{N}(a)$, $b \neq a$. The quantity

$$m^{**}(\mathcal{G}) = \max\big\{m_h : h \text{ is } (\exists, \forall)\text{–coloring of } \mathcal{G}\big\}$$

is hard to analyse in general.

4. Asymptotic results via graph coloring of type $(\exists, \exists)$

The following Theorems are all obtained as consequences of Theorem 1. We solely have to estimate D and N in the graphs associated with the various configurations.

For the maximal cardinality of families with the relations "comparable", "disjoint", and "intersecting" we choose the letters C, D , and I , respectively. The types of problems such as $(\forall, \forall)$ etc. appear as an argument and the cardinality of Ω appears as an index n . In addition, in the restricted case a k appears in the argument. It indicates that the ground set is $\binom{\Omega}{k}$.

We consider now $C_n(\exists, \exists)$ and $D_n(\exists, \exists)$. $I_n(\cdot, \cdot)$ is determined *exactly* in all cases in Section 6. Then we go to the restricted cases $D_n(\exists, \exists, k)$ and $I_n(\exists, \exists, k)$. Trivially, $C_n(\exists, \exists, k) = 1$.

Theorem 3. *For n sufficently large*

(a) $(n^2 \cdot e^4 \cdot \log 3)^{-\frac{1}{2}} (2 \cdot 3^n)^{\frac{1}{2}} \leq C_n(\exists, \exists) \leq (2 \cdot 3^n)^{\frac{1}{2}}$ *(for all n).*

(b) $\lim\limits_{n \to \infty} \dfrac{1}{n} \log C_n(\exists, \exists) = \dfrac{1}{2} \log 3$.

Proof: Define $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows:

$$\mathcal{V} = \mathcal{P}(\Omega), \mathcal{E} = \big\{\{A, B\} : A, B \in \mathcal{V}, A \neq B, A \subset B\big\}. \tag{4.1}$$

Then $N = \sum_{k=0}^{n} \binom{n}{k} (2^{n-k} - 1 + 2^k - 1) = 2 \cdot 3^n - 2 \cdot 2^n$ and by (a) in Theorem 1

$$C_n(\exists, \exists) = m'(\mathcal{G}) \leq (2 \cdot 3^n)^{\frac{1}{2}}. \tag{4.2}$$

However, since $D = \max_{0 \leq k \leq n} (2^{n-k} - 1 + 2^k - 1) = 2^n - 1$, condition (2.11) does not hold and (b) is not directly applicable. We circumvent this difficulty by restricting the vertex set to $\mathcal{V}_{k,\ell} = \mathcal{P}_k(\Omega) \cup \mathcal{P}_{k+\ell}(\Omega)$ for suitable k and ℓ , and by choosing the edge set $\mathcal{E}_{k,\ell}$ induced by $\mathcal{E}$ on $\mathcal{V}_{k,\ell}$. For the graph $\mathcal{G}_{k,\ell} = (\mathcal{V}_{k,\ell}, \mathcal{E}_{k,\ell})$ we have

$$N = 2 \binom{n}{k} \binom{n-k}{\ell} \quad \text{and} \quad D = \max\left(\binom{n-k}{\ell}, \binom{k+\ell}{k} \right). \tag{4.3}$$

Not bothering about integrality we can choose $k = \ell = \frac{1}{3} n$ and obtain

$$N = 2 \cdot \binom{n}{\frac{1}{3}n} \binom{\frac{2}{3}n}{\frac{1}{3}n} = 2 \frac{n!}{\left((\frac{1}{3}n)!\right)^3} \geq \frac{2}{n} \, 3^n, \tag{4.4}$$

$$D = \binom{\frac{2}{3}n}{\frac{1}{3}n} \leq 2^{\frac{2}{3}n}. \tag{4.5}$$

Since $2^{\frac{2}{3}} \leq 3^{\frac{1}{2}}$, (2.11) holds for n large enough and we get from (b) in Theorem 1

$$C_n(\exists, \exists) \geq m(\mathcal{G}_{\ell,k}) \geq \left(\frac{2 \cdot 3^n}{n \cdot e^4 \cdot n \log 3} \right)^{\frac{1}{2}}.$$

Thus (a) holds and a fortiori also (b).

Theorem 4.

(a) $(n^2 e^4 \cdot \log 3)^{-\frac{1}{2}} 3^{\frac{n}{2}}$ *(for n large)* $\leq D_n(\exists,\exists) \leq 3^{\frac{n}{2}}$ *(for all n)*

(b) $\lim\limits_{n\to\infty} \frac{1}{n} \log D_n(\exists,\exists) = \frac{1}{2} \log 3$.

Proof: Define $\mathcal{V} = \mathcal{P}(\Omega)$, $\mathcal{E} = \big\{\{A,B\} : A,B \in \mathcal{V}\ ,\ A \neq B\ ,\ A \cap B = \varnothing\big\}$ and notice that

$$N = 2|\mathcal{E}| = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 3^n. \tag{4.6}$$

The upper bound follows with Theorem 1 (a).

To get the lower bound with Theorem 1 (b) we choose the graph $\mathcal{G}_k = \big(\mathcal{P}_k(\Omega), \mathcal{E}_k\big)$, where $\mathcal{E}_k$ is induced by $\mathcal{E}$ on $\mathcal{P}_k(\Omega)$.

Furthermore, we choose $k = \frac{1}{3}n$ and obtain

$$N = \binom{n}{\frac{1}{3}n}\binom{\frac{2}{3}n}{\frac{1}{3}n} > \frac{1}{n}\, 3^n, \tag{4.7}$$

$$D = \binom{\frac{2}{3}n}{\frac{1}{3}n} \leq 2^{\frac{2}{3}n}. \tag{4.8}$$

Compairing these quantities with those in the proof of Theorem 3 we notice that only N is decreased by a factor 2 and thus the result holds.

