ON INTERACTIVE COMMUNICATION [3]

Rudolf Ahlswede [1], Ning Cai [1] and Zhen Zhang [2]

Keywords:

communication complexity, abstract sources, hypergraph covering and coloring, worst case complexity, average degree bound.

1.
Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

2.
Communication Sciences Institute
University of Southern California
Los Angeles, CA 90089–2565
Research supported in part by
NSF Grant NCR–920 5265 and
SFB 343, Diskrete Strukturen
in der Mathematik, Bielefeld

3. The paper was presented at the IEEE International Symposium on Information Theory, Trondheim, Norway, June 27 – July 1, 1994.

Abstract

Almost two decades ago Ahlswede introduced an abstract correlated source $(\mathcal{V}\times\mathcal{W},\mathcal{S})$ with outputs $(v,w)\in\mathcal{S}\subset\mathcal{V}\times\mathcal{W}$, where persons $P_\mathcal{V}$ and $P_\mathcal{W}$ observe v and w, resp. More recently Orlitsky considered the minimal number C_m of bits to be transmitted in m rounds to "inform $P_\mathcal{W}$ about v over 1 channel". He showed that $C_2\leq 4C_\infty+3$ and that generally $C_2\not\sim C_\infty$. We give a simpler example than Zhang and Xia for $C_3\not\sim C_\infty$. However, for the new model "inform $P_\mathcal{W}$ over 2 channels" 4 rounds are optimal for this example — a result, we conjecture in general. If both, $P_\mathcal{V}$ and $P_\mathcal{W}$, are to be informed over 2 channels about the other outcome, we determine asymptotically the complexities for all sources. In our last model "inform $P_\mathcal{V}$ and $P_\mathcal{W}$ over 1 channel" for all sources the total number T_2 of required bits is known asymptotically and T_∞ is bounded from below in terms of average degrees. There are exact results for several classes of regular sources. An attempt is made to discuss the methods of the subject systematically.

1. Introduction

The study of channels with several senders and receivers was initiated in 1961 by Shannon [1] and the first multi–user coding theorem was proved ten years later in [2]. This led to a great research activity during the seventies in an area, which is usually called Multi–user Information Theory. On the source coding side a strong impetus came in 1973 from the paper [3], which concerns a probabilistic model of correlated sources. In the same year, independently and almost unnoticed, in [4] a hypergraph coloring lemma was presented (see Lemma 1 below), which yields asymptotically optimal list codes for abstract (purely combinatorial) correlated sources. This work was continued in [5]. There it is shown that hypergraph coloring concepts are at the root of several probabilistic, semi–probabilistic, and non–probabilistic multi–user source coding problems.

Whereas in [4] correlated sources are modelled as a *sequence* of i.i.d. pairs of RV's $(X_t, Y_t)_{t=1}^{\infty}$, in [5] also just one pair (X, Y) is considered.

Again simultaneously and independently this more abstract view was also taken in [6] in a model for information exchange in distributed computing: We are given a function $f : \mathcal{V} \times \mathcal{W} \to \mathcal{Z}$, where $\mathcal{V}, \mathcal{W}$, and $\mathcal{Z}$ are usually finite, $\mathcal{V}$ outputs v and $\mathcal{W}$ outputs w . A person $P_{\mathcal{V}}$ observes v and another person $P_{\mathcal{W}}$ observes w . They can transmit messages to each other alternately over a binary noiseless channel and their goal is to find out the value $f(v, w)$ with minimal worst case transmission time. We denote this quantity by $C(f; P_{\mathcal{V}}, P_{\mathcal{W}})$. Specific ingredients here are:

1) No probabilistic assumptions on the source $(\mathcal{V}, \mathcal{W}, f)$ are made.
2) Correct decoding for all source outputs is required.
3) Both persons send messages according to a protocol. Each message is based on the input known to the transmitter and on his previously received messages.

   When a communicator transmits a message, the other knows when it ends, and when the last message ends, both communicators know that communication has ended.
4) Both persons use the same channel for transmitting their bits.
5) Both persons compute $f(v, w)$.

There are some basic variations of this model. We start with the work of [7], which was performed again independently of Yao's work.

I. We keep here assumptions 1) and 2) and otherwise assume that

3_I) $P_{\mathcal{V}}$ sends bits to $P_{\mathcal{W}}$.

5_I) $P_{\mathcal{W}}$ computes f .

The one–way communication complexity $C_1(f; P_{\mathcal{W}})$ is the minimal number of bits to be transmitted from $P_{\mathcal{V}}$ to $P_{\mathcal{W}}$ so that $P_{\mathcal{W}}$ can compute f .

Example 1

Choose $(\mathcal{V},\mathcal{W})=\big(\{1,2,\dots,\alpha\}^n,\{1,2,\dots,\alpha\}^n\big)$ and consider the parity of the Hamming distance φ_n , that is, for $(x^n,y^n)\in\mathcal{V}\times\mathcal{W}$

$$\varphi_n(x^n,y^n)=\begin{cases}1 & \text{if } d_H(x^n,y^n) \text{ is odd}\\ 0 & \text{if } d_H(x^n,y^n) \text{ is even.}\end{cases}$$

Then

$$C_1(\varphi_n;P_{\mathcal{W}})=\lceil n\cdot\log_2\alpha\rceil,$$

because for any $x^n,x'^n\in\mathcal{V}$ there is a $y^n\in\mathcal{W}$ with $\varphi_n(x^n,y^n)\neq\varphi_n(x'^n,y^n)$ and therefore the outputs of $\mathcal{V}$ have to be encoded differently.

II. Problems studied in [7] include the following. Suppose that $(X_t,Y_t)_{t=1}^\infty$ is a discrete memoryless source, where $X^n=X_1\dots X_n$ (resp. $Y^n=Y_1\dots Y_n$) takes values in $\mathcal{X}^n$ (resp. $\mathcal{Y}^n$), and suppose that $f_n:\mathcal{X}^n\times\mathcal{Y}^n\to\mathbb{N}$ is to be computed correctly with a probability tending to 1 as $n\to\infty$ by $P_{\mathcal{W}}$, who observes Y^n . How many bits must $P_{\mathcal{V}}$ transmit to $P_{\mathcal{W}}$?

Example 2

Let X_t and Y_t be independent for $t=1,2,\dots$ and let them take their values with equal probabilities. The so–called 1–Bit Theorem ([7],[8]) implies that a computation in the sense described requires for the parity function φ_n a one–way transmission rate $\log_2\alpha$ for $\alpha\geq 3$. Even though 1 bit of information about the value of f is obtained by $P_{\mathcal{W}}$, $P_{\mathcal{V}}$ has to give full information about X^n !

III. A yet unsolved problem is concerned with the situation, where both $P_{\mathcal{V}}$ and $P_{\mathcal{W}}$ can inform a third person, say P_f , about their source outputs. What are the necessary rates of transmission to enable P_f to compute $f_n:\mathcal{X}^n\times\mathcal{Y}^n\to\mathbb{N}$ with probability approaching 1 as $n\to\infty$? This problem can be considered in a purely combinatorial setting. An encoding in this case is a product of partitions of $\mathcal{X}^n$ and $\mathcal{Y}^n$, with monochromatic members. This in turn is a special case of what has been called in [5] a strict coloring of orthogonal hypergraphs. A special case of such hypergraphs is a triple $(\mathcal{V},\mathcal{W},S)$ with $S\subset\mathcal{V}\times\mathcal{W}$, which describes which values for (v,w) are possible.

An exchange of ideas from abstract source coding theory [5] and distributed computing [6] has been very fruitful for the advancement of other models and new methods. In particular in 1983, during a visit of the first author to the Information System Laboratory in Stanford, where Yao's ideas were popular, there were stimulating discussions with Abbas El Gamal, King F. Pang, Tom Cover, and Alon Orlitsky. The effect was that starting with [9], [10] there was great interest in Bielefeld in determining communication complexity for specific functions. This gave impetus also to combinatorial extremal theory ([11] – [15]) and, as one of the highlights, led to the 4–words inequality [16].

Whereas the papers [17] – [19] concern $C(f; P_{\mathcal{V}}, P_{\mathcal{W}})$ a seemingly essential progress in the study of $C(f; P_{\mathcal{W}})$ (only $P_{\mathcal{W}}$ computes f) was made in [20].

On the other side the Stanford group picked up the idea of abstract source coding in [21] – [23] and combined it with Yao's idea of exchanging messages in several rounds.

(Cum grano salis it can be said, that this can be traced back to [1]. See also the discussion in [26]).

It is very interesting to see what has happened in the last decade in this direction, called interactive data communication (see [21] – [25] and [27] – [29]), mainly in several contributions of Alon Orlitsky ([22] – [25]).

IV. We recall first in our terminology the model studied. Given are the (abstract) correlated source $(\mathcal{V}, \mathcal{W}, S)$ and two communicators $P_{\mathcal{V}}$, an informant, and $P_{\mathcal{W}}$, a recipient. $P_{\mathcal{V}}$ knows output $v \in \mathcal{V}$ and $P_{\mathcal{W}}$ knows output $w \in \mathcal{W}$. Both communicators want the recipient $P_{\mathcal{W}}$ to learn v without error, whereas the informant $P_{\mathcal{V}}$ may or may not learn w . They alternately transmit messages with finite sequences of bits over the same binary noiseless channel by a predetermined protocol. The total number of bits are counted. $C_m\big((\mathcal{V}, \mathcal{W}, S), P_{\mathcal{W}}\big)$ is then the minimal number of bits to be exchanged in m rounds in the worst case for a best protocol.

Since the hypergraph terminology is more established we write $C_m(\mathcal{H}, P_{\mathcal{E}})$ instead of $C_m\big((\mathcal{V}, \mathcal{W}, S), P_{\mathcal{W}}\big)$. The main known results about $C_m(\mathcal{H}, P_{\mathcal{E}})$ are mentioned in the Appendix. After Orlitsky [23] has shown that C_2 may be much smaller than C_∞ and Zhang and Xia [29] have shown that even C_3 is not optimal, that is, may be smaller than C_∞ , we *conjecture* that C_4 is optimal. In so far we have the following results:

Already for a nice structure such as the uniform k–regular hypergraph $\left([n], \binom{[n]}{k}\right)$ $C_2 \sim C_3$ and $C_3 > C_4$. Thus we have now a simpler example for "4 messages are better than 3" than the one of [29]. We actually determine C_2 .

In another direction we study a question of [28] about C_2 for product hypergraphs.

V. As in the contexts of [4] and [5], hypergraph coloring and covering play a key role also in establishing achievability results (direct parts) for "interactive communication". Again, as in [5], often they have to be used in combination. Key roles are played by the following three lemmas.

Coloring Lemma 1. *([4])*

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ *be a hypergraph with* $D_{\mathcal{E}} = \max_{E \in \mathcal{E}} |E| \leq L$ *. If for some* $t \in \mathbb{N}$

$$|\mathcal{E}| \binom{D_{\mathcal{E}}}{t} \cdot L^{1-t} < 1, \tag{1.1}$$

then a coloring $\varphi : \mathcal{V} \to \{1, 2, \ldots, L\}$ *exists with*

$$|\varphi^{-1}(i) \cap E| < t \ \text{ for all } \ i = 1, \ldots, L \ \text{ and all } \ E \in \mathcal{E}.$$

We give the proof, because it is very simple.

Proof: For every $A \subset \mathcal{V}$ define

$$\mathcal{F}(A) = \{\varphi : \mathcal{V} \to \{1, \dots, L\} : \varphi \text{ is constant on } A\}.$$

So the set $\mathcal{F}_E^t$ of colorings that are bad on edge E (that is, have the same color on t vertices in E) is given by

$$\mathcal{F}_E^t = \bigcup_{\substack{A \subset E \\ |A|=t}} \mathcal{F}(A)$$

and the set $\mathcal{F}^t$ of all bad colorings equals $\bigcup\limits_{E \in \mathcal{E}} \mathcal{F}_E^t$.

The set $\mathcal{F}$ of all colorings $\varphi : \mathcal{V} \to \{1, \dots, L\}$ has the cardinality $|\mathcal{F}| = L^{|\mathcal{V}|}$ and we also have

$$|\mathcal{F}_E^t| \le \binom{D_\mathcal{E}}{t} \cdot L \cdot L^{|\mathcal{V}|-t}, \tag{1.2}$$

because $|E| \le D_\mathcal{E}$ for all $E \in \mathcal{E}$, $\binom{|E|}{t} \le \binom{D_\mathcal{E}}{t}$, one of L colors is needed to color the vertices in A , and there are $L^{|\mathcal{V}|-t}$ possible colorings of the vertices outside A .

The rough union bound gives now

$$|\mathcal{F}^t| \le |\mathcal{E}| \binom{D_\mathcal{E}}{t} L \cdot L^{|\mathcal{V}|-t}. \tag{1.3}$$

Therefore

$$\frac{|\mathcal{F}^t|}{|\mathcal{F}|} \le |\mathcal{E}| \binom{D_\mathcal{E}}{t} L^{1-t} < 1 \quad \text{(by assumption)}$$

and there exists a good coloring.

In [5] several other coloring concepts were introduced.

For (L, λ)–colorings of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ in every edge $E \in \mathcal{E}$ at least $(1-\lambda)|E|$ colors occur only once. An $(L, 0)$–coloring is called strict coloring or simply L–coloring . Their analysis reduces to coloring the associated graph $(\mathcal{V}, \mathcal{E}^*)$, where $\mathcal{E}^* = \big\{\{v, v'\} : v, v' \in \mathcal{V} \text{ and } \{v, v'\} \in E \text{ for some } E \in \mathcal{E}\big\}$.

The greedy algorithm gives the following simple result.

Coloring Lemma 2.

For a graph $(\mathcal{V}, \mathcal{E})$ *with maximum degree* $D_\mathcal{V} \triangleq \max\limits_{v \in \mathcal{V}} \deg(v) \le D$ *there exists a strict* L*–coloring , if* $L \ge D + 1$.

For our results mentioned under IV we also use another old result of general nature.