Theorem 5. *For $k < n-k$ and $\binom{n}{k} > \binom{n-k}{k} e^4 \cdot n \log 3$*

(a) $(n\, e^4 \log 3)^{-\frac{1}{2}} \big[\binom{n}{k}\binom{n-k}{k}\big]^{\frac{1}{2}} \leq D_n(\exists,\exists,k) \leq \big[\binom{n}{k}\binom{n-k}{k}\big]^{\frac{1}{2}}$ *(for all n)*

(b) $\lim\limits_{n\to\infty} \frac{1}{n} \log D_n(\exists,\exists,\varepsilon n) = \frac{1}{2}\big(h(\varepsilon) + (1-\varepsilon) h\big(\frac{\varepsilon}{1-\varepsilon}\big)\big)$.

Proof: For the "canonical" graph $(\mathcal{P}_k, \mathcal{E}_k)$ $N = \binom{n}{k}\binom{n-k}{k}$ and thus the upper bound follows in the usual way. Since $D = \binom{n-k}{k}$ and $N \leq 3^n$ we conclude with Theorem 1 that $D_n(\exists,\exists,k) \geq \left(\frac{N}{e^4 \log N}\right)^{\frac{1}{2}} \geq (n\, e^4 \log 3)^{-\frac{1}{2}} \big[\binom{n}{k}\binom{n-k}{k}\big]^{\frac{1}{2}}$ provided that (2.11) holds. But that is guaranteed by our assumption. An elementary calculation gives also (b).

Theorem 6.

(a) $\log^{-1}\binom{n}{k}\big[\binom{n}{k} - \binom{n-k}{k}\big] \leq I_n(\exists,\exists,k) \leq \big[\binom{n}{k}\big(\binom{n}{k} - \binom{n-k}{k}\big)\big]^{\frac{1}{2}}$

(b) $\lim\limits_{n\to\infty} \frac{1}{n} \log I_n(\exists,\exists,\varepsilon n) = h(\varepsilon)$.

Proof: In the canonical graph

$$N = \binom{n}{k}\left(\binom{n}{k} - \binom{n-k}{k} - 1\right), D = \binom{n}{k} - \binom{n-k}{k} - 1.$$

Condition (2.11) does not hold!

Since $I_n(\exists,\exists,k) \geq I_n(\forall,\exists,k)$ we get the lower bound in (a) from Theorem 8 in the next Section.

5. Asymptotic results via graph coloring of type $(\forall,\exists)$

Theorem 7.

(a) $\max\bigl((k\log n)^{-1}, n^{-1}\bigr)\binom{n-k}{k} \leq D_n(\forall,\exists,k) \leq \binom{n-k}{k} + 1$

(b) $\lim_{n\to\infty} \frac{1}{n}\log D_n(\forall,\exists,\varepsilon n) = (1-\varepsilon)h\left(\frac{\varepsilon}{1-\varepsilon}\right)$.

Proof: In the associated graph we have $d = \min_{x\in\mathcal{P}_k}\deg(x) = D = \binom{n-k}{k}$. Since $D_n(\forall,\exists,k) \geq m^*(\mathcal{G}_k)$, Theorem 2 implies

$$D_n(\forall,\exists,k) \geq \max\bigl((k\log n)^{-1}, n^{-1}\bigr)\binom{n-k}{k}.$$

From (b) in Theorem 2 we conclude that

$$D_n(\forall,\exists,k) \leq D + 1 = \binom{n-k}{k} + 1.$$

Theorem 8.

(a) $\log^{-1}\binom{n}{k}\left[\binom{n}{k} - \binom{n-k}{k}\right] \leq I_n(\forall,\exists,k) \leq \binom{n}{k} - \binom{n-k}{k}$

(b) $\lim_{n\to\infty} \frac{1}{n} I_n(\forall,\exists,\varepsilon n) = h(\varepsilon)$.

Proof: In the associated graph we have $d = D = \binom{n}{k} - \binom{n-k}{k} - 1$ and again the results follow from Theorem 2.

6. Exact solutions for intersecting systems

Recall the definitions of Section 4. The following equations are easy to derive.

Theorem 9.

(i) $I_n(\forall, \forall) = 2^{n-1}$

(ii) $I_n(\exists, \forall) = 2^{n-1}$

(iii) $I_n(\forall, \exists) = 2^{n-1}$

(iv) $I_n(\exists, \exists) = 2^{n-1} + 2^{n-2} - 1$

Proof of (i), (ii), and (iii):

Consider the family of clouds $\big\{\{A\} : x \in A\big\}$ for some fixed $x \in \Omega$. It has the $(\forall, \forall, I)$–property. Therefore $I_n(\forall, \exists) \geq I_n(\exists, \forall) \geq I_n(\forall, \forall) \geq 2^{n-1}$. Conversely, suppose that for a family of clouds $(\mathcal{A}_i)_{i=1}^{I_n(\forall,\exists)}$ for some j $|\mathcal{A}_j| = 1$, i.e. $\mathcal{A}_j = \{A_j\}$. Then A_j^c cannot occur in any other cloud, so we may as well count it for $\mathcal{A}_j$. Thus all clouds can be counted with at least two elements and $I_n(\forall, \exists) \leq 2^{n-1}$.

Proof of (iv): Observe that the empty set can be ignored, because it does not add anything to any cloud. Also, by the foregoing arguments at most 2^{n-1} clouds can have exactly one element.