Covering Lemma 3. ([5])

For a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ *there is a covering* $\mathcal{C}, \mathcal{C} \subset \mathcal{E}$ *, of the vertex set* $\mathcal{V}$ *(that is,* $\bigcup_{E \in \mathcal{C}} E = \mathcal{V}$ *) with*

$$|\mathcal{C}| \leq \left\lceil |\mathcal{E}| d_{\mathcal{V}}^{-1} ln |\mathcal{V}| \right\rceil ,$$

where $d_{\mathcal{V}} = \min_{v \in \mathcal{V}} \deg(v)$ *(defined in (2.2)).*

To get an idea of the scope of this Covering Lemma we discuss several applications.

Many problems in interactive communication reduce to covering a hypergraph $\mathcal{H}^* = (\mathcal{V}^*, \mathcal{E}^*)$ by a family $\tilde{\mathcal{F}}$ of subhypergraphs of $\mathcal{H}^*$, with minimal size and/or certain required properties in the following sense. "For all $E^* \in \mathcal{E}^*$ there is an $\tilde{\mathcal{H}} = (\mathcal{V}^*, \tilde{\mathcal{E}}) \in \tilde{\mathcal{F}}$, such that $E^* \in \tilde{\mathcal{E}}$." It is clear that to such a covering of $\mathcal{H}^*$ by its subhypergraphs corresponds a covering of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ in the sense of [5], if we define $\mathcal{H}$ by $\mathcal{V} = \mathcal{E}^*$, $\mathcal{E} = \left\{ E(\tilde{\mathcal{H}}) : \tilde{\mathcal{H}} \in \mathcal{F}^* \right\}$, and $E(\tilde{\mathcal{H}}) = \left\{ E^* \in \mathcal{V} : E^* \in \tilde{\mathcal{E}} \right\}$, where $\mathcal{F}^*$ is the family of subhypergraphs of $\mathcal{H}^*$ with the required properties.

1. It seems to us that among the concepts, not already contained in [5], the most basic is "perfect hashing", which comes from computer science (c.f. [30]). It can be looked at it in several ways. We have the following in mind.

   Strict colorings of hypergraphs often require a large number of colors. In many cases it suffices to work with several functions $f_1, \ldots, f_k : \mathcal{V} \to B$, $B = \{1, 2, \ldots, b\}$.

   $f = (f_1, \ldots, f_k)$ is a *perfect hashing* of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, if for every $E \in \mathcal{E}$ there is an f_i with
   $$f_i(v) \neq f_i(v') \quad \text{for} \quad v, v' \in E; v \neq v'.$$

   This means that $\mathcal{H}$ can be decomposed into hypergraphs $\mathcal{H}_i = (\mathcal{V}, \mathcal{E}_i)$, $\bigcup_i \mathcal{E}_i = \mathcal{E}$ such that f_i is a strict coloring of $\mathcal{H}_i$. In other words, a perfect hashing of $\mathcal{H}$ is just a covering of $\mathcal{H}$ by its b–colorable subhypergraphs. Thus the Covering Lemma is applicable. The proof of Theorem 10 (i) in Section 7 uses the Covering Lemma along these lines in order to get a perfect hashing for $\mathcal{H} = \left([n], \binom{[n]}{k} \right)$.

   In applications it is relevant that kb is usually much smaller than $\chi(\mathcal{H})$.

2. "The football league problem", of [22] means covering the edges of a complete graph by the edge–sets of bipartite graphs.

   Actually, here the result of [22] is better, because it is constructive.

3. In this last example we show how to get the major part of Lemma 1 in [28] from the Covering Lemma:

   For $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $D_{\mathcal{E}} \geq 31$ and $J = \lceil \theta D_{\mathcal{E}} \log |\mathcal{V}| \rceil$ (for a constant θ), there is a family $\{f_j\}_{j=1}^{J}$ of functions $f_j : \mathcal{V} \to \{1, 2, \ldots, D_{\mathcal{E}}\}$ such that for all E there exists a $j \in \{1, \ldots, J\}$ with

   $$|f_j^{-1}(\alpha) \cap E| \leq \log D_{\mathcal{E}}$$

for all $\alpha \in \{1, \dots, D_\mathcal{E}\}$.

Indeed, let $\mathcal{V}^* = \mathcal{E}$ and let $\mathcal{H}^* = (\mathcal{V}^*, \mathcal{E}^*)$ be defined such that to each $f : \mathcal{V} \to \{1, 2, \dots, D_\mathcal{E}\}$ corresponds an edge

$$E_f = \left\{E \in \mathcal{V}^* : |f^{-1}(\alpha) \cap E| \leq \log D_\mathcal{E} \text{ for all } \alpha \in \{1, 2, \dots, D_\mathcal{E}\}\right\} \in \mathcal{E}^*,$$

and conversely. Then the above follows from the Covering Lemma for $\mathcal{H}^*$.

We have made our point that we are dealing with a covering problem. Special properties can sometimes yield better bounds than the general bound in the Covering Lemma.

Finally we mention that Lovász's Local Lemma has been used successfully by Orlitsky in coloring problems, for instance in [24]. It yields the following improvement of Coloring Lemma 1.

Strengthened Coloring Lemma.

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ *be a hypergraph with* $D_\mathcal{E} = \max_{E \in \mathcal{E}} |E| \leq L$. *If for some* $t \in \mathbb{N}$

$$\frac{e \cdot D_\mathcal{E}^2 D_\mathcal{V}}{t!} \leq 1, \tag{1.4}$$

then a coloring $\varphi : \mathcal{V} \to \{1, 2, \dots, L\}$ *exists with*

$$|\varphi^{-1}(i) \cap E| < t \quad \textit{for all} \quad i = 1, 2, \dots, L$$

and all $E \in \mathcal{E}$.

Proof: A consequence of the Local Lemma states (see [31, pp. 53–55]):

Let $A_1, \dots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of at least $n - d$ other events A_j . If

$$\max_{1 \leq i \leq n} Pr(A_i) \leq \frac{1}{e(d+1)}, \quad \text{then} \quad Pr\left(\bigwedge_{i=1}^{n} \overline{A}_i\right) > 0. \tag{1.5}$$

Imagine now that a coloring function Φ is choosen according to the uniform distribution on $\mathcal{F}$. Then the event $A_E = \{\Phi \in \mathcal{F}_E^t\}$ has by (1.2)

$$\text{Prob}(A_E) = \frac{|\mathcal{F}_E^t|}{|\mathcal{F}|} \leq \binom{D_\mathcal{E}}{t} L^{1-t}. \tag{1.6}$$

Instead of the rough union bound leading to (1.3) we use now that by (1.5) for $L = D_\mathcal{E}$ $\text{Prob}\left(\bigwedge_{E \in \mathcal{E}} \overline{A}_E\right) > 0$ (and there exists the desired coloring), if

$$\frac{D_\mathcal{E}}{t!} \leq \frac{1}{e D_\mathcal{E} \cdot D_\mathcal{V}}. \tag{1.7}$$

The Covering Lemma can be strengthened in the same way. Since it is not used in this paper, we omit the details and refer to the analogous version for perfect hashing in Lemma 3 of [24].

VI. We recall previous work on interactive communication (see [21] – [25] and [27] – [29]) addresses the case in which $P_{\mathcal{E}}$ is to be informed about the vertex $v \in \mathcal{V}$ and the case in which $P_{\mathcal{V}}$ and $P_{\mathcal{E}}$ communicate over the same binary noiseless channel and the total number of bits are counted.

In [5] on page 236 we point out that it makes a difference, if the transmission in different directions runs over different channels (distinction between actual and potential rate!).

Here it is not the sum of the number of bits that matters, but the "capacity" of each channel to cope with the possible bits in each direction. We indicate the two channels by writing now $\overline{\overline{C}}_m(\mathcal{H}, P_{\mathcal{E}})$ for the m–round communication complexity.

Moreover, we denote by $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_{\mathcal{E}})$ the region of possible pairs of bit numbers. Our results for this model are almost complete. In particular, we have a nice general lower bound, which is for instance tight for the uniform k–regular hypergraph.

VII. To our surprise the study of other models was even more rewarding. As already in Yao's model we consider situations where both persons, $P_{\mathcal{V}}$ and $P_{\mathcal{E}}$, have to inform each other about there outputs. We write here $C_m(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$ for the communication complexity.

The corresponding region for two channels is denoted by $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$ and its minimal rate sum is denoted by $\overline{\overline{C}}_m(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$.

It turns out that Coloring Lemmas 1 (and its strengthening) and 2 are already sufficient tools for deriving upper bounds.

More specifically, Coloring Lemma 1 alone already gives us

$$\overline{\overline{\mathcal{C}}}_m^{\ell_{\mathcal{V}}, \ell_{\mathcal{E}}}(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}),$$

that is, the resp. communication complexities, where E is known to $P_{\mathcal{V}}$ and v is known to $P_{\mathcal{E}}$ only as a member of lists of sizes $\ell_{\mathcal{V}}$ and $\ell_{\mathcal{E}}$, resp.

The reduction to list sizes 1 uses Coloring Lemma 2. It essentially uses that both persons are to be informed!

List size 1 complexities and complexities for small list sizes are essentially equal here. This changes drastically, if only $P_{\mathcal{E}}$ is to be informed. (Compare [4] for a similar phenomenon for zero–error problems in channel coding). Here the hashing idea becomes relevant.

We obtain complete results for $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$. We actually show that $\overline{\overline{\mathcal{C}}}_3(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}) = \overline{\overline{\mathcal{C}}}_\infty(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$. We also give a lower bound on $\mathcal{C}_\infty(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}})$, which we expect to be tight. It uses an idea of decomposing $\mathcal{H}$, which was motivated by the decomposition of [5], page 225.

Finally we mention that we have constructions based on Baranyai's Theorem ([32] and [33, pp. 50–56]) and also on Vizing's Theorem [34]. Thus more classical combinatorics has entered the field.

The presentation of our results and proofs proceeds as follows. We begin with the model for two recipients and two channels. It has not been considered in the literature. However, mathematically there is a very close connection to work in [24]. Nevertheless, for the benefit of the reader we give complete proofs, because they help building up a certain intuition for the analysis of the other models. Also these proofs in Sections 2 and 3 are short and the examples are new by all standards.

We start from first principles by considering first the easier case of list decision in Section 2.

The result appears in Theorem 1. It is based on Coloring Lemma 1. In Section 3 we get via Coloring Lemma 2 the bit numbers regions for exact decisions and $m \geq 3$ rounds (Theorem 2). We also settle the case $m = 2$ (and thus all cases) with Theorem 3.

Next we consider $C_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$, that is, the smallest worst case bit number achievable in m rounds for two recipients. For $m = 2$ we give first essentially exact bounds for two special cases, the general graph (using Vizing's Theorem) and the complete k–uniform hypergraph (using Baranyai's Theorem).

In the same Section 4 we then settle the general case with Theorem 6.

Next we show in Section 5 that $C_\infty(\mathcal{H}; P_\mathcal{E}, P_\mathcal{V}) \geq \log \overline{D}_\mathcal{E} + \log \overline{D}_\mathcal{V}$ (Average–degree Lemma).

The study of the model with one recipient starts in Section 6 for two channels. Again we investigate the complete k–regular hypergraph and derive a lower bound on the bit number $b_\mathcal{E}$ on the channel $\mathcal{E} \to \mathcal{V}$, if the bit number $b_\mathcal{V}$ on the channel $\mathcal{V} \to \mathcal{E}$ is fixed (Theorem 8). This bound is for $m = \infty$. It coincides with the upper bound already for $m = 4$ (Theorem 9).

We did not yet succeed to derive an analogous result, if only one channel is used. However, we determined C_2 for the same hypergraph (Theorem 10). Whereas the upper bound uses a familiar Covering Lemma from [5], the lower bound is based on new ideas of some independent combinatorial interest. Moreover, we establish $C_2 \sim C_3$ for $k = 0(\log n)$ and demonstrate that $C_3 < C_4$. Thus we have obtained a simpler hypergraph than that of Zhang and Xia, for which three "messages" don't suffice.

In Section 8 we study a question of M. Noar, A. Orlitsky and P. Shor ([28]) concerning their "amortized complexity" A_2 and take first steps towards a characterization of this two rounds quantity.

Finally, in an Appendix some earlier basic results of Alon Orlitsky are stated for the orientation of the reader.

2. A CHARACTERIZATION OF $\overline{\overline{\mathcal{C}}}_m^{\ell_\mathcal{V},\ell_\mathcal{E}}(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$ FOR SMALL LIST SIZES $\ell_\mathcal{V}, \ell_\mathcal{E}$.

The essential parameters here are

$$D_\mathcal{V} = \max_{v\in\mathcal{V}} \deg(v) \quad \text{and} \quad D_\mathcal{E} = \max_{E\in\mathcal{E}} \deg(E), \tag{2.1}$$

where

$$\deg(v) = |\{E \in \mathcal{E} : v \in E\}| \quad \text{and} \quad \deg(E) = |\{v \in \mathcal{V} : v \in E\}| = |E|. \tag{2.2}$$

We begin with $m = 2$. Clearly, on the channel $\mathcal{V} \to \mathcal{E}$ the number of bits $b_\mathcal{V}$ to be transmitted in the worst case satisfies

$$b_\mathcal{V} \geq \lceil \log D_\mathcal{E} \rceil. \tag{2.3}$$

Analogously, the number of bits $b_\mathcal{E}$ to be transmitted in the worst case over the channel $\mathcal{E} \to \mathcal{V}$ satisfies

$$b_\mathcal{E} \geq \lceil \log D_\mathcal{V} \rceil. \tag{2.4}$$

Moreover, if we require v to be on a list $\mathcal{V}(v, E)$ and E to be on a list $\mathcal{E}(v, E)$, where

$$|\mathcal{V}(v, E)| \leq \ell_\mathcal{V} \quad \text{and} \quad |\mathcal{E}(v, E)| \leq \ell_\mathcal{E}, \tag{2.5}$$

then the numbers of bits needed satisfy

$$b_\mathcal{V}(\ell_\mathcal{V}) \geq \lceil \log \frac{D_\mathcal{E}}{\ell_\mathcal{V}} \rceil, \tag{2.6}$$

$$b_\mathcal{E}(\ell_\mathcal{E}) \geq \lceil \log \frac{D_\mathcal{V}}{\ell_\mathcal{E}} \rceil. \tag{2.7}$$

Also, these bounds hold for any $m \geq 2$.