Therefore

$$I_n(\exists, \exists) \leq 2^{n-1} + \left\lfloor \frac{2^{n-1} - 1}{2} \right\rfloor = 2^{n-1} + 2^{n-2} - 1.$$

On the other hand the following construction shows that the upper bound is tight.

In case $n = 2m+1$ use the elements in $\bigcup_{k=m+1}^{n} \mathcal{P}_k$ as the single members of clouds. The elements in $\bigcup_{k=1}^{m} \mathcal{P}_k$ can be paired to clouds, so that in every cloud $\{A, A'\}$ we have $|A \cup A'| \geq m+1$ and exactly one element is left over. Since $2(m+1) > n$ we have the $(\exists, \exists, I)$–property. Notice that $|\bigcup_{k=m+1}^{n} \mathcal{P}_k| + \frac{1}{2}\big(|\bigcup_{k=1}^{m} \mathcal{P}_k| - 1\big) = 2^{n-1} + \frac{1}{2}(2^{n-1} - 2) = 2^{n-1} + 2^{n-2} - 1$. In case $n = 2m$ choose the singletons from $\bigcup_{k=m+1}^{n} \mathcal{P}_k$ and from $\mathcal{P}_m$, but without choosing complementary sets. Pair then the remaining sets such that again $|A \cup A'| \geq m+1$.

7. Excess problems

A prototype of an excess problem is the determination of

$$\max_{\mathcal{A} \subset \mathcal{P}(\Omega_n), |\mathcal{A}| = N} \Big|\big\{(A, B) : A, B \in \mathcal{A} \text{ with } |A \triangle B| = 1\big\}\Big|. \tag{7.1}$$

It was conjectured by Harper [23] and proved by Lindsey [30] that generalized cylinders are optimal.

Another excess problem is the determination of

$$I_1(n,N,k) = \max_{\mathcal{A}\subset\mathcal{P}_k(\Omega_n),|\mathcal{A}|=N} \big|I(\mathcal{A},\mathcal{A})\big|, \tag{7.2}$$

where here and later we use the notation

$$I(\mathcal{A},\mathcal{B}) = \big|\{(A,B) : A\in\mathcal{A}, B\in\mathcal{B}, A\cap B\neq\varnothing\}\big|, \tag{7.3}$$

$$D(\mathcal{A},\mathcal{B}) = \big|\{(A,B) : A\in\mathcal{A}, B\in\mathcal{B}, A\cap B=\varnothing\}\big|. \tag{7.4}$$

$I_1(n,N,2)$ was characterized by Ahlswede and Katona [27]. The quantity

$$I_1(n,N) = \max_{\mathcal{A}\subset\mathcal{P}(\Omega_n),|\mathcal{A}|=N} \big|I(\mathcal{A},\mathcal{A})\big| \tag{7.5}$$

is assumed for a certain quasi–sphere (Ahlswede [26]).

For $\mathcal{A}\subset\mathcal{P}_k(\Omega_n)$ we use the complement

$$\mathcal{A}^{c_k} = \mathcal{P}_k(\Omega_n)\smallsetminus\mathcal{A} \tag{7.6}$$

and analyse here the quantities

$$I(n,k) = \max_{\mathcal{A}\subset\mathcal{P}_k(\Omega_n)} \big|I(\mathcal{A},\mathcal{A}^{c_k})\big|, \tag{7.7}$$

$$D(n,k) = \max_{\mathcal{A}\subset\mathcal{P}_k(\Omega_n)} \big|D(\mathcal{A},\mathcal{A}^{c_k})\big|. \tag{7.8}$$

Since

$$\big|I(\mathcal{A},\mathcal{A}^{c_k})\big| + \big|D(\mathcal{A},\mathcal{A}^{c_k})\big| = |\mathcal{A}|\left(\binom{n}{k} - |\mathcal{A}|\right)$$

there are relations between these quantities. However, they are not obvious, because $|\mathcal{A}|$ is not fixed. Our first result is of asymptotic nature.

Theorem 10. *For fixed* $k\geq 2$

$$\lim_{n\to\infty}\frac{1}{n^{2k-1}}I(n,k) = \frac{1}{4}\left(\frac{1}{(k-1)!}\right)^2.$$

Next we give exact answers for $k=2$.

Theorem 11. *For* $n=4m+\ell>2$

$$I(n,2) = \begin{cases} n\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil; & \ell=0,1,2\\ n\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil-1; & \ell=3.\end{cases}$$

Theorem 12.

$$D(n,2)=\binom{a^*}{2}\frac{(n-a^*)(n+a^*-5)}{2},$$

where a^* *is the largest integer* a *with*

$$\binom{a-2}{2}\le\frac{1}{2}\binom{n-2}{2}.$$

Proof of Thoerem 10:

We establish first auxiliary results concerning approximations of graphs by bipartite graphs.

For a graph $G=(\mathcal{V},\mathcal{E})$ and a vertex coloring (or bipartition) $\varphi:\mathcal{V}\to\{0,1\}$ let $L(G,\varphi)$ be the number of edges in $\mathcal{E}$ connecting vertices differently coloured under φ . We study $L(G)=\max_\varphi L(G,\varphi)$.

It is convenient to use the abbreviation $\overline{\alpha}=\lfloor\frac{\alpha}{2}\rfloor\lceil\frac{\alpha}{2}\rceil$.

Lemma 1. *(Folklore) For any graph* $G=(\mathcal{V},\mathcal{E})$

$$\frac{1}{4}\sum_{v\in\mathcal{V}}\deg(v)\le L(G).$$

Proof: Let $\{X_v:v\in\mathcal{V}\}$ be independent, identically distributed random variables with $\text{Prob}(X_v=0)=\text{Prob}(X_v=1)=\frac{1}{2}$.

Define

$$Y_{v,v'}=\begin{cases}X_v+X_{v'}\mod 2 & \text{, if } (v,v')\in\mathcal{E}\\ 0 & \text{, if } (v,v')\notin\mathcal{E}\end{cases}$$

and

$$Z=\frac{1}{2}\sum_v\sum_{v'}Y_{v,v'}=\sum_{(v,v')\in\mathcal{E}}Y_{v,v'},\quad\text{that is,}$$

the number of edges with differently colored vertices in the "random coloring $\{X_v: v\in\mathcal{V}\}$ ". Therefore

$$\begin{aligned}L(G)\ge\mathbb{E}\,Z&=\frac{1}{2}\sum_{v\in\mathcal{V}}\sum_{v':(v,v')\in\mathcal{E}}\mathbb{E}\;Y_{v,v'}\\&=\frac{1}{2}\sum_{v\in\mathcal{V}}\sum_{v':(v,v')\in\mathcal{E}}\frac{1}{2}=\frac{1}{4}\sum_{v\in\mathcal{V}}\deg(v).\end{aligned}$$

Now we derive an upper bound on $L(G)$. A family of graphs $\{G_j=(\mathcal{V}_j,\mathcal{E}_j):j\in J\}$ covers $G=(\mathcal{V},\mathcal{E})$, if $\mathcal{V}=\bigcup_{j\in J}\mathcal{V}_j$ and $\mathcal{E}\subset\bigcup_{j\in J}\mathcal{E}_j$.

Now clearly

$$L(G) \le \sum_j L(G_j)$$

and since

$$L(G_j, \varphi_j) \le |\varphi_j^{-1}(0)||\varphi_j^{-1}(1)| \le \overline{\frac{|\mathcal{V}_j|}{2}},$$

we have

$$L(G) \le \sum_j \overline{\frac{|\mathcal{V}_j|}{2}}.$$

We state this for the ease of reference.

Lemma 2. *For any graph* $G = (\mathcal{V}, \mathcal{E})$

$$L(G) \le \min_{\{G_j : j \in J\} \text{ covers } G} \sum_j \left\lceil \frac{|\mathcal{V}_j|}{2} \right\rceil \left\lfloor \frac{|\mathcal{V}_j|}{2} \right\rfloor .$$

We show first that

$$\underline{\lim}_{n\to\infty} \frac{1}{n^{2k-1}} I(n,k) \ge \frac{1}{4} \left(\frac{1}{(k-1)!} \right)^2 . \tag{7.9}$$

For this we define a graph G with vertex–set $\mathcal{V} = \mathcal{P}_k(\Omega_n)$ and edge–set $\mathcal{E} = \big\{(A,B) : A, B \in \mathcal{V}, A \cap B \neq \varnothing\big\}$. By Lemma 1

$$\begin{aligned} I(n,k) = L(G) &\ge \frac{1}{4} \sum_{v \in V} \deg(v) = \frac{1}{4} \binom{n}{k} \left[\binom{n}{k} - \binom{n-k}{k} - 1 \right] \\ &= \frac{1}{4} \binom{n}{k} \big[\frac{1}{k!} \big[n(n-1)\dots(n-k+1) - (n-k)(n-k-1)\cdot(n-2k+1) \big] - 1 \big] \\ &= \frac{1}{4} \binom{n}{k} \frac{1}{k!} \big[\big(n^k - n^{k-1} \sum_{i=0}^{k-1} i \big) - \big(n^k - n^{k-1} \sum_{j=k}^{2k-1} j \big) + o(n^{k-1}) \big] \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{n^{2k-1}} I(n,k) &\ge \frac{1}{4} \frac{\binom{n}{k}}{n^k} \frac{n^{k-1}}{n^{k-1} \cdot k!} \left(\sum_{j=k}^{2k-1} j - \sum_{i=0}^{k-1} i \right) + o(1) \\ &= \frac{1}{4} \left(\frac{1}{k!} \right)^2 \left[\frac{k(3k-1)}{2} - \frac{k(k-1)}{2} \right] + o(1) \\ &= \frac{1}{4} \left(\frac{1}{k!} \right)^2 k^2 + o(1). \end{aligned}$$

Finally, define for all $j \in \Omega_n$ G_j as the complete graph with the vertex set $\mathcal{V}_j = \{A \subset \mathcal{P}_k(\Omega_n) : j \in A\}$. Then $\{G_j : j = 1, 2, \ldots, n\}$ covers G and by Lemma 2, applied with $|\mathcal{V}_j| = \binom{n-1}{k-1}$,

$$I(n,k) = L(g) \le n \left\lfloor \frac{1}{2}\binom{n-1}{k-1} \right\rfloor \left\lceil \frac{1}{2}\binom{n-1}{k-1} \right\rceil \tag{7.10}$$

and therefore

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{2k-1}} I(n,k) \le \frac{1}{4}\left(\frac{1}{(k-1)!}\right)^2 .$$

Proof of Theorem 11:

From (7.10) we conclude for $k = 2$

$$I(n,2) \le n \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil . \tag{7.11}$$

In case $n = 4m+3$ we can do slightly better. The bound (7.11) is based on the inequality

$$\begin{aligned} I(n,2) &\le \sum_j \big|\{(A,A') : A, A' \in \mathcal{P}_2(\Omega_n), A \cap A' = \{j\}\}\big| \\ &= n \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil . \end{aligned}$$