Next we show that they are essentially optimal.

Actually, this readily follows from Coloring Lemma 1, one of the most basic tools in this area. There are only very few methods of comparable significance for the subject.

We choose $L = D_\mathcal{E}$ and $t = \ell_\mathcal{V}$. The coloring φ of Coloring Lemma 1 serves as encoding function $f_\mathcal{V}$ and the list for $P_\mathcal{E}$ is then

$$\mathcal{V}(v, E) = \big\{v' : v' \in E, f_\mathcal{V}(v') = f_\mathcal{V}(v)\big\}.$$

Any list size $\ell_\mathcal{V}$ with

$$\ell_\mathcal{V}! > |\mathcal{E}| D_\mathcal{E} \tag{2.8}$$

is achievable, because sufficient for (1.1) is $|\mathcal{E}| L < t!$.

Since $\ell_\mathcal{V}! > \left(\frac{\ell_\mathcal{V}}{e}\right)^{\ell_\mathcal{V}}$, we can choose $\ell_\mathcal{V}$ such that

$$\ell_\mathcal{V}(\log \ell_\mathcal{V} - \log e) > \log |\mathcal{E}| D_\mathcal{E}$$

and certainly such that

$$\ell_{\mathcal{V}} = \lceil \log |\mathcal{E}| D_{\mathcal{E}} \rceil + 6. \tag{2.9}$$

We need

$$\lceil \log L \rceil = \lceil \log D_{\mathcal{E}} \rceil \tag{2.10}$$

bits on the channel $\mathcal{V} \to \mathcal{E}$.

Symmetrically, there is an encoding $f_{\mathcal{E}}$ which requires $\lceil \log D_{\mathcal{V}} \rceil$ bits on the channel $\mathcal{E} \to \mathcal{V}$ and leaves $P_{\mathcal{V}}$ with a list

$$\mathcal{E}(v, E) = \big\{E' : v \in E', f_{\mathcal{E}}(E') = f_{\mathcal{E}}(E)\big\}$$

of size

$$\ell_{\mathcal{E}} = \lceil \log |\mathcal{V}| D_{\mathcal{V}} \rceil + 6. \tag{2.11}$$

To see this, just apply the former proof to the dual hypergraph

$$\big(\mathcal{V}^*, \mathcal{E}^*\big), \mathcal{V}^* = \mathcal{E}, \mathcal{E}^* = \{E_v = \{E : v \in E\} : v \in \mathcal{V}\}\big). \tag{2.12}$$

We summarize our findings.

Here and later we use the notation $a \sim c$, if $a = c + o(c)$ and

$$(a_1, a_2) \sim (c_1, c_2), \quad \text{if} \quad (a_1, a_2) = (c_1, c_2) + \big(o(c_1), o(c_2)\big). \tag{2.13}$$

We say that pairs of numbers of bits $(b_{\mathcal{V}}, b_{\mathcal{E}})$ are achievable with list sizes $(\ell_{\mathcal{V}}, \ell_{\mathcal{E}})$ iff there are encoding functions $(f_{\mathcal{V}}, f_{\mathcal{E}})$ with these list sizes and

$$b_{\mathcal{V}} \sim \log \|f_{\mathcal{V}}\|, b_{\mathcal{E}} \sim \log \|f_{\mathcal{E}}\|. \tag{2.14}$$

Theorem 1. *For the list size pairs*

$$(\ell_{\mathcal{V}}, \ell_{\mathcal{E}}) \sim \big(\log |\mathcal{V}| D_{\mathcal{V}}, \log |\mathcal{E}| D_{\mathcal{E}}\big)$$

the set of achievable pairs of bit numbers in m *rounds is given by*

$$\bar{\bar{\mathcal{C}}}_m^{\ell_{\mathcal{V}}, \ell_{\mathcal{E}}}(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \big\{(b_{\mathcal{V}}, b_{\mathcal{E}}) : b_{\mathcal{V}} \geq \log D_{\mathcal{V}}, b_{\mathcal{E}} \geq \log D_{\mathcal{E}}\big\}$$

for $m \geq 2$.

Remark 1: This result can be improved by using the strengthened Coloring Lemma. For this see [24] with an analogous situation. We demonstrate the effect in Section 3 while improving Theorem 2A to Theorem 2B.

3. A CHARACTERIZATION OF $\overline{\overline{\mathcal{C}}}_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$: THREE ROUNDS SUFFICE

Now we use the simple Coloring Lemma 2 in conjunction with the previous result. Suppose that $\big(f_{\mathcal{V}}(v), f_{\mathcal{E}}(E)\big) = (i,j)$, then both, $P_{\mathcal{V}}$ and $P_{\mathcal{E}}$, know that $v \in \mathcal{V}_i = \big\{v' : f_{\mathcal{V}}(v') = i\big\}$ and $E \in \mathcal{E}_j = \big\{E' : f_{\mathcal{E}}(E') = j\big\}$. Thus the hypergraph $(\mathcal{V}, \mathcal{E})$ has been reduced to $\mathcal{H}_{i,j} = (\mathcal{V}_i, \mathcal{E}_j)$. Furthermore, we know that

$$D_{\mathcal{V}_i} \le \ell_{\mathcal{E}} = \lceil \log |\mathcal{V}| D_{\mathcal{V}} \rceil, \tag{3.1}$$

$$D_{\mathcal{E}_j} \le \ell_{\mathcal{V}} = \lceil \log |\mathcal{E}| D_{\mathcal{E}} \rceil. \tag{3.2}$$

We introduce now the associated graph $\mathcal{G}_{i,j} = (\widetilde{\mathcal{V}}_i, \widetilde{\mathcal{E}}_j)$, where $\widetilde{\mathcal{V}}_i = \mathcal{V}_i$

$$\widetilde{\mathcal{E}}_j = \big\{(v,v') : v, v' \in \mathcal{V}_i \text{ and for some } F \in \widetilde{\mathcal{E}}_j \quad \{v,v'\} \subset F\big\}.$$

Notice that

$$D_{\widetilde{\mathcal{V}_i}} \le D_{\mathcal{V}_i}(D_{\mathcal{E}_j} - 1) \le \ell_{\mathcal{E}}(\ell_{\mathcal{V}} - 1) \le \lceil \log |\mathcal{V}| D_{\mathcal{V}} \rceil \cdot \lceil \log |\mathcal{E}| D_{\mathcal{E}} \rceil. \tag{3.3}$$

By Coloring Lemma 2 $P_{\mathcal{V}}$ can inform now $P_{\mathcal{E}}$ about v with $\log(\ell_{\mathcal{V}} \cdot \ell_{\mathcal{E}}) + 1$ bits via an encoding function $g_{\mathcal{V}}$. $P_{\mathcal{E}}$ can then inform $P_{\mathcal{E}}$ about E with $\lceil \log \ell_{\mathcal{V}} \rceil$ bits via an encoding function $g_{\mathcal{E}}$.

The whole protocol is then to send first $f_{\mathcal{V}}(v)$ over channel $\mathcal{V} \to \mathcal{E}$, then $f_{\mathcal{E}}(E)$ over channel $\mathcal{E} \to \mathcal{V}$, then $g_{\mathcal{V}}(v)$ over channel $\mathcal{V} \to \mathcal{E}$, and finally $g_{\mathcal{E}}(v)$ over channel $\mathcal{E} \to \mathcal{V}$. This uses 4 rounds.

However, there is a better way! Just follow the order $f_{\mathcal{E}}(E)$, then $f_{\mathcal{V}}(v)$ *and* $g_{\mathcal{V}}(v)$, and then $g_{\mathcal{E}}(E)$. This requires 3 rounds. We state the result.

Theorem 2A. *For* $m \ge 3$
$\overline{\overline{\mathcal{C}}}_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \big\{(b_{\mathcal{V}}, b_{\mathcal{E}}) : b_{\mathcal{V}} \ge \log D_{\mathcal{E}}, b_{\mathcal{E}} \ge \log D_{\mathcal{V}}\big\}$ *, if*

$$\log\big(\log |\mathcal{V}| + \log |\mathcal{E}|\big) = o\big(\log \min(D_{\mathcal{V}}, D_{\mathcal{E}})\big). \tag{3.4}$$

What happens in case $m = 2$?

The key parameters are here $\chi(\mathcal{H})$, the chromatic number (vertex coloring), and $\operatorname{ind}(\mathcal{H})$, the chromatic index (edge coloring), of $\mathcal{H}$. In a protocol we have either $f_{\mathcal{V}}$ and then $f_{\mathcal{E}}$ to be transmitted or $g_{\mathcal{E}}$ and then $g_{\mathcal{V}}$. In the first case necessarily $\|f_{\mathcal{V}}\| \ge \chi(\mathcal{H})$ and in the second case necessarily $\|g_{\mathcal{E}}\| \ge \operatorname{ind}(\mathcal{H})$. This is a simple consequence of the conditions 2.) and 3.) in the Introduction, and an analogous proof appeared in Lemma 2 of [22]. Similar inequalities will be used in the sequel without saying. Furthermore, for instance in the first case necessarily $\|f_{\mathcal{E}}(v,\cdot)\| \ge \deg(v)$.

So in the worst case

$$\max_{v\in\mathcal{V}} \|f_{\mathcal{E}}(v,\cdot)\| \geq D_{\mathcal{V}}. \tag{3.5}$$

On the other hand a coloring $f_{\mathcal{V}}$ with $\chi(\mathcal{H})$ colors and a labeling $f_{\mathcal{E}}(v,E)=j$, if E is the j–th set in $\mathcal{E}_v \subset \{E_1,\dots,E_{|\mathcal{V}|}\}$ achieve the lower bounds.

Theorem 3. *For every hypergraph* $\mathcal{H}=(\mathcal{V},\mathcal{E})$

$$\bar{\bar{\mathcal{C}}}_2(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}) = \big\{(b_{\mathcal{V}},b_{\mathcal{E}}) : b_{\mathcal{V}} \geq \log\chi(\mathcal{H}), b_{\mathcal{E}} \geq \log D_{\mathcal{V}} \quad or \quad b_{\mathcal{V}} \geq \log D_{\mathcal{E}}, b_{\mathcal{E}} \geq \log ind(\mathcal{H})\big\}.$$

From Theorems 2, 3 we conclude that for many hypergraphs

$$\bar{\bar{\mathcal{C}}}_2(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}) \neq \bar{\bar{\mathcal{C}}}_3(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}) \sim \bar{\bar{\mathcal{C}}}_\infty(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}).$$

We also notice that condition (3.4) is rather restrictive. Before we present an improved condition in Theorem 2B below, we discuss some examples.

Example 3: $\mathcal{H}=\Big([n],\binom{[n]}{k}\Big)$.

Clearly $D_{\mathcal{E}}=k$, $D_{\mathcal{V}}=\binom{n-1}{k-1}$, $|\mathcal{V}|=n$, $|\mathcal{E}|=\binom{n}{k}$, $\chi(\mathcal{H})=n$ and Baranyai's Theorem in its generalized form (see [32] and [33, pp. 50–56]) asserts that $\mathrm{ind}(\mathcal{H})=\lceil\binom{n}{k}/\lfloor\frac{n}{k}\rfloor\rceil$. The two minimal pairs in Theorem 3 are

$\Big(\log n, \log\binom{n-1}{k-1}\Big)$ and $\big(\log k, \log\lceil\binom{n}{k}/\lfloor\frac{n}{k}\rfloor\big)\rceil$.

If $k|n$, then $\mathrm{ind}(\mathcal{H})=\binom{n-1}{k-1}$ and we actually have only one minimal pair, namely

$\Big(\log k, \log\binom{n-1}{k-1}\Big)$.

The optimum is achieved already for two rounds.

Example 4: Consider any graph $\mathcal{G}=(\mathcal{V},\mathcal{E}), \mathcal{V}=\{v_1,\dots,v_n\}$.

By Vizing's Theorem [34], a deep improvement of Shannon's edge coloring result for multigraphs [35],

$$D_{\mathcal{V}} \leq \mathrm{ind}(\mathcal{G}) \leq D_{\mathcal{V}}+1. \tag{3.6}$$

Therefore by Theorem 3 we have

$$(b_{\mathcal{V}},b_{\mathcal{E}}) = \big(1,\lceil\log\mathrm{ind}(\mathcal{G})\rceil\big).$$

One may analyse whether the pair $\big(\lceil\log\chi(\mathcal{G})\rceil,\lceil\log D_{\mathcal{V}}\rceil\big)$ can be smaller in the second component, if $\mathrm{ind}(\mathcal{G})=D_{\mathcal{V}}+1$.

Example 5: Consider a k–uniform hypergraph $\mathcal{H}=(\mathcal{V},\mathcal{E})$ with $\mathcal{E}\subset\binom{\mathcal{V}}{k}$. Notice that

$$D_{\mathcal{V}} \leq \mathrm{ind}(\mathcal{H}) \leq k\cdot D_{\mathcal{V}}+1. \tag{3.7}$$

Example 6: A protocol for the complete graph $\mathcal{K}_n = \left([n], \binom{[n]}{2}\right)$.

We present the vertices by binary strings

$$v = (v_1, \dots, v_m), m = \lceil \log |\mathcal{V}| \rceil$$

and encode the edge $\{v, v'\}$ as

$$(v_1, \dots, v_m) + (v'_1, \dots, v'_m) = (w_1, \dots, w_m) \text{ (binary addition mod 2).}$$

$P_\mathcal{V}$, knowing $V = (v_1, \dots, v_m)$, calculates

$$(w_1, \dots, w_m) + (v_1, \dots, v_m) = (v'_1, \dots, v'_m).$$

Thus he knows also v' and he tells $P_\mathcal{E}$ whether v or v' has the smaller label. Thus $(b_\mathcal{V}, b_\mathcal{E}) = (1, \lceil \log |\mathcal{V}| \rceil)$ is achievable and $|\mathcal{V}| = n = D_\mathcal{V} + 1$.