However, when $n = 2m+3$, for any $\mathcal{A}$ there is at least one element of Ω_n , say j_0 , with

$$\big|\{A : A \in \mathcal{A}, j_o \in A\}\big| \ne 2m+1 = \frac{n-1}{2}, \tag{7.12}$$

because otherwise we would have

$$|\mathcal{A}| = \frac{1}{2} n \big(\frac{n-1}{2}\big) = \frac{1}{2}(4m+3)(2m+1)$$

in contradiction to the fact that $|\mathcal{A}|$ is integral. Since $\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil = 2m+1$, we conclude that

$$\begin{aligned} I(4m+3,2) &\le (n-1)\big(\frac{n-1}{2}\big)^2 + \big(\frac{n-1}{2} - 1\big)\big(\frac{n-1}{2} + 1\big) \\ &= n\big(\frac{n-1}{2}\big)^2 - 1 = (4m+3)(2m+1)^2 - 1. \end{aligned}$$

We turn now to the proof of achievability of the bound. We need a very special case $(k=2)$ of a well–known result.

General form of Baranyai's Theorem.

Let $n_1, n_2, \ldots, n_t$ be natural numbers such that $\sum_{i=1}^t n_i = \binom{n}{k}$. Then $\mathcal{P}_k(\Omega_n)$ can be partitioned into disjoint sets $P_1, \ldots, P_t$ such that $|P_i| = n_i$ and each $\ell \in \Omega_n$ is contained in exactly $\lceil \frac{n_i \cdot k}{n} \rceil$ or $\lfloor \frac{n_i \cdot k}{n} \rfloor$ members of P_i .

Choose $t = 2$, $k = 2$, $n_1 = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$, $n_2 = \left\lceil \frac{n(n-1)}{4} \right\rceil$, $\mathcal{A} = P_1$ and $\mathcal{A}^{c_2} = P_2$. Now one verifies that

$$\frac{\left\lfloor \frac{n(n-1)}{4} \right\rfloor \cdot 2}{n} = \begin{cases} \frac{n-1}{2} & , \text{ if } n = 4m \text{ or } 4m+2 \\ \frac{n-1}{2} - \frac{1}{n} & , \text{ if } n = 4m+2 \text{ or } 4m+3, \end{cases}$$

which implies $\left\lceil \frac{2|\mathcal{A}|}{n} \right\rceil = \lceil \frac{n-1}{2} \rceil$, $\left\lfloor \frac{2|\mathcal{A}|}{n} \right\rfloor = \lfloor \frac{n-1}{2} \rfloor$, where $n \neq 4m+3$.

Thus there are exactly $\lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil$ pairs $(A, B) \in \mathcal{A} \times \mathcal{A}^{c_2}$ with $A \cap B = \{j\}$ for all $j \in \Omega_n$. That means, $I(n,2) \geq n \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil$ when $n \equiv 0, 1, 2 \mod 4$. On the other hand, when $n = 4m+3$, again by Baranyai's Theorem ([34]), one can partition $\mathcal{P}_2(\Omega_n)$ into $\{\mathcal{A}, \mathcal{A}^{c_2}\}$ such that $|\mathcal{A}| = \frac{(4m+3)(2m+1)-1}{2}$ and there exists a $j_0 \in \Omega_n$ with $|\{A \in \mathcal{A} : j_0 \in A\}| = 2m = \left\lfloor \frac{2|\mathcal{A}|}{n} \right\rfloor$ and $|\{A \in \mathcal{A} : j \in A\}| = 2m+1 = \left\lceil \frac{2|\mathcal{A}|}{n} \right\rceil$ for $j \in \Omega_n \smallsetminus \{j_0\}$. Thus

$$\begin{aligned} I(4m+3, 2) &\geq (4m+2)(2m+1)^2 + 2m(2m+2) \\ &= (4m+3)(2m+1)^2 - 1. \end{aligned}$$

Proof of Theorem 12:

First we provide auxiliary results.

Lemma 3. *If* $\mathcal{A}, \mathcal{A}' \subset \mathcal{P}_2(\Omega_n)$ *has the properties*

(i) $|\mathcal{A}| = |\mathcal{A}'| = N$

(ii) $I(\mathcal{A}, \mathcal{A}) \geq I(\mathcal{A}', \mathcal{A}')$

then

(iii) $D(\mathcal{A}, \mathcal{A}^{c_2}) \geq D(\mathcal{A}', \mathcal{A}'^{c_2})$.

Proof: Observe that

$$N \cdot 2(n-2) = I(\mathcal{A}, \mathcal{A}) + I(\mathcal{A}, \mathcal{A}^{c_2}) = I(\mathcal{A}', \mathcal{A}') + I(\mathcal{A}', \mathcal{A}'^{c_2}) \tag{7.13}$$

and that

$$N \cdot \Big(\binom{n}{2} - N\Big) = D(\mathcal{A}, \mathcal{A}^{c_2}) + I(\mathcal{A}, \mathcal{A}^{c_2}) = D(\mathcal{A}', \mathcal{A}'^{c_2}) + I(\mathcal{A}', \mathcal{A}'^{c_2}). \tag{7.14}$$

Clearly, (ii), (7.13) and (7.14) imply (iii).

As in [27] we define the quasi–complete graph C_n^N with N edges and n vertices in the following way: i and j are connected for $i,j \leq s (i \neq j)$ and $s+1$ is connected with $1,2,\dots,t$, where s and t are determined by the unique representation

$$N = \binom{s}{2} + t, 0 \leq t < s.$$

Now notice that

$$D(n,2) = \max_{\mathcal{A} \subset \mathcal{P}_2(\Omega_n)} |D(\mathcal{A},\mathcal{A}^{c_2})| = \max_N D(n,N,2), \tag{7.15}$$

where

$$D(n,N,2) = \max_{\mathcal{A} \subset \mathcal{P}_2(\Omega_n), |\mathcal{A}|=N} |D(\mathcal{A},\mathcal{A}^{c_2})|.$$

By Lemma 3 here the maximum is assumed for a family $\mathcal{C}$ assuming the maximum in $\max_{\mathcal{A}:|\mathcal{A}|=N} |I(\mathcal{A},\mathcal{A})|$ and, since $|D(\mathcal{A},\mathcal{A}^{c_2})| = |D(\mathcal{A}^{c_2},\mathcal{A})|$, also $\mathcal{C}^{c_2}$ assumes $D(n, \binom{n}{2} - N, 2) = D(n,N,2)$. By Theorem 2 of [27] we can restrict the maximization to quasi–complete graphs, that is, $\mathcal{C}$ is the family of edges of such a graph. With the next Lemma it readily follows that the "quasi" can be dropped.

Lemma 4. *Let $\mathcal{C}$ be the edge set of a quasi–complete graph C_n^N with $N = \binom{s}{2} + t$, $1 \leq t \leq s$. Then with $A = \{s+1, t+1\}$, $B = \{s+1, t\}$ we have*

$$\big|D\big(\mathcal{C} \cup \{A\}, (\mathcal{C} \cup \{A\})^c\big)\big| - \big|D(\mathcal{C},\mathcal{C}^c)\big| = \big|D(\mathcal{C},\mathcal{C}^c)\big| - \big|D\big(\mathcal{C} \setminus \{B\}, (\mathcal{C} \setminus \{B\})^c\big)\big|. \tag{7.16}$$

Proof:

Since $\big|D(\mathcal{A},\mathcal{A}^c)\big| = |\mathcal{A}|\binom{n-2}{2} - \big|D(\mathcal{A},\mathcal{A})\big|$ the equation (7.16) is equivalent to

$$\big|D\big(\mathcal{C} \cup \{A\}, \mathcal{C} \cup \{A\}\big)\big| - \big|D(\mathcal{C},\mathcal{C})\big| = \big|D(\mathcal{C},\mathcal{C})\big| - \big|D\big(\mathcal{C} \setminus \{B\}, \mathcal{C} \setminus \{B\}\big)\big|. \tag{7.17}$$

This, however, is equivalent to

$$2\big|D(\{A\},\mathcal{C})\big| = 2\big|D(\{B\},\mathcal{C})\big|,$$

which obviously is true by the definitions of $\mathcal{C}$, A, and B.

Let now $\mathcal{C}$ be of maximal cardinality among the quasi–complete graphs with $\big|D(\mathcal{C},\mathcal{C}^c)\big| = D(n,2)$.

Then clearly

$$\big|D(\mathcal{C},\mathcal{C}^c)\big| \geq \big|D(\mathcal{C} \setminus \{B\}, (\mathcal{C} \setminus \{B\})^c)\big|$$

and (7.16) implies

$$\big|D\big(\mathcal{C} \cup \{A\}, (\mathcal{C} \cup \{A\}^c)\big| \geq \big|D(\mathcal{C},\mathcal{C}^c)\big|.$$

We can assume therefore that $\mathcal{C}$ is a complete graph and therefore necessarily for some s

$$|\mathcal{C}| = N = \binom{s}{2} \tag{7.18}$$

and $\big|D(\mathcal{C},\mathcal{C}^c)\big| = \binom{s}{2}\big[\binom{n-2}{2} - \binom{s-2}{2}\big]$.

Notice that for $C \in \mathcal{C}$ $\big|D(\{C\},\mathcal{C})\big| \le \frac{1}{2}\binom{n-2}{2}$, because otherwise $D\big(\mathcal{C} \smallsetminus \{C\}, (\mathcal{C} \smallsetminus \{C\})^c\big) > D(\mathcal{C},\mathcal{C}^c)$. Therefore

$$\big|D(\mathcal{C},\mathcal{C}^c)\big| \ge |\mathcal{C}|\frac{1}{2}\binom{n-2}{n} = \binom{s}{2}\frac{1}{2}\binom{n-2}{2}$$

and hence $\frac{1}{2}\binom{n-2}{2} \ge \binom{s-2}{2}$. If now also $\binom{s-1}{2} \le \frac{1}{2}\binom{n-2}{2}$, then

$$\begin{aligned}
&\binom{s+1}{2}\left[\binom{n-2}{2} - \binom{s-1}{2}\right] - \binom{s}{2}\left[\binom{n-2}{2} - \binom{s-2}{2}\right] \\
&= s\binom{n-2}{2} - \frac{(s+1-(s-3))s(s-1)(s-2)}{4} \\
&= s\binom{n-2}{2} - 2s\binom{s-1}{2} \ge 0
\end{aligned}$$

and this proves that the choice in Theorem 12 is best.

8. Open problems and conjectures

The reader can see at the chart in the Introduction where there are open problems. Systems of type $(\exists,\forall)$ are understood the least.

The famous Erdös/Ko/Rado Theorem says that

$$I_n(\forall,\forall,k) = \binom{n-1}{k-1} \quad \text{for} \quad n \ge 2k. \tag{8.1}$$

The determination of $I_n(\exists,\forall,k)$ seems to be a formidable task. In [20] it was shown that

$$I_n(\exists,\forall,2) = \begin{cases} n-1 & \text{for } n \in \mathbb{N} - \{3,5\} \\ n & \text{for } n = 3,5. \end{cases} \tag{8.2}$$

In the range $n \ge 2k$ only for $n = 5$ we have $I_n(\exists,\forall,2) > I_n(\forall,\forall,2)$. There we can partition $\mathcal{P}_2(\Omega_5)$ into $\{P_1, P_2, \dots, P_5\}$ with two disjoint two element sets in each P_i . The partition has the $(\exists,\forall)$–property and therefore $I_5(\exists,\forall,2) \ge 5 > 4 = I_n(\forall,\forall,2)$.