We see that the pair must be optimal, if $\lceil \log(D_\mathcal{V} + 1) \rceil = \lceil \log D_\mathcal{V} \rceil$.

Moreover, the following argument shows that it is always optimal. In the remaining cases we have $|\mathcal{V}| = n = D_\mathcal{V} + 1 = 2^m + 1$ for some $m \in \mathbb{N}$. The protocol provides a tight bound too, because for all $\ell \in \mathbb{N}$, $\operatorname{ind}(\mathcal{K}_{2\ell+1}) \geq 2\ell + 1 \quad (= D_\mathcal{V} + 1)$ holds. In fact, assume the $\binom{2\ell+1}{2} = \ell(2\ell+1)$ edges of $\mathcal{K}_{2\ell+1}$ are already colored by 2ℓ colors, then there must be $\left\lceil \frac{\ell(2\ell+1)}{2\ell} \right\rceil = \ell + 1$ edges with the same color. However, there are only $2\ell + 1$ vertices. So there must be a pair of edges among them with a common vertex.

Finally, we improve Theorem 2A by using the strengthened Coloring Lemma.

Since $t! \geq \left(\frac{t}{e}\right)^t$, it suffices to choose

$$2 \log D_\mathcal{E} + \log D_\mathcal{V} + \log e \leq t \log \frac{t}{e}$$

or

$$t = \log \max(D_\mathcal{E}, D_\mathcal{V}), t \geq e^4. \tag{3.8}$$

Theorem 2B. *For* $m \geq 3$

$$\overline{\overline{\mathcal{C}}}_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \sim \big\{(b_\mathcal{V}, b_\mathcal{E}) : b_\mathcal{V} \geq \log D_\mathcal{E}, b_\mathcal{E} \geq \log D_\mathcal{V}\big\},$$

if

$$\log(\log D_\mathcal{E} + \log D_\mathcal{V}) = o\big(\log \min(D_\mathcal{V}, D_\mathcal{E})\big). \tag{3.9}$$

Proof: Just follow the proof for Theorem 2A and replace (3.1) and (3.2) by the sharper bounds (for $D_\mathcal{E}, D_\mathcal{E} \geq e^4$)

$$D_{\mathcal{V}_i} \leq \ell_\mathcal{E} = \lceil \log \max(D_\mathcal{E}, D_\mathcal{V}) \rceil, \tag{3.10}$$

$$D_{\mathcal{E}_j} \leq \ell_\mathcal{V} = \lceil \log \max(D_\mathcal{E}, D_\mathcal{V}) \rceil \tag{3.11}$$

which follow from (3.8).

Remarks:

2. By a different approach estimates for probabilities of events with small dependencies were also found in [5]. However, they appear to be not as sharp.

3. Since $\overset{=}{C}_m(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E}) \le C_{m-1}(\mathcal{H}; P_\mathcal{E}) + \log D_\mathcal{V}$, for $m \ge 4$, the same upper bound (after ignoring the constant term) can be obtained from the upper bound for $C_3(\mathcal{H}; P_\mathcal{E})$ in Theorem 1 of [24]. However, for $m = 3$ there is no direct implication between the two theorems.

4. $C_2(\mathcal{H}; P_\mathcal{V}, P_\mathcal{E})$: Sharp estimates in two special cases and asymptotically sharp bounds in general

We begin with the special cases, because this way we build up an understanding for the general case.

Theorem 4. *For any graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$C_2(\mathcal{G}; P_\mathcal{V}, P_\mathcal{E}) = \lceil \log \mathit{ind}(\mathcal{G}) \rceil + \varepsilon, \varepsilon \in \{0,1\}$$

and

$$D_\mathcal{V} \le \mathit{ind}(\mathcal{G}) \le D_\mathcal{V} + 1.$$

Proof: By Vizing's Theorem there is an edge coloring $f_\mathcal{E} : \mathcal{E} \to \mathbb{N}$ with $\|f_\mathcal{E}\| - 1 \le D_\mathcal{V}$. We can use the protocol:

For given (v, E) with $v \in E$, $P_\mathcal{E}$ sends $f_\mathcal{E}(E)$ with $\lceil \log \mathrm{ind}(\mathcal{G}) \rceil$ bits to $P_\mathcal{V}$, who can recover E .

With one bit he informs $P_\mathcal{E}$ about his vertex. Then we have shown that

$$C_2(\mathcal{G}; P_\mathcal{V}, P_\mathcal{E}) \le \lceil \log \mathrm{ind}(\mathcal{G}) \rceil + 1. \tag{4.1}$$

Now conversely, even if $P_\mathcal{E}$ knows the outcomes v , he has to transmit $\lceil \log D_\mathcal{V} \rceil$ bits in the worst case and even if $P_\mathcal{V}$ knows the outcome E , he has to transmit 1 bit in the worst case. Therefore

$$C_2(\mathcal{G}; P_\mathcal{V}, P_\mathcal{E}) \ge \lceil \log D_\mathcal{V} \rceil + 1 \ge \lceil \log\big(\mathrm{ind}(\mathcal{G}) - 1\big) \rceil + 1.$$

The upper and lower bound coincide for $2^r < D_\mathcal{V} < 2^{r+1}$.

Example 7: If $\mathcal{G} = \left([3], \binom{[3]}{2}\right)$ is a triangle, then $D_\mathcal{V} = 2$ and inspection shows that $C_2(\mathcal{G}; P_\mathcal{V}, P_\mathcal{E}) = 3$ (the upper bound (4.1)).

Remark 4: Again we can use the nice protocol in Example 6. Can this idea be generalized to k larger than 2?

Next we consider again the complete k–regular hypergraph $\left([n], \binom{[n]}{k}\right)$. The lower bound is derived as before. In any case $P_{\mathcal{E}}$ has to transmit $\left\lceil \log \binom{n-1}{k-1} \right\rceil$ bits (even if he knows v) and $P_{\mathcal{V}}$ has to transmit $\lceil \log n \rceil$ bits (even if he knows E). Formally, this follows from the Average–degree Lemma in Section 5. Thus we get

$$C_m\left(\left([n], \binom{[n]}{k}\right); P_{\mathcal{V}}, P_{\mathcal{E}}\right) \geq \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} \right\rceil, m \geq 2.$$

Baranyai's Theorem in its generalized form (see [33, pp. 50–56) asserts

$$\binom{n-1}{k-1} \leq \operatorname{ind}\left(\mathcal{V}, \binom{\mathcal{V}}{k}\right) = \left\lceil \binom{n}{k} \Big/ \left\lfloor \frac{n}{k} \right\rfloor \right\rceil .$$

Now $P_{\mathcal{E}}$ uses an edge coloring with $\operatorname{ind}\left([n], \binom{[n]}{k}\right)$ colors and sends the color with $\left\lceil \log \operatorname{ind}\left([n], \binom{[n]}{k}\right) \right\rceil$ bits. $P_{\mathcal{V}}$ encodes in a binary string the position of $v = v_i$ in E . This requires $\lceil \log k \rceil$ bits.

Theorem 5. *For the k–uniform complete hypergraph* $\mathcal{H} = \left([n], \binom{[n]}{k}\right)$

$$C_2(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) = \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} + \varepsilon \right\rceil, \varepsilon \in \{0, 1\}.$$

Furthermore,

$$C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) = \lceil \log k \rceil + \left\lceil \log \binom{n-1}{k-1} + \varepsilon \right\rceil \quad \text{for all} \quad m \geq 2.$$

Remark 5: For $m = 2$, if $P_{\mathcal{V}}$ starts sending, then necessarily $b_{\mathcal{V}} = \lceil \log n \rceil$ and $b_{\mathcal{E}} = \left\lceil \log \binom{n-1}{k-1} \right\rceil$. This, of course, is not as good as the reverse order.

We turn now to $C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$ for general hypergraphs $\mathcal{H}$.

The situation for one channel is quite different from the one for two channels studied in earlier sections.

For a strategy we cannot just add $\max\limits_{v} b_{\mathcal{V}}(v)$ and $\max\limits_{v,E} b_{\mathcal{E}}(v, E)$, because, as in our earlier work [18], [19], now a small $b_{\mathcal{V}}(v)$ may be coupled with a large $\max\limits_{E} b_{\mathcal{E}}(v, E)$, and, vice versa.[1]

On the other hand, if

$$\deg(v) = D_{\mathcal{V}} \quad \text{for all} \quad v \in \mathcal{V} \tag{4.2}$$

and

$$\deg(E) = D_{\mathcal{E}} \quad \text{for all} \quad E \in \mathcal{E}, \tag{4.3}$$

[1]We have been informed that this idea appeared already in Orlitsky [23].

or these equations "almost" hold, then $\log D_{\mathcal{V}}$ and $\log D_{\mathcal{E}}$ are the crucial parameters again and they can be added! (See again the Average–degree Lemma in Section 5.)

Example 8: $|\mathcal{V}| = |\mathcal{E}| = n$, (4.2) and (4.3) hold, and, particularly, $D_{\mathcal{V}} = D_{\mathcal{E}} = k$; $k = k(n)$ with $\lim\limits_{n\to\infty} k(n) = \infty$.

Then (3.9) holds and by Theorem 2B and the previous observation

$$C_m(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \log D_{\mathcal{E}} + \log D_{\mathcal{V}}.$$

On the other hand further insight is gained from the next case.

Example 9: Consider the multi–hypergraph $(\mathcal{V}, \mathcal{W}, S)$ with $S = \big\{(v_1, w_j) : 1 \leq j \leq n\big\} \cup \big\{(v_i, w_1) : 1 \leq i \leq m\big\}$. Then $D_{\mathcal{V}} = n, D_{\mathcal{W}} = m$, and if $m \geq n$, then $C_2\big((\mathcal{V}, \mathcal{W}, S); P_{\mathcal{V}}, P_{\mathcal{W}}\big) = \lceil \log \max(n, m-1)\rceil + 1$.

Protocol: If $v \neq v_1$ then necessarily $w = w_1$ and $P_{\mathcal{V}}$ knows this. $P_{\mathcal{W}}$ does not act before $P_{\mathcal{V}}$ acts. $P_{\mathcal{V}}$ sends a string of length $1 + \lceil \log(m-1)\rceil$. He begins with 0 . Then $P_{\mathcal{E}}$ knows v . If $v = v_1$, then $P_{\mathcal{V}}$ sends 1 and $P_{\mathcal{E}}$ sends j with $\lceil \log n\rceil$ bits. Thus the result.

Notice that $\overset{=}{C}_2 = \lceil \log m\rceil + \lceil \log n\rceil$.

From Example 9 we learn that the coloring numbers $\chi(\mathcal{H}), \operatorname{ind}(\mathcal{H})$ and maximal cross–section numbers $D_{\mathcal{V}}$ and $D_{\mathcal{E}}$ alone don't suffice for a characterization of C_2 . We give a somewhat refined analysis.

For $s = 1, 2, \ldots, \lceil \log D_{\mathcal{E}}\rceil$ define

$$\mathcal{E}_s = \big\{E \in \mathcal{E} : 2^{-s} D_{\mathcal{E}} < |E| \leq 2^{-s+1} D_{\mathcal{E}}\big\} \tag{4.4}$$

and $\mathcal{H}_s = (\mathcal{V}_s, \mathcal{E}_s)$ with $\mathcal{V}_s = \mathcal{V}$.

Let $f_s : \mathcal{E}_s \to \mathbb{N}$ be an edge coloring of $\mathcal{H}_s$ with $\|f_s\| = \operatorname{ind}(\mathcal{H}_s)$.

For given (v, E) $P_{\mathcal{E}}$ sends s , if $E \in \mathcal{E}_s$, and also $f_s(E)$. $P_{\mathcal{V}}$ sends the position $g(v, s, E)$ of v in E .

How many bits are needed for this? Let s be represented by the string $\alpha(s)$ and let $f_s(E)$ be represented by the string $\beta_s(E)$. Finally, let $g(v, s, E)$ be encoded as $\gamma(v, s, E)$. Thus

$$b_E + b_v(E) = |\alpha(s)| + |\beta_s(E)| + |\gamma(v, s, E)|.$$

Clearly, we have

$$\begin{aligned} &\lceil \log D_{\mathcal{E}}\rceil + \max_{1\leq s\leq D_{\mathcal{E}}} \big\lceil \big(\log \operatorname{ind}(\mathcal{H}_s)\big\rceil - s\big) \leq \max_{v,E}\big(b_E + b_v(E)\big) \\ &\leq \big\lceil \log\lceil \log D_{\mathcal{E}}\rceil\big\rceil + \max_s \big\lceil\big(\log \operatorname{ind}(\mathcal{H}_s)\big\rceil + \lceil \log D_{\mathcal{E}}\rceil - s + 1\big) \\ &\sim \log D_{\mathcal{E}} + \max_{1\leq s\leq D_{\mathcal{E}}} \big(\log \operatorname{ind}(\mathcal{H}_s) - s + 1\big), \end{aligned} \tag{4.5}$$

because all edges in $\mathcal{E}_s$ have up to the factor 2 the same size.

Of course, we can exchange the roles of $\mathcal{V}$ and $\mathcal{E}$.

Theorem 6 A. *For any hypergraph* $\mathcal{H}$ *we have*

(i) $C_2(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) \leq$

$$\min\Bigg(\max_{1\leq s\leq D_\mathcal{E}} \lceil \log ind(\mathcal{V},\mathcal{E}_s) - s + 1\rceil + \lceil \log D_\mathcal{E}\rceil + \lceil \log\log D_\mathcal{E}\rceil,$$

$$\max_{1\leq t\leq D_\mathcal{V}} \lceil \log \chi(\mathcal{V}_t,\mathcal{E}) - t + 1\rceil + \lceil \log D_\mathcal{V}\rceil + \lceil \log\log D_\mathcal{V}\rceil\Bigg)$$

(ii) $C_2(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) \geq$

$$\min\Bigg(\max_{1\leq s\leq D_\mathcal{E}} \lceil \log ind(\mathcal{V},\mathcal{E}_s) - s\rceil + \lceil \log D_\mathcal{E}\rceil, \max_{1\leq f\leq D_\mathcal{V}} \lceil \log \chi(\mathcal{V}_t,\mathcal{E}) - t\rceil + \log D_\mathcal{V}\rceil\Bigg) .$$

The bounds differ by at most

$$\max\big(\lceil \log\log D_\mathcal{E}\rceil, \lceil \log\log D_\mathcal{V}\rceil\big) + 1.$$

Now we give an even better protocol and achieve a difference of the upper and lower bound by two bits!

For any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ we weigh subsets of vertices and edges by the degrees

$$D_\mathcal{U} = \max_{v\in\mathcal{U}} \deg(v),\ \mathcal{U} \subset \mathcal{V}, \tag{4.6}$$

$$D_\mathcal{F} = \max_{E\in\mathcal{F}} \deg(E),\ \mathcal{F} \subset \mathcal{E}. \tag{4.7}$$

To a given c–coloring $\varphi : \mathcal{V} \to \{1, 2, \dots, c\}$ of $\mathcal{V}$, we associate a weight

$$W_\varphi = \sum_{i=1}^{c} D_{\varphi^{-1}(i)} \tag{4.8}$$

and we define the weighted chromatic number $\tilde{\chi}(\mathcal{H})$ as the minimal weight among the vertex colorings of $\mathcal{H}$. Analogously we define the weighted index $\widetilde{\mathrm{ind}}(\mathcal{H})$ of $\mathcal{H}$.

Now we consider a 2 rounds protocol (φ, χ) starting with $P_\mathcal{V}$. Obviously φ must be a vertex coloring and also, if $v \in \varphi^{-1}(j)$, then after knowing v $P_\mathcal{E}$ has to use a $D_{\varphi^{-1}(j)}$–size code in the second round to inform $P_\mathcal{V}$ about his E .

This means that the protocol must use at least W_φ sequences and therefore at least $\log W_\varphi$ bits. We have therefore

$$C_2(\mathcal{H}, P_\mathcal{V}, P_\mathcal{E}) \geq \lceil \log \tilde{\chi}(\mathcal{H})\rceil. \tag{4.9}$$

On the other hand we use again our old adaptive coding idea ([18], [19]). [1]

Let the coloring $\varphi : \mathcal{V} \to \{1, \dots, c\}$ achieve $\tilde{\chi}(\mathcal{H})$ and set

$$\ell_i = \left\lceil \log \frac{\tilde{\chi}(\mathcal{H})}{D_{\varphi^{-1}(i)}} \right\rceil \quad \text{for} \quad i = 1, \dots, c. \tag{4.10}$$

By Kraft's inequality one can code φ with a prefix code $\mathcal{C} = \{u_1, \dots, u_c\}$ such that u_i has length ℓ_i and $\varphi(v)$ is encoded by u_i , if $\varphi(v) = i$.

The following protocol works:

1.) For given $v \in \mathcal{V}$ $P_{\mathcal{V}}$ sends $u_{\varphi(v)}$

2.) After having received $u_{\varphi(v)}$ $P_{\mathcal{E}}$ recovers v and informs $P_{\mathcal{V}}$ about the given E via a block code of a length not exceeding $\lceil \log D_{\varphi^{-1}(i)} \rceil$, if $v \in \varphi^{-1}(i)$.

By (4.10) the total number of bits is always bounded by $\log \tilde{\chi}(\mathcal{H}) + 1$. Introducing

$$\tilde{\sigma}(\mathcal{H}) = \min\big(\tilde{\chi}(\mathcal{H}), \widetilde{\text{ind}}(\mathcal{H})\big) \tag{4.11}$$

and exchanging the roles of $\mathcal{V}$ and $\mathcal{E}$ we can draw the following conclusion.

Theorem 6 B. *For any hypergraph $\mathcal{H}$ we have*

$$\lceil \log \tilde{\sigma}(\mathcal{H}) \rceil \leq C_2(\mathcal{H}, P_{\mathcal{V}}, P_{\mathcal{E}}) \leq \lceil \log \tilde{\sigma}(\mathcal{H}) \rceil + 2.$$

5. A lower bound on $C_\infty(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$

We know already that under condition (3.9)

$$\overline{\overline{C}}_3(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \sim \log D_{\mathcal{V}} + \log D_{\mathcal{E}}. \tag{5.1}$$

Clearly, $C_3(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}}) \leq \overline{\overline{C}}_3(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$ and Example 9 shows that already $C_2(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$ can be much smaller than $\log D_{\mathcal{V}} + \log D_{\mathcal{E}}$. $C_4(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$ can be much smaller in general.

We give here a lower bound on $C_\infty(\mathcal{H}; P_{\mathcal{V}}, P_{\mathcal{E}})$, which we conjecture to be tight. We also believe that it can be achieved in 4 rounds. The bound uses a decomposition idea.

We start with a purely combinatorial problem. We call a 0–1–matrix a quasi–permutation matrix, if at most one 1 occurs in every row and in every column. Thus a permutation matrix is a square quasi–permutation matrix such that there is exactly one 1 in every row and in every column. A quasi–permutation *decomposition* $\{A_i = i \in I\}$ of

M , is a partition of M into submatrices A_i , which are quasi–permutation matrices. Partition here means that every entry of M occurs in exactly one member of $\{A_i : i \in I\}$.

We also introduce the average degrees for $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ by

$$\overline{D}_{\mathcal{V}} = \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \deg(v), \overline{D}_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \sum_{E \in \mathcal{E}} \deg(E), \tag{5.2}$$

and similarly for $M = (\mathcal{V}, \mathcal{W}, S)$ $\overline{D}_{\mathcal{V}}$ is the average number of 1's in the rows and $\overline{D}_{\mathcal{W}}$ is the average number of 1's in the columns.

Obviously, we have for the total number of 1's

$$|\mathcal{V}| \cdot \overline{D}_{\mathcal{V}} = |\mathcal{W}| \cdot \overline{D}_{\mathcal{W}}. \tag{5.3}$$

Decomposition Lemma.

If $\{A_i : i \in I\}$ is a quasi–permutation decomposition of the 0–1–matrix M , then

$$|I| \geq \overline{D}_{\mathcal{V}} \cdot \overline{D}_{\mathcal{W}}. \tag{5.4}$$

Proof: Assume that (5.4) does not hold for $\{A_i : i \in I\}$ and that A_i contains r_i rows, c_i columns, and λ_i 1's.

By the definition of a quasi–permutation matrix

$$\lambda_i \leq \min(r_i, c_i) \text{ for } i \in I. \tag{5.5}$$

Suppose now that M has $|\mathcal{V}|$ rows and $|\mathcal{W}|$ columns, then by the partition property

$$\sum_{i \in I} r_i c_i = |\mathcal{V}||\mathcal{W}| \tag{5.6}$$

and

$$\sum_{i \in I} \lambda_i = |\mathcal{V}|\overline{D}_{\mathcal{V}} = |\mathcal{W}|\overline{D}_{\mathcal{W}}. \tag{5.7}$$

Consequently, by (5.7)

$$\begin{aligned}\left(\sum_{i\in I}\frac{1}{|I|}\lambda_i\right)^2 &= \frac{1}{|I|^2}|\mathcal{V}|\overline{D}_{\mathcal{V}}|\mathcal{W}|\overline{D}_{\mathcal{W}}\\ &= \left(\sum_{i\in I}\frac{1}{|I|}r_ic_i\right)\left(\frac{1}{|I|}\overline{D}_{\mathcal{V}}\overline{D}_{\mathcal{W}}\right)\quad \text{(by (5.6))}\\ &> \left(\sum_{i\in I}\frac{1}{|I|}r_ic_i\right)\quad \text{(by assumption)}\\ &\geq \sum_{i\in I}\frac{1}{|I|}\lambda_i^2\quad \text{(by (5.5)).}\end{aligned}$$

This contradicts the Cauchy–Schwarz inequality. We readily derive now a basic lower bound.

Average–degree Lemma.

For any multi–hypergraph $\mathcal{H}=(\mathcal{V},\mathcal{E})$ *(or, equivalently, 0–1–matrix* $M=(\mathcal{V},\mathcal{W},S)$ *)*

$$C_\infty(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}})\geq \log\overline{D}_{\mathcal{V}}+\log\overline{D}_{\mathcal{E}}. \tag{5.8}$$

Proof: Each protocol partitions M into quasi–permutation submatrices (just as in Yao's case $f:\mathcal{V}\times\mathcal{W}\to\mathcal{Z}$ it partitions M_f into "monochromatic" submatrices) and the number of bits used in the protocol is bounded by the logarithm of $I(M)\triangleq$ the usual size of a quasi–permutation decomposition of M .

By the Decomposition Lemma

$$\log I(M)\geq \log\overline{D}_{\mathcal{V}}+\log\overline{D}_{\mathcal{W}}=\log\overline{D}_{\mathcal{V}}+\log\overline{D}_{\mathcal{E}},$$

the desired inequality.

Example 9 shows that (5.8) is not tight. However, it can be used to derive a much better bound. For this we write

$M'\prec M$ iff M' is a submatrix of M and $\mathcal{H}'\prec\mathcal{H}$ iff $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$.

Also with $M'=(\mathcal{V}',\mathcal{W}',S')$ we associate

$$\sigma(M')\triangleq\frac{|S'|^2}{|\mathcal{V}'||\mathcal{W}'|}=\overline{D}_{\mathcal{V}'}\cdot\overline{D}_{\mathcal{W}'}\quad\text{(by (5.3)).}$$

Since obviously

$$C_m(M;P_{\mathcal{V}},P_{\mathcal{E}})\geq C_m(M';P_{\mathcal{V}'},P_{\mathcal{W}'})$$

or in terms of hypergraphs

$$C_m(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}) \geq C_m(\mathcal{H}';P_{\mathcal{V}'},P_{\mathcal{E}'}),$$

we get the following consequence of the Average–degree Lemma in hypergraph language.

Theorem 7. *For any multi–hypergraph* $\mathcal{H}=(\mathcal{V},\mathcal{E})$

$$C_\infty(\mathcal{H};P_{\mathcal{V}},P_{\mathcal{E}}) \geq \max_{\mathcal{H}'\prec\mathcal{H}} \sigma(\mathcal{H}').$$

6. A lower bound on $\bar{\bar{C}}_\infty\left(\left([n],\binom{[n]}{k}\right),P_{\mathcal{E}}\right)$

Consider successful protocols with $b_{\mathcal{V}}=\beta$ and let $\nu_\infty(n,k;\beta)$ be the minimal $b_{\mathcal{E}}$ among these protocols. Let Ω be a protocol with this property and set $\alpha = \nu_\infty(n,k;\beta)$.

For any (v,E) with $v\in E$

$$\Omega(v,E)=x_1(E)y_1(v,x_1)x_2(E,y_1)\dots$$

(here x_1 can be the empty word), and we can form the strings

$$x_1(E)x_2(E,y_1)\dots \quad \text{and} \quad y_1(v,x_1)y_2(v,x_1,x_2),$$

which we extend in any way to strings of length α and β , respectively.

Let $X_v\subset\{0,1\}^\alpha$ be the set of all strings of length α produced this way by $P_{\mathcal{E}}$ under Ω , if v is given to $P_{\mathcal{V}}$. Every $v\in\mathcal{V}$ defines then a function $F_v: X_v\to\{0,1\}^\beta$, because $P_{\mathcal{V}}$'s operations dependent only on the messages from $P_{\mathcal{E}}$ and on v .

Clearly, there are in total at most $2^{\beta\cdot 2^\alpha}$ such functions and if $2^{\beta\cdot 2^\alpha}<n$ or, equivalently, if

$$\alpha < \log\log n - \log\beta, \tag{6.1}$$

then at least two functions must be the same, that is,

$$F_v\equiv F_{v'} \quad \text{for some} \quad v,v'\in\mathcal{V}. \tag{6.2}$$

But then $P_{\mathcal{E}}$ receives the same strings if v or v' are given to $P_{\mathcal{V}}$. Therefore for any $E\supset\{v,v'\}$ there is no way for $P_{\mathcal{E}}$ to separate v and v' . Obviously, in a complete k–uniform hypergraph there is such an edge E . Therefore (6.1) cannot hold and we have proved the following inequality.

Lemma 1. *For all* $k \le n$ *and* $\beta \le \lceil \log n \rceil$

$$\nu_\infty(n, k; \beta) \ge \log\log n - \log \beta. \tag{6.3}$$

This simple fact will yield inductively the main result of this section.

Theorem 8. *For* $\beta < \frac{1}{\ell} \log n$ *and* $\ell^2 \le \log n$

$$\nu_\infty(n, 2^\ell; \beta) \ge \log\log n + \ell - 1 - \log \beta, \tag{6.4}$$

Furthermore, for $2^{\ell-1} < k < 2^\ell$

$$\nu_\infty(n, k, \beta) \ge \log\log n + \ell - 2 - \log \beta.$$

A successful strategy for the hypergraph $\left([n], \binom{[n]}{k}\right)$ must have $\beta \ge \log k$. Therefore we get the following consequence of Theorem 8.

Corollary 1. *For* $\lceil \log k \rceil^2 \le \log n$

$$\bar{\bar{C}}_\infty\left(\left([n], \binom{[n]}{k}\right), P_\mathcal{E}\right) \ge \log\log n + 2\log k - 1 - \log\log k.$$

Proof of Theorem 8: We proceed by induction on ℓ . Lemma 1 gives $\nu_\infty(n, 2, \beta) \ge \log\log n - \log \beta$, which settles the case $\ell = 1$. By induction hypothesis it holds for $\ell - 1$.

We call a protocol Ω an (α, β, ℓ) protocol, if $b_\mathcal{E} \le \alpha$, $b_\mathcal{V} \le \beta$ and it is successful (that is, $P_\mathcal{E}$ can separate the vertices in his 2^ℓ–size edges).

Even case: $P_\mathcal{E}$ sends first.

Suppose that Ω is an (α, β, ℓ) protocol violating (6.4), that is,

$$\alpha < \log\log n + \ell - 1 - \log \beta. \tag{6.5}$$

After having sent his first bit $P_\mathcal{E}$ divides the protocol Ω into two subprotocols Ω_0 and Ω_1 .