This construction can be generalized. By Baranyai's Theorem (stated in Section 7) for $n = 2k + m$, $0 \leq m < 2$, we can let $P_i (1 \leq i \leq M)$ have two disjoint k element sets. Here $M = \frac{1}{2}\binom{n}{k}$ and

$$I_{2k+m}(\exists, \forall, k) = \frac{1}{2}\binom{n}{k} = \frac{1}{2}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] \geq \binom{n-1}{k-1}$$

for $m \geq 0$ and equality holds exactly if $m = 0$.

<u>Problem 1:</u> Determine $I_n(\exists, \forall, k)$. We *conjecture* that $I_n(\exists, \forall, k) = \binom{n-1}{k-1}$ for $n \geq 3k$. This would, if true, be a stronger statement than the Erdös/Ko/Rado Theorem in the range specified.

For the $(\forall, \exists, I)$–problem Baranyai's partition can be used even more generally for all $n = \ell\, k + m,\ m < k$. There obviously

$$I_n(\forall, \exists, k) \geq \frac{1}{\ell}\binom{n}{k}. \tag{8.3}$$

Notice that $\frac{1}{\ell}\binom{n}{k} = \frac{1}{\ell}\frac{n}{k}\binom{n-1}{k-1} \geq \binom{n-1}{k-1}$, with equality iff $m = 0$.

We know from [20] that

$$I_n(\forall, \exists, 2) = \begin{cases} n & \text{for } \ n \in \mathbb{N} - \{1, 2, 4\} \\ n-1 & \text{for } \ n = 1, 2, 4. \end{cases}$$

In the range $n \geq 2k = 4$ for $n = 2\ell + 1$ the bound (8.3) gives $I_n(\forall, \exists, 2) \geq 2\ell + 1$; so it is tight for odd n . However, for even n we get $\frac{1}{\ell}\binom{2\ell}{2} = 2\ell - 1 = n - 1$; so here the bound is not tight.

<u>Problem 2:</u> Determine $I_n(\forall, \exists, k)$. We *conjecture* that $\lim_{n\to\infty} I_n(\forall, \exists, 3)\binom{n}{2}^{-1} = \frac{5}{4}$. We also have the lower bound

$$I_n(\exists, \exists, k) \geq \binom{n-1}{k-1} + \frac{1}{\ell}\binom{n-1}{k}. \tag{8.4}$$

This bound is obtained by choosing as clouds Baranyai partitions of $\mathcal{P}_k(\Omega_{n-1})$ and in addition all clouds containing exactly one k–element subset of Ω_n containing the element n .

However, the result of [20]

$$I_n(\exists, \exists, 2) \sim n^{\frac{3}{2}} \tag{8.5}$$

shows that for $k = 2$ the linear bound (8.4) is far from being optimal even in growth.

<u>Problem 3:</u> Determine $I_n(\exists, \exists, k)$.

We *conjecture* that $\lim_{n\to\infty} I_n(\exists, \exists, k)\binom{n}{k-1}^{-1} n^{-\frac{1}{2}} = 1$.

In [20] we proved that

$$\lim_{n\to\infty} D_n(\exists,\exists,2)n^{-2} = \lim_{n\to\infty} D_n(\forall,\exists,2)n^{-2} = \frac{1}{4}$$

and that $\lim_{n\to\infty} D_n(\exists,\forall,2)n^{-2} = \frac{1}{6}$.

<u>Problems 4–6:</u>[1] Determine $D_n(\exists,\forall,k)$, $D_n(\forall,\exists,k)$, and $D_n(\exists,\exists,k)$. We *conjecture* that $\lim_{n\to\infty} D_n(\exists,\forall,k)\binom{n}{k}^{-1} = \frac{1}{k+1}$ and that $\lim_{n\to\infty} D_n(\forall,\exists,k)\binom{n}{k}^{-1} = \frac{1}{2}$.

We turn now to comparable systems. It is easy to show that $C_n(\forall,\forall) = n+1$. Here maximal chains in $\mathcal{P}(\Omega_n)$ are optimal, that is, for any such chain every cloud contains exactly one of its members.

Borden [31] proved that

$$\lim_{n\to\infty} \frac{1}{n}\log C_n(\forall,\exists) = \log\frac{\sqrt{5}+1}{2}. \tag{8.6}$$

<u>Problem 7:</u> Determine $C_n(\exists,\forall)$.

There is an easier problem. Suppose that for any two clouds $\mathcal{A}_i,\mathcal{A}_j$ the requirement is that there is an $A_{ij}\in\mathcal{A}_i$ with $A_{ij}\subset A_j$ for *all* $A_j\in\mathcal{A}_j$ *or* $A_{ij}\supset A_j$ for *all* $A_j\in\mathcal{A}_j$.

If $C_n^*(\exists,\forall)$ denotes the maximal cardinality of such a system, then clearly

$$n+1 \le C_n^*(\exists,\forall) \le C_n(\exists,\forall) \tag{8.7}$$

and it is not hard to show that actually $C_n^*(\exists,\forall) = n+1$.