Here Ω_i is the protocol, if $i \in \{0, 1\}$ was sent. However, by induction hypothesis and (6.5) both, Ω_0 and Ω_1 , are not $(\alpha - 1, \beta, \ell - 1)$ protocols. So for $i = 0, 1$ there must be a subset $T_i \subset \mathcal{V}$ of cardinality $2^{\ell-1}$, which is not separated by Ω_i . Define $T^* = T_0 \cup T_1$. Then $|T^*| \le 2^\ell$, but T^* cannot be separated by Ω !

Odd case: $P_\mathcal{V}$ sends first.

Suppose that $P_\mathcal{V}$ uses in his first round a prefix code $\{w_1, \dots, w_\gamma\}$ to send his message. Namely, for $v \in \mathcal{V}_i$ he sends w_i of length $\ell_i < \beta$ and $\bigcup_{i=1}^{\gamma} \mathcal{V}_i = \mathcal{V}$.

This reduces our problem to that of finding an $(\alpha, \beta - \ell_i, \ell)$ protocol in the even case for the hypergraph $\mathcal{H}_i = \left(\mathcal{V}_i, \binom{\mathcal{V}_i}{2^\ell}\right)$.

If there is an $i \in \{1, \dots, \gamma_1\}$ with

$$\frac{\log |\mathcal{V}_i|}{\beta - \ell_i} \geq \frac{\log n}{\beta}, \tag{6.6}$$

then with the already established (6.4) in the even case

$$\alpha \geq \log\log |\mathcal{V}_i| + \ell - 1 - \log(\beta - \ell_i)$$

as $\beta < \frac{1}{\ell} \log n$ and (6.6) imply $\beta - \ell_i < \frac{1}{\ell} \log |\mathcal{V}_i|$ and $|\mathcal{V}_i| > 2^\ell$.

We conclude that

$$\alpha \geq \log\left(\frac{\beta - \ell_i}{\beta} \log n\right) + \ell - 1 - \log(\beta - \ell_i) = \log\log n + \ell - 1 - \log \beta,$$

and we are done.

Otherwise,

$$\frac{\log |\mathcal{V}_i|}{\beta - \ell_i} < \frac{\log n}{\beta} \quad \text{for} \quad i = 1, \dots, \gamma, \tag{6.7}$$

which is equivalent to

$$|\mathcal{V}_i| < n^{1 - \frac{\ell_i}{\beta}} \quad \text{for} \quad i = 1, \dots, \gamma. \tag{6.8}$$

By summation of both sides

$$n = \sum_{i=1}^{\gamma} |\mathcal{V}_i| < n \sum_{i=1}^{\gamma} n^{-\frac{\ell_i}{\beta}}$$

or

$$1 < \sum_{i=1}^{\gamma} 2^{-\frac{\ell_i}{\beta} \log n}. \tag{6.9}$$

Since $\beta < \log n$, this implies

$$1 < \sum_{i=1}^{\gamma} 2^{-\ell_i},$$

which contradicts Kraft's inequality.

In Theorem 1 of [28] it is established that

$$C_4\left(\left([n],\binom{[n]}{k}\right),P_{\mathcal{E}}\right)\lesssim 2\log k+\log\log n. \tag{6.10}$$

Actually, the protocol used requires always to send $\sim\log k$ bits for $P_{\mathcal{V}}$ and $\sim$ $\log k+\log\log n$ bits for $P_{\mathcal{E}}$. It is therefore an upper bound also in case of two channels.

In conjunction with Corollary 1 we have therefore a complete characterization:

Theorem 9. *For* $\lceil\log k\rceil^2<\log n$ *and* $m\geq 4$

$$\bar{\bar{C}}_m\left([n],\binom{[n]}{k},P_{\mathcal{E}}\right)\sim 2\log k+\log\log n.$$

Remarks:

6. The pair (v,E) may determine who sends first.
7. We conjecture that also (6.10) is tight. In fact we tend to believe that four rounds suffice for all $\mathcal{H}$ in the one channel model with one recipient.
8. In Section 7 we show that for $\mathcal{H}(n,k)=\left([n],\binom{[n]}{k}\right)$

$$C_2\big(\mathcal{H}(n,k),P_{\mathcal{E}}\big)\backsimeq C_3\big(\mathcal{H}(n,k),P_{\mathcal{E}}\big)>C_4\big(\mathcal{H}(n,k),P_{\mathcal{E}}\big).$$

   It was shown first by Zhang and Xia [29] that three rounds may not suffice. The present example is simpler.
9. With similar ideas we obtained also tight bounds for $\bar{\bar{C}}_m\left(\binom{[n]}{L},\binom{[n]}{K},P_{\mathcal{E}}\right)$, where $K>L$ and an edge is a K–element subset with its L–element subsets as its vertices.

7. Characterization of $C_2\left(\left([n],\binom{[n]}{k}\right),P_{\mathcal{E}}\right)$

We use here the abbreviations

$$\mathcal{H}(n,k)=\left([n],\binom{[n]}{k}\right)\quad\text{and}\quad\mu_m(n,k)=C_m\big(\mathcal{H}(n,k),P_{\mathcal{E}}\big). \tag{7.1}$$

The following upper bound (i) on $\mu_2(n,k)$ is due to Orlitsky and appears as Theorem 4 in [22]. It is included here, because we want to demonstrate that it is a simple consequence of the Covering Lemma, and also in order to show that our deeper lower bound is “almost optimal”.

Theorem 10.

(i) $\mu_2(n,k) \leq (3\log k + \log\log n)$ *for* $k \geq k_0$ *(suitable).*

(ii) *For all* $\varepsilon > 0$, $\mu_2(n,k) \geq (1-\varepsilon)(3\log k + \log\log n)$, *if* $k = 0(\log n)$ *and* n,k *are large.*

a. The upper bound

We make use of a consequence of the Covering Lemma stated in the Introduction.

It is formulated for every hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{E}' \subset \binom{[n]}{k}$, but will be used here only for $\mathcal{H}(n,k)$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{H}')$ be the automorphism group of $\mathcal{H}'$, that is, the maximal subgroup of the group Σ_n of all permutations on $\mathcal{V}' = [n]$, for whose elements σ

$$\sigma E = \{\sigma v : v \in E\} \in \mathcal{E} \quad \text{for all} \quad E \in \mathcal{E}. \tag{7.2}$$

$\mathcal{A}(\mathcal{H}')$ is transitive, if for any $E_1, E_2 \in \mathcal{E}$ a $\sigma \in \mathcal{A}(\mathcal{H}')$ exists with $\sigma E_1 = E_2$.

Corollary 2. *For a hypergraph* $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ *with transitive automorphism group* $\mathcal{A}(\mathcal{H}')$ *and any subhypergraph* $\mathcal{H}'' = (\mathcal{V}', \mathcal{E}'')$, $\mathcal{E}'' \subset \mathcal{E}'$, *there is a subset* $\mathcal{B}'' \subset \mathcal{A}(\mathcal{H}')$ *with the properties*

(i) *For every* $E \in \mathcal{E}'$ *there is a* $\sigma \in \mathcal{B}''$ *with* $E \in \sigma\mathcal{E}'' = \{\sigma E'' : E'' \in \mathcal{E}''\}$,

(ii) $|\mathcal{B}''| \leq \left\lceil \frac{|\mathcal{E}'|}{|\mathcal{E}''|} \ell n |\mathcal{E}'| \right\rceil$.

Proof: Choose in the Covering Lemma

$$\mathcal{V} \triangleq \mathcal{E}', \mathcal{E} \triangleq \{\sigma\mathcal{E}'' : \sigma \in \mathcal{A}(\mathcal{H}')\}.$$

Thus

$$|\mathcal{V}| = |\mathcal{E}'|, |\mathcal{E}| = |\mathcal{A}(\mathcal{H}')|, d_\mathcal{V} = D_\mathcal{V} = \frac{|\mathcal{A}(\mathcal{H}')||\mathcal{E}''|}{|\mathcal{E}'|},$$

and the result follows.

Proof of (i) in Theorem 10: Choose $\mathcal{H}' = (\mathcal{V}', \mathcal{E}') = \mathcal{H}(n,k)$ and partition $\mathcal{V}'$ into k^2 parts $U_1, \dots, U_{k^2}$ of sizes $\lfloor \frac{n}{k^2} \rfloor$ or $\lceil \frac{n}{k^2} \rceil$ and define $\mathcal{H}'' = (\mathcal{V}', \mathcal{E}'')$ with

$$\mathcal{E}'' = \{E'' \in \mathcal{E}' : |E'' \cap U_j| \leq 1 \quad \text{for} \quad j = 1, \dots, k^2\}.$$

Let $\mathcal{B}''$ be the set of permutations with the properties (i), (ii) in Corollary 2.

Since

$$\frac{|\mathcal{E}'|}{|\mathcal{E}''|} \leq \frac{\binom{n}{k}}{\binom{k^2}{k} \lfloor \frac{n}{k^2} \rfloor^k} \leq \frac{1}{\prod_{i=0}^{k-1} \left(1 - \frac{i}{k^2}\right)} = \exp\left\{ -\sum_{i=1}^{k-1} ln \left(1 - \frac{i}{k^2}\right) \right\}$$

$$\leq \exp\left\{ \sum_{i=1}^{k-1} \frac{i}{k^2} + \theta \left(\frac{i}{k^2}\right)^2 \right\} \leq \exp\left\{ \frac{k(k-1)}{2k^2} + \theta \frac{1}{k} \right\} \leq c$$

for suitable constants θ and c, we get

$$|\mathcal{B}''| \le c \cdot ln \binom{n}{k}. \tag{7.3}$$

Protocol: $P_{\mathcal{E}}$ tells $P_{\mathcal{V}}$ the σ with $E \in \sigma\mathcal{E}''$ and then $P_{\mathcal{V}}$ tells $P_{\mathcal{E}}$ the index j with $v \in \sigma U_j$.

This requires exactly $\lceil \log|\mathcal{B}''|\rceil + 2\log k$ bits and not more than $\log\log n + 3\log k$ bits for $k \ge k_0(c)$.

This establishes (i).

b. The lower bound

Since there are 2 rounds, if $P_{\mathcal{V}}$ starts in a protocol he has to send $\lceil \log n \rceil$ bits so that $P_{\mathcal{E}}$ knows v. We can assume therefore that always $P_{\mathcal{E}}$ starts. We split the proof into two parts.

Lemma 2. *For a successful strategy $\Omega = (f,g)$, the set of edges with "few answers"*

$$\mathcal{E}_1 \triangleq \big\{E \in \mathcal{E} : |\big\{g(v, f(E)) : v \in \mathcal{V}\big\}| \le k^{2-\delta}\big\}$$

requires either "many questions", that is,

$$ln|\big\{f(E) : E \in \mathcal{E}_1\big\}| > \frac{\theta_1}{8}k^{\delta} \tag{7.4}$$

for some constant $\theta_1 \in (0,1)$, or else there is a set

$$F = \{v_1, \dots, v_{k_1}\} \subset \mathcal{V}, k_1 = \left\lfloor \frac{k}{2} \right\rfloor, \text{ such that for all } E \in \mathcal{E}_1 \text{ there are } v_i, v_j \in F$$

with

$$g\big(v_i, f(E)\big) = g\big(v_j, f(E)\big). \tag{7.5}$$

Proof: Let us write $\ell \triangleq k^{2-\delta}$. For each fixed $f(E), E \in \mathcal{E}_1$, there are at most $\binom{\ell}{k_1}\lceil\frac{n}{\ell}\rceil^{k_1}$ edges E' in $\Big(\mathcal{V}, \binom{\mathcal{V}}{k_1}\Big)$ such that

$$g\big(v, f(E)\big) \ne g\big(v', f(E)\big) \text{ for all } v, v' \in E'(v \ne v'). \tag{7.6}$$

Therefore the number of $E' \in \binom{\mathcal{V}}{k_1}$ satisfying (7.6) for some $E \in \mathcal{E}_1$ cannot exceed

$$\binom{\ell}{k_1}\left\lceil\frac{n}{\ell}\right\rceil^{k_1} \cdot |\big\{f(E) : E \in \mathcal{E}_1\big\}|. \tag{7.7}$$

We evaluate and upper bound this number

$$\begin{aligned}
&\binom{\ell}{k_1}\left\lceil\frac{n}{\ell}\right\rceil^{k_1} \sim \binom{\ell}{k_1}\left(\frac{n}{\ell}\right)^{k_1} = \frac{\ell^{k_1}}{k_1!}\left(\frac{n}{\ell}\right)^{k_1}\prod_{i=0}^{k_1-1}\left(1-\frac{i}{\ell}\right)\\
&= \binom{n}{k_1}\frac{\prod_{i=0}^{k_1-1}(1-\frac{i}{\ell})}{\prod_{i=0}^{k_1-1}(1-\frac{i}{n})} \sim \binom{n}{k_1}\exp\left\{-\frac{k_1(k_1-1)}{2\ell}+\frac{k_1(k_1-1)}{2n}\right\}\\
&\leq \binom{n}{k_1}\exp\left\{-\frac{k_1\theta_1}{\ell}\right\},
\end{aligned}$$

if n and k are large enough, $\frac{n}{\ell}\to\infty \quad (n\to\infty)$, and $\theta_1\in(0,1)$ is a constant. Further,

$$|\{f(E):E\in\mathcal{E}_1\}| = \exp\{ln|\{f(E):E\in\mathcal{E}_1\}|\}$$

and thus the upper bound

$$\binom{n}{k_1}\exp\left\{ln|\{f(E):E\in\mathcal{E}_1\}|-\frac{\theta_1}{8}k^\delta\right\}.$$

Therefore there is the claimed F unless (7.4) holds.

Lemma 3.

Suppose that for a successful strategy $\Omega=(f,g)$ *for all possible pairs* $(x^{\alpha_1},y^{\beta_2})$

$$\beta_2\leq\lceil 3\log k+\log\log n\rceil = \log\gamma, \quad \textit{say},$$

then for the set $\mathcal{X}=\{x^{\alpha_1}=f(E):E\in\mathcal{E}_2=\mathcal{E}\setminus\mathcal{E}_1\}$

$$|\mathcal{X}|\geq\big(k_2\log n/\log\gamma\big)(1-\varepsilon') \tag{7.8}$$

for $k_2=\lfloor\frac{k_1}{2}\rfloor=\left\lfloor\frac{\lfloor\frac{k}{2}\rfloor}{2}\right\rfloor$ *, for all* $\varepsilon'\in(0,1)$ *, and all large* n,k *.*

Proof: Assume that (7.8) does not hold. Partition $\mathcal{X}$ into $\mathcal{X}_1,\dots,\mathcal{X}_{k_2}$ such that for $i=1,\dots,k_2$

$$|\mathcal{X}_i|\leq\left\lceil\frac{|\mathcal{X}|}{k_2}\right\rceil<\frac{\log n}{\log\gamma}. \tag{7.9}$$

Equivalent is

$$\gamma^{|\mathcal{X}_i|}<n=|\mathcal{V}| \quad \text{for} \quad i=1,2,\dots,k_2. \tag{7.10}$$

Since our assumption implies that $g(\cdot,E)$ takes at most γ values for all $E\in\mathcal{E}$, for every $i\in\{1,\dots,k_2\}$ one can find a pair $\{u_i,u_i'\}\subset\mathcal{V}$ such that for all $x^{\alpha_1}\in\mathcal{X}_i$

$$g(u_i, x^{\alpha_1}) = g(u_i', x^{\alpha_1}).$$

Let $E^* = \left(\bigcup_{i=1}^{k_2} \{u_i, u_i'\}\right) \cup \{v_1, \dots, v_{k_1}\}$ for above pairs and $\{v_1, \dots, v_{k_1}\}$ as in Lemma 2. Then $|E^*| \le k$, so that exists an $E \in \mathcal{E}$ with $E^* \subset E$.

However, when $E \in \mathcal{E}_1$, then by (7.5) $P_{\mathcal{E}}$ cannot distinguish some v_i and v_j and when $E \in \mathcal{E}_2$, there exists an i such that $f(E) \in \mathcal{X}_i$ and therefore $P_{\mathcal{E}}$ cannot distinguish u_i and u_i' . This contradiction proves the Theorem.

Remark 10: The strategy yielding our upper bound can be viewed as a perfect hashing: there are strict colorings for every $(\mathcal{V}, \sigma\mathcal{E}'')$.

Corollary 3. *$\mu_3(n,k) \ge \mu_2(n,k)$ if $k = 0(\log n)$ and n, k are large.*

Proof: Write the possible sequences under a strategy Ω as $(y^{\beta_1}, x^{\alpha_2}, y^{\beta_3})$ with $y^{\beta_1} = f(v)$.

Now, if $\left|\left\{y^{\beta_1} = f(v) : v \in \mathcal{V}\right\}\right| \le n^{\frac{1}{2}}$ then for some y^{β_1} $\mathcal{V}' = \left\{v \in \mathcal{V} : f(v) = y^{\beta_1}\right\}$ has cardinality at least $n^{\frac{1}{2}}$.

When y^{β_1} has been sent we are in the case of 2 rounds for the hypergraph $\mathcal{H}' = \left(\mathcal{V}', \binom{\mathcal{V}'}{k}\right)$ and have the lower bound $(1-\varepsilon)(3\log k + \log\log n - 1)$.

We can assume therefore that there are at least $n^{\frac{1}{2}}$ possible strings y^{β_1} and one must have length at least $\frac{1}{2}\log n \gg 3\log k + \log\log n$ by our assumption.

Remark 11: Is it true that $\mu_3(n,k) = \mu_2(n,k)$?

8. On Cartesian products of hypergraphs

One can come to the present hypergraphs as follows.

Consider a discrete memoryless correlated source $(X^n, Y^n)_{n=1}^{\infty}$, where $X^n = X_1 \dots X_n$, $Y^n = Y_1 \dots Y_n$.

Person $P_{\mathcal{X}}$ observes the output $X^n = x^n$ and person $P_{\mathcal{Y}}$ observes $Y^n = y^n$. They exchange knowledge with zero probability of error. How many bits do they have to exchange in the worst case until $P_{\mathcal{Y}}$ knows x^n ?

We construct the multi–hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, where $\mathcal{E} = \{E_y : y \in \mathcal{Y}\}$ and

$$E_y = \left\{x \in \mathcal{X} : \mathrm{Prob}(X = x, Y = y) > 0\right\}, \tag{8.1}$$

and its n-th Cartesian product $\mathcal{H}^n = (\mathcal{X}^n, \mathcal{E}^n)$, where

$$\mathcal{X}^n = \textstyle\prod_1^n \mathcal{X} \text{ and } \mathcal{E}^n = \prod_1^n \mathcal{E} = \left\{E_{y^n} = \displaystyle\prod_{t=1}^{n} E_{y_t} : y^n = (y_1, \dots, y_n) \in \mathcal{Y}^n\right\}.$$

Clearly,

$$\deg(x^n) = |\{E^n \in \mathcal{E}^n : x^n \in E^n\}| = \prod_{t=1}^{n} \deg(x_t) \tag{8.2}$$

and

$$\deg(E_{y^n}) = |E_{y^n}| = \prod_{t=1}^{n} |E_{y^t}|. \tag{8.3}$$

$$D_{\mathcal{X}^n} = \max_{x^n} \deg(x^n) = \prod_{t=1}^{n} D_{\mathcal{X}} = (D_{\mathcal{X}})^n,$$

$$D_{\mathcal{E}^n} = \max_{E^n} \deg(E^n) = \prod_{t=1}^{n} D_{\mathcal{E}} = (D_{\mathcal{E}})^n.$$

For $C_m(\mathcal{H}^n, P_{\mathcal{E}^n})$ we define

$$A_m = \lim_{n\to\infty} \frac{1}{n} C_m(\mathcal{H}^n, P_{\mathcal{E}^n}). \tag{8.4}$$

Then obviously

$$A_2 \geq A_3 \geq A_4 \geq \cdots \geq A_\infty \geq \log D_{\mathcal{E}}.$$

It was shown in [28] that $A_4 = A_\infty = \log D_{\mathcal{E}}$ and that

$$A_2 \leq 2 \log D_{\mathcal{E}}. \tag{8.5}$$

We mention in the Appendix that for "balanced" hypergraphs ([24])

$$C_3 \leq \log D_{\mathcal{E}} + 3 \log\log \max(D_{\mathcal{E}}, D_{\mathcal{V}}) + 11. \tag{8.6}$$

For our product hypergraph $\mathcal{H}^n$ this implies

$$C_3(\mathcal{H}^n; P_{\mathcal{E}^n}) \leq n \log D_{\mathcal{E}} + 3 \log n + 3 \log\log \max(D_{\mathcal{E}}, D_{\mathcal{V}}) + 11$$

and thus

$$A_3 \leq \log D_{\mathcal{E}}. \tag{8.7}$$

Remark 12: In [28] only $A_4 \leq \log D_{\mathcal{E}}$ was proved.

Our main concern is to determine A_2. We succeeded to derive bounds for special classes of hypergraphs.

For $\mathcal{H}^n = (\mathcal{X}^n, \mathcal{E}^n)$ as above let $\eta_n(\nu)$ be the maximal number of edges in $\mathcal{H}^n$ which can be properly colored with a vertex coloring $\varphi : \mathcal{X}^n \to \{1, 2, \ldots, \nu\}$.

When $\mathcal{H}^n$ has a transitive automorphism group, then by Corollary 2 in Section 7 one can use the set of properly colored edges $\mathcal{E}'$ to cover $\mathcal{E}^n$ with at most

$$n(ln|\mathcal{E}|)\cdot\frac{|\mathcal{E}|^n}{\eta_n(\nu)}+1 \quad \text{“copies”}.$$

Thus, similarly to Theorem 10

$$C_2(\mathcal{H}^n;P_{\mathcal{E}^n})\le\min_\nu\left[\log\nu+\log n+\log\log|\mathcal{E}|+n\log\frac{|\mathcal{E}|}{(\eta_n(\nu))^{\frac{1}{n}}}\right](1+o(1)). \tag{8.8}$$

Suppose on the other hand that $P_{\mathcal{E}^n}$ decomposes $\mathcal{E}^n$ into t parts, informs $P_{\mathcal{V}}$ about the part E_{y^n} is in, and then $P_{\mathcal{V}}$ uses at most ν colors for the vertices.

To guarantee that $P_{\mathcal{E}^n}$ knows x^n at the end of the communication for all $E^n\in\mathcal{E}^n$ there has to be one of the parts in which E^n is properly colored. Therefore

$$t\ \eta_n(\nu)\ge|\mathcal{E}|^n. \tag{8.9}$$

or

$$\frac{1}{n}\log t\ge\log|\mathcal{E}|-\frac{1}{n}\log\eta_n(\nu). \tag{8.10}$$

Therefore

$$\frac{1}{n}C_2(\mathcal{X}^n;P_{\mathcal{E}^n})\ge\frac{1}{n}\min\Big\{\log t+\log\nu:\frac{1}{n}\log t\ge\log|\mathcal{E}|-\frac{1}{n}\log\eta_n(\nu)\Big\}$$

and this and (8.8) imply the following result.

Lemma 4. *If the product hypergraph $\mathcal{H}^n=(\mathcal{X}^n,\mathcal{Y}^n)$ has a transitive automorphism group, then*

$$A_2=\lim_{n\to\infty}\min_\nu\left(\frac{1}{n}\log\nu+\log|\mathcal{E}|-\frac{1}{n}\log\eta_n(\nu)\right). \tag{8.11}$$

Here the limit exists, since

$$\eta_{n_1+n_2}(\nu_1\times\nu_2)\ge\eta_{n_1}(\nu_1)\eta_{n_2}(\nu_2).$$

Example 10: (see also [36])

$\mathcal{H}=(\mathcal{X},\mathcal{E})=\big(\{0,1,2\},\{e_1,e_2,e_3\}\big)$ where $e_i=\{i,i+1\mod 3\}$ for $i=0,1,2$.

Obviously, choosing $\nu=3$ in (8.11), we see that $A_2\le\log 3<2\log D_{\mathcal{E}}=2\log 2=2$.

Actually we can do even better and show that

$$A_2 \leq \log\sqrt{6}. \tag{8.12}$$

Consider the 4–coloring of $\mathcal{H}^2$:

$$\mathcal{X}_1^2 = \{(0,1),(1,0)\}, \mathcal{X}_2^2 = \{(1,2),(2,1)\}, \mathcal{X}_3^2 = \{(2,0),(0,2)\}, \mathcal{X}_4^2 = \{(0,0),(1,1),(2,2)\}. \tag{8.13}$$

It is proper for all edges

$$\mathcal{E}^2 \smallsetminus \{\{0,1\}\times\{0,1\},\{1,2\}\times\{1,2\},\{2,0\}\times\{2,0\}\}.$$

Therefore (8.11) implies (8.12).

Example 11: For graphs $\mathcal{H}=(\mathcal{X},\mathcal{E})$, $|E|=2$ for all $E\in\mathcal{E}$, (8.5) is also not tight. It suffices to consider $\mathcal{H}=\left([\ell],\binom{[\ell]}{2}\right)$. Moreover, as adding vertices does not decrease the complexity, we assume $3|\ell$.

Partition $[\ell]$ into the parts $\mathcal{L}_0,\mathcal{L}_1$, and L_2 and let $g(v)=i$ iff $v\in\mathcal{L}_i=\{m: 1\leq m\leq\ell, m\equiv i \mod 3\}$ $(i=0,1,2)$.

On $\mathcal{H}^2$ define $g_2(v_1,v_2)=\big(g(v_1),g(v_2)\big)$. Replacing (v_1,v_2) in (8.13) by $g_2(v_1,v_2)$ one obtains a 4–coloring of $\mathcal{H}^2$. Write the edges of $\mathcal{H}^2$ in the form

$$\{v_1,v_1'\}\times\{v_2,v_2'\}.$$

As before we can see that an edge is properly colored, if

$g(v_i)\neq g(v_i')$ for $i=1,2$ and

$$\{g(v_1),g(v_1')\}\neq\{g(v_2),g(v_2')\}.$$

Therefore the number of properly colored edges is at least

$$6\cdot\left(\frac{\ell}{3}\right)^4=\frac{2}{27}\ell^4$$

and consequently by definition

$$\eta_2(4)\geq\frac{2\ell^4}{27}.$$

However,

$$\frac{\left|\binom{[\ell]}{2}\right|}{\eta_2(4)}\leq\frac{\left(\frac{\ell(\ell-1)}{2}\right)^2}{\frac{2\ell^4}{27}}<\frac{27}{8}$$

and thus by (8.11)

$$A_2\leq\frac{1}{2}\log\left(4\cdot\frac{27}{8}\right)=\frac{1}{2}\log\frac{27}{2}=1.877\dots<2. \tag{8.14}$$

(8.14) can be improved.

Using Hamming codes, we show next that (8.12) is not tight either for the triangle.

Our bound (8.12) was derived with the 4–coloring of $\mathcal{H}^2$, say $f : \mathcal{X}^2 \to \{1,2,3,4\}$ described in (8.13) and its n–fold product

$$f_n = (f, \dots, f) : (\mathcal{X}^2)^n \to \{1,2,3,4\}^n.$$

We call it the *first* coloring.

Now we introduce a second coloring. For this we extend the sets $\mathcal{X}_1^2, \mathcal{X}_2^2, \mathcal{X}_3^2$ to the sets $\mathcal{Y}_1^2 = \mathcal{X}_1^2 \cup \{y_1\}$, $\mathcal{Y}_2^2 = \mathcal{X}_2^2 \cup \{y_2\}$, $\mathcal{Y}_3^2 = \mathcal{X}_3^2 \cup \{y_3\}$ when y_1, y_2, y_3 are new symbols. We also set $\mathcal{Y}_4^2 = \mathcal{X}_4^2$. Thus the four new sets have all 3 elements and the set

$$\mathcal{Z}^n = \prod_{t=1}^n \mathcal{Z}_i \ \text{ with } \ \mathcal{Z}_i \in \{\mathcal{Y}_j^2 : 1 \leq j \leq 4\}$$

has 3^n elements.