An important relation is that of Hamming distance δ for two words. We have shown via Theorem 1 that, in obvious notation, for a constant c , any $\varepsilon>0$ and n large

$$\frac{c}{n^{\frac{1}{2}+\varepsilon}}2^{\frac{n}{2}}\binom{n}{\delta}^{\frac{1}{2}} \le H_n(\forall,\forall,\delta) \le 2^{\frac{n}{2}}\binom{n}{\delta}^{\frac{1}{2}}. \tag{8.8}$$

<u>Problem 8:</u> Determine $H_n(\exists,\exists,k,\delta)$. We *conjecture* that $H_n(\exists,\exists,3,2)n^{-2} = \frac{1}{\sqrt{2}}$.

<u>Problem 9:</u> Determine $H_n(\forall,\exists,k,\delta)$. We *conjecture* that $H_n(\forall,\exists,3,2)n^{-1} = 2$.

[1] These conjectures have been proved by N. Alon and B. Sudakov in their paper "Disjoint Systems", which is to appear in Random Structures & Algorithms.

9. Discussion of further directions for higher level extremal problems

We mention now some directions of research. There are various combinations of these directions, which are left to the imagination of the reader.

I. The binary relations in (1.1) of the Introduction can be replaced by basic functions such as

$$h(A,B) = |A \triangle B|,\ m(A,B) = |A \setminus B|, u(A,B) = |A \cup B| \ \text{ or } \ i(A,B) = |A \wedge B|. \tag{9.1}$$

In particular the Hamming distance h can be studied under constraints such as in case $(\forall, \forall)$

$$h(A,B) = \delta \ \text{ for all } \ A, B \in \mathcal{A} \tag{9.2}$$

(equidistant code) or

$$h(A,B) \geq \delta \ \text{ for all } \ A, B \in \mathcal{A} \tag{9.3}$$

(code) or

$$h(A,B) \leq \delta \ \text{ for all } \ A, B \in \mathcal{A} \tag{9.4}$$

(specified diameter).

We get a new chart by allowing other types of cloud families with binary relations specified in (9.2) or (9.3) or (9.4).

Also, in the spirit of our first question in the Introduction one can fix $N = 2$ and look at extremal $\big(|\mathcal{A}_1|, |\mathcal{A}_2|\big)$. Then we come via (9.2) in case $(\forall, \forall)$ to the constant distance code pairs ([1], [2], [3], [4], [5], [6], [7]). Furthermore, via (9.4) we come to the vertex isoperimetric theorem in Hamming space (see [15], [25]).

Instead of binary relations one can study for instance the 4–words property of [32]. It includes the parity function for the Hamming distance. As in [20] one can also consider 1–sided conditions.

II. In all cases the conditions on clouds can be distinctness or disjointness.

III. A global condition can be replaced by local conditions. Viewing sets as 0–1–sequences one can introduce

— k–codes as set of words with pairwise distance $\geq \ell$ in any k components

— a k–diameter problem in a similar spirit

— k–antichains as set of words such that any two words are incomparable already on suitable k components

— there is a similar notion for chains.

IV. We have worked in the Boolean lattice. The work should be extended to other lattices such as

— multi sets

— the cubical poset

— the lattice of subspaces over $GF(q)$

— the projective space lattice

— the modular geometric lattice.

V. In case $N = 2$ their are 2 family problems, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are in different sets. For instance $\mathcal{A}_1 \subset \binom{\Omega}{k}$ and $\mathcal{A}_2 \subset \binom{\Omega}{k-1}$.

It seems that $(\forall, \exists)$–type problems are interesting here. Let us require a 1–sided condition, that is, for *all* $A \in \mathcal{A}_2$ there is a $B \in \mathcal{A}_1$ in proper relation with A . Let this relation be "incomparable". For fixed cardinality N_1 of $\mathcal{A}_1$ we maximize the cardinality N_2 of $\mathcal{A}_2$. This is equivalent to minimizing $|\mathcal{A}_2|$ for the relation "comparable". Kruskal [14] and subsequently Katona solved this problem. Notice that our maximization problem has the same answer for the 2–sided condition.

Now we choose the relation "intersect" or, equivalently, the relation "disjoint" with minimization. This in turn is equivalent for minimization under the relation "Containment" on the $n - (k-1)$'s level. Again we arrived at the classical Kruskal shadow problem. Now we *maximize* under the relation "comparable" or minimize under the relation "incomparable". This is a new shadow problem! The last case, maximization under "disjoint", is equivalent to maximization under "comparable" on level $n - (k-1)$ again.

VI. There are many *excess problems.* For instance as an extension of Sperner's Lemma, for $N > \binom{n}{\lfloor \frac{n}{2} \rfloor}$ find $\mathcal{A} \subset \mathcal{P}(\Omega)$ with $|\mathcal{A}| = N$ and maximal cardinality

$$\big|\{(A, A') : A, A' \in \mathcal{A}, A \supset\!|\!\subset A'\}\big|.$$

One can ask analogous questions for this problem and those in Section 7 for two families. In particular for N_1, N_2 fixed

$$\max_{\substack{\mathcal{A} \subset \mathcal{P}_k, \mathcal{B} \subset \mathcal{P}_{k-1} \\ |\mathcal{A}| = N_1, |\mathcal{B}| = N_2}} \big|\{(A, B) : A \in \mathcal{A}, B \in \mathcal{B}, A \supset\!|\!\subset B\}\big|.$$

VII. We have introduced notions of colorings for graphs in Sections 2,3. These notions are parallelled by new notions of independence (or stability) numbers. In particular they can be studied for product graphs occuring in Shannon's zero error capacity problem. For instance the pentagon has $(\exists, \exists)$–independence number 3 for disjoint clouds.

VIII. We wonder whether our work has any bearing on non–determinism in computing.

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