For this *fixed* $\mathcal{Z}^n$ let $g_i : \mathcal{Z}_i \to \{0,1,2\}$ be bijective.

Next we color the vertices in $\{0,1,2\}^n$ according to the cosets of an $[n,k]$ (where $k \triangleq n-r$, $n \triangleq \frac{3^r-1}{2}$) ternary Hamming code $H_{n,k}$:

two elements have the same color iff they are in the same coset. Let h_n be this coloring. Finally we color the elements in $\mathcal{Z}^n$ through

$$G_{\mathcal{Z}^n} = h_n(g_1, \dots, g_n). \tag{8.15}$$

This is a coloring with 3^r colors. Combination with the coloring f_n gives a $4^n \times 3^n$ coloring F_n .

In order to derive a new bound on A_2 we need an estimate for the number of good edges under F_n . Denote the edges of the 2n–th product of triangles as $U^n = U_1 \times \dots \times U_n$, where U_i is a 2–dimensional edge (a "square").

Notice that under our first coloring difficulties arise only at the U_i's equal to $E_j = e_j \times e_j \quad (j = 0,1,2)$.

Considering that any two elements in (the fixed) $\mathcal{Z}^n$ with Hamming distance smaller than 2 have different second color, because the Hamming code has minimal distance 3, it is not hard to see that all $U^n = U_1 \times \dots \times U_n$ with

$$\left| \{U_j : 1 \leq i \leq n\} \cap \left(\bigcup_{i=0}^2 E_i \right) \right| < 3 \tag{8.16}$$

are well–colored.

Let now $\nu = 4^n \cdot 3^r$, then

$$\eta_{2n}(\nu) \geq \sum_{j=0}^2 \binom{n}{j} 6^{n-j} 3^j.$$

Analogous to the previous derivation for the product of triangles, we get now for $n = \frac{3^r-1}{2}, r = 2, 3, \dots$

$$\begin{aligned}
A_2 &\leq \frac{1}{2n} \log \frac{3^{2n} \cdot 4^n \cdot 3^r}{\sum_{j=0}^{2} \binom{n}{j} 6^{n-j} 3^j} \\
&= \frac{1}{2n} \log \frac{6^n \cdot 3^r}{\sum_{j=0}^{2} \binom{n}{j} \left(\frac{1}{2}\right)^j} \\
&= \log \sqrt{6} + \frac{1}{2n} \log \frac{2n+1}{1 + \frac{n}{2} + \frac{n(n-1)}{8}} \\
&= \log \sqrt{6} + \gamma_n, \quad \text{say.}
\end{aligned}$$

Choosing a large value for $n = \frac{3^r-1}{2}$ γ_n becomes negative, which results in a bound better than $\log \sqrt{6}$. Choosing n very large makes this bound less good, because of the factor $\frac{1}{2n}$.

Remark 13: It should be possible to improve the bound with other linear codes.

An obvious lower bound on A_2 for graphs is $\log D_{\mathcal{E}} = 1$. The simplest open question is to determine A_2 for the triangle, for which we know that

$$1 \leq A_2 < \log \sqrt{6}.$$

Appendix: Known results for $C_m(\mathcal{H}, P_{\mathcal{E}})$.

We feel that our sketch is helpful for the orientation of the reader. The shorthand C_m is used for $C_m(\mathcal{H}, P_{\mathcal{E}})$. It was shown in [22] that for every hypergraph

$$C_1 = \lceil \log \chi(\mathcal{H}) \rceil. \tag{A.1}$$

and

$$C_\infty \geq \lceil \log C_1 \rceil + 1, \tag{A.2}$$

if $\chi(\mathcal{H}_1) > 1$, that is, $\mathcal{H}$ is non–trivial.

Such a bound (without the extra bit) appears for communication complexity already in [37].

The complete hypergraph $\mathcal{H} = \left([n], \binom{[n]}{2}\right)$ (League Problem in [22]) is an interesting example. It was shown in [22] that

$$C_2 = \dots = C_\infty = \lceil \log \log n \rceil + 1, C_1 = \lceil \log n \rceil. \tag{A.3}$$

The idea generalizes to all graphs $(D_{\mathcal{E}} = 2)$:

$$C_\infty = \lceil \log \lceil \log \chi(\mathcal{H}) \rceil \rceil + 1 = \lceil \log C_1 \rceil + 1. \tag{A.4}$$

For every hypergraph an obvious bound is

$$C_\infty \geq \log D_{\mathcal{E}}. \tag{A.5}$$

Using a perfect hashing $f = (f_1, \ldots, f_k)$ with c colors, that is, the k functions $f_i : \mathcal{V} \to \{1, \ldots, c\}$ are such that for every $E \in \mathcal{E}$ some f_j is bijective on E , one obtains

$$C_2 \leq \lceil \log k \rceil + \lceil \log c \rceil.$$

Actually, knowing E $P_{\mathcal{E}}$ encodes i and $P_{\mathcal{V}}$ encodes $f_i(v)$.

By random choice it is shown in [22] that

$$C_2 \leq \log\log |\mathcal{V}| + 3 \log D_{\mathcal{E}}.$$

In the same argument $\mathcal{V}$ can be replaced by $\{1, \ldots, \chi\}$:

$$C_2 \leq \log\log \chi + 3 \log D_{\mathcal{E}}.$$

This and the relations (A1), (A2), and (A5) imply

$$C_2 \leq 4\ C_\infty + 3. \tag{A.6}$$

For communication complexity it was noticed in [37] that there can be an exponential gap between $C_1(f)$ and $C_2(f)$.

In [38] it was shown that for every m there is a function g with an exponential gap between $C_m(g)$ and $C_{m+1}(g)$. (A6) shows that this does not happen for interactive communication.

How large can the gaps between C_m and C_∞ be?

The main result of [23] is, that for the "generalized league problem"

$$C_2 \geq (2 - \varepsilon) C_\infty \tag{A.7}$$

can happen for every $\varepsilon > 0$.

Therefore two rounds ("messages") are not optimal.

It was shown in [29] that three rounds are not optimal either! In the present paper we have shown that already a simply structured hypergraph such as the k–uniform hypergraph $\left([n], \binom{[n]}{k}\right)$ has this property. Moreover in this case $C_2 = C_3$.

For balanced hypergraphs $(D_{\mathcal{V}} = D_{\mathcal{E}})$ it is an immediate consequence of Coloring Lemma 2 that (see [24])

$$C_1 \leq 2 \log D_{\mathcal{E}} + 1 \leq 2C_\infty + 1. \tag{A.8}$$

A clever construction in projective planes gives an example of a balanced hypergraph ([24] with J. Kahn) with

$$C_1 \geq 2C_\infty - 6. \tag{A.9}$$

Finally the hashing idea and Coloring Lemma 1 in Conjunction with Lovász's Local Lemma give in the balanced case ([24])

$$C_3 \leq \log D_{\mathcal{E}} + 3 \log\log D_{\mathcal{E}} + 11 \tag{A.10}$$

or by (A5)

$$C_3 \leq C_\infty + 3 \log\log C_\infty + 11.$$

Presently there is no example in the balanced case (only an existence proof in [24]) for

$$C_2 \geq (2 - \varepsilon) C_\infty \geq c \;\; \text{(any const.)} \tag{A.11}$$

References

[1] C.E. Shannon, "Two–way communication channels", Proc. 4th Berkeley Symp. Math. Statist. and Prob., Univ. of California Press, Berkeley, vol. 1, pp. 611–614, 1961.

[2] R. Ahlswede, "Multi–way communication channels", Proc. of 2nd Inf. Symp. on Inform. Theory, Tsakkadsor, Armenian SSR, pp. 23–52, 1971.

[3] D. Slepian and J.K. Wolf, "Noiseless coding of correlated information sources", IEEE Trans. Inform. Theory, vol. IT–19, pp. 471–480, 1973.

[4] R. Ahlswede, "Channel capacities for list codes", J. Appl. Prob., vol. 10, pp. 824–836, 1973.

[5] R. Ahlswede, "Coloring hypergraphs: a new approach to multi–user source coding", J. Comb. Inform. & System Sc., Part I: vol. 4, pp. 76–115, 1979; Part II: vol. 5, pp. 220–268, 1980.

[6] A.C. Yao, "Some complexity questions related to distributive computing", Proc. 11th Annual ACM Symp. on the Theory of Computing, pp. 209–213, 1979.

[7] R. Ahlswede and I. Csiszár, "To get a bit of information may be as hard as to get full information", IEEE Trans. Inform. Theory, vol. IT–27, pp. 398–408, 1980.

[8] A. El Gamal, "A simple proof of the Ahlswede–Csiszár 1–bit theorem", IEEE Trans. Inform. Theory, vol. IT–29, pp. 931–933, 1983.

[9] A. El Gamal and K.F. Pang, "Communication complexity of computing the Hamming distance", SIAM J. Comput., vol. 15, pp. 932–947, 1986.

[10] R. Ahlswede, A. El Gamal, and K.F. Pang, "A two–family extremal problem in Hamming Space", Discrete Math., vol.49, pp. 1–5, 1984.

[11] P. Delsarte and P. Piret, "An extension of an inequality by Ahlswede, El Gamal, and Pang for pairs of binary codes", Discrete Math., vol. 55, pp. 313–315, 1985.

[12] J.I. Hall and J.H. van Lint, "Constant distance code pairs", Proc. Kon. Nederl. Akad. Wetensch. Ser. A, vol. 88, pp. 41–45, 1985.

[13] R. Ahlswede and M. Moers, "Inequalities for code pairs", European J. of Comb., vol. 9, pp. 175–181, 1987.

[14] N. Cai, "A bound of sizes of code pairs satisfying the strong 4–words property for Lee distance", J. System Sc. & Math. Sc., vol. 6, pp. 129–135, 1986.

[15] R. Ahlswede, "On code pairs with specified Hamming distance", Colloquia Mathematica Societatis János Bolyai, vol. 52, Combinatorics, Eger, pp. 9–47, 1987.

[16] R. Ahlswede, N. Cai, and Z. Zhang, "A general 4–words inequality with consequences for 2–way communication complexity", Advances in Applied Mathematics, vol. 10, pp. 75–94, 1989.

[17] U. Tamm, "Communication complexity of sum–type functions invariant under translation", Inform. and Computation, vol. 116, pp. 162–173, 1995.

[18] R. Ahlswede and N. Cai, "On communication complexity of vector–valued functions", IEEE Trans. Inform. Theory, vol. IT–40, pp. 2062–2067, 1994.

[19] R. Ahlswede, N. Cai, and U. Tamm, "Communication complexity in lattices", Applied Math. Lett., vol. 6, pp. 53–58, 1993.

[20] R. Ahlswede and N. Cai, "2–way communication complexity of sum–type functions for one processor to be informed", Problemy Peredachi Informatsii, vol. 30, pp. 3–12, 1994.

[21] A. El Gamal and A. Orlitsky, "Interactive data compression", Proc. 25th Annual Symp. on Foundations of Computer Sc., pp. 100–108, 1984.

[22] A. Orlitsky, "Worst–case interactive communication I: two messages are almost optimal", IEEE Trans. Inform. Theory, vol. IT–36, pp. 1111–1127, 1990.

[23] A. Orlitsky, "Worst–case interactive communication II: two messages are not optimal", IEEE Trans. Inform. Theory, vol. IT–37, pp. 995–1005, 1991.

[24] A. Orlitsky, "Interactive communication, balanced distributions, correlated files, and average case complexity", Proc. 23rd Annual Symp. Foundations of Computer Sc., pp. 228–238, 1991.

[25] A. Orlitsky, "Average–case interactive communication", IEEE Trans. Inform. Theory, vol. IT–38, pp. 1534–1547, 1992.

[26] R. Ahlswede and T.S. Han, "On source coding with side information via a multiple–access channel and related problems in multi–user information theory", IEEE Trans. Inform. Theory, vol. IT–29, pp. 396–412, 1983.

[27] T. Feder, E. Kushilevitz, and M. Noar, "Amortized communication complexity", Proc. of the 32nd Annual Symp. on Foundations of Computer Sc., pp. 239–248, 1991.

[28] M. Noar, A. Orlitsky, and P. Shor, "Three results on interactive communication", IEEE Trans. Inform. Theory, vol. IT–39, pp. 1608–1615, 1993.

[29] Z. Zhang and X.G. Xia, "Three messages are not optimal in worst–case interactive communication", Preprint Jan. 1993.

[30] M.L. Fredman and J. Komlós, "On the size of separating systems and families of perfect hash functions", SIAM J. on Algebraic and Discrete Methods, vol. 5, pp. 61–68, 1984.

[31] N. Alon and J.H. Spencer, "The Probabilistic Method", Wiley–Interscience Series in Discrete Math. and Optimization, John Wiley & Sons, Inc., New York, 1992.

[32] Z. Baranyai, "On the factorization of the complete uniform hypergraph", Colloquia Mathematica Societatis, Janos Bolyai 10. Infinite and finite sets, Kezthely, Hungary, pp. 91–108, 1975.

[33] A. Schrijver, "Packing and Covering in Combinatorics", Math. Centre Tracts 106, Mathematisk Centrum Amsterdam, 1979.

[34] V.G. Vizing, "On an estimate of the chromatic class of a p–graph" (Russian), Diskret. Analiz., vol. 3, pp. 25–30, 1964.

[35] C.E. Shannon, "A theorem on coloring the lines of a network", J. Math. Phys., vol. 28, pp. 148–151, 1949.

[36] R. Ahlswede, "On set coverings in Cartesian product spaces", SFB 343 "Diskrete Strukturen in der Mathematik", Bielefeld, Ergänzungsreihe 92–005, 1992.

[37] C.H. Papadimitriou and M. Sipser, "Communication complexity", Proc. 14th Annual ACM Symp. on the Theory of Computing, pp. 196–200, 1982.

[38] P. Duris, Z. Galil, and G. Schnitger, "Lower bounds on communication complexity", Proc. 16th Annual ACM Symp. on the Theory of Computing, pp. 81–91, 1984.