MESSY BROADCASTING IN NETWORKS

BY

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Dedicated to James L. Massey on the occasion of his sixtieth birthday.

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An apology to all friends of J.L. M.

Among the many discoveries Jim has made until now one observation (Taschkent 1984) is that R.A. seldom does what most people expect him to do. Here we have made an attempt to give a contribution which has no relation to anything J.L. M. ever did.

However, it relates to him. It is in the spirit of the following question: "Which travel time to Zürich can an absentminded mathematician guarantee, if at any station he chooses any train in any available direction without going to the same city twice?"

1. Introduction

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one of the members, is transmitted to all members of the network.

A communication network is a connected graph $G = (V, E)$, where V is a set of vertices (members) and E is a set of edges. Transmission of the message from the originator to all members is said to be broadcasting, if the following conditions hold:

1) Any transmission of information requires a unit of time.
2) During one unit of time every informed vertex (member) can transmit information to one of its neighboring vertices (members).

The classical model.

For a $u \in V$ we define the broadcast time $t(u)$ of vertex u as the minimum number of time units required to complete broadcasting starting from vertex u. We denote by $t(G) = \max_{u \in V} t(u)$ the broadcast time of graph G. It is easy to see that for any connected graph G $t(G) \geq \lceil \log_2 n \rceil$, where $n = |V|$, since during each time unit the number of informed vertices can at most be doubled.

A minimal broadcast graph (MBG) is a graph with n vertices, in which a message can be broadcasted in $\lceil \log_2 n \rceil$ time units.

The broadcast function β assigns to n as value $\beta(n)$ the minimum number of edges in a MBG on n vertices. Presently exact values of $\beta(n)$ are known only for two infinite sets of parameters of MBG's, namely, for $\{n = 2^m : m = 1, 2, 3, \dots\}$ ([1]) and $\{n = 2^m - 2 : m = 2, 3, \dots\}$ ([2] and independently [3]). Known are also the exact values of $\beta(n)$ for some $n \leq 63$ ([1], [4–7]). We recommend [8] as a survey of results on classical broadcasting and related problems.

New models.

In this paper we consider three new models of broadcasting, which we call "Messy broadcasting". We refer to them as M_1, M_2, and M_3.

In the classical broadcast model it is tacitly assumed that every node (member) of the scheme produces the broadcasting in the most clever way. For this it is assumed either, that there is a leader, who coordinates the actions of all members during the whole broadcasting process (which seems to be practivally not realistic) or the members must have a coordinated set of protocols with respect to any originator, enough storage space, timing, and they must know the originator and its starting time.

Now we assume that there is no leader, that the state of the whole scheme is secret for the members, the members do not know the starting time and the originator, and their protocols are not coordinated.

Moreover, even if the starting time and originator are known, and the scheme is public, it is possible that the nodes of the scheme are primitive. They have only a simple memory, which is not sufficient to keep the set of coordinated protocols. Technically it is much easier to build such a network. It is very rebust and reliable.

In all models M_1 , M_2 , and M_3 in any unit of time every vertex can receive information from several of its neighbors simultaneously, but can transmit only to one of its neighbors.

Model M_1 .

In this model in any unit of time every vertex knows the states of its *neighbors*, i.e. which are informed and which are not. We require that in any unit of time every informed vertex must transmit information to one of its noninformed neighbors.

Model M_2 .

In this model we require that in any unit of time every informed vertex u must transmit the information to one of those of its neighbors that did not send the information to u and did not receive it from u before.

Model M_3 .

In this model we require that in any unit of time every informed vertex u must transmit the information to one of those neighbors that did not receive the information from u before.

For an originator $u \in G$ the sequence of calls $\sigma(u)$ is said to be a *strategy* for the model $M_i (i = 1, 2, 3)$ if

a) every call in $\sigma(u)$ is not forbidden in model $M_i (i = 1, 2, 3)$

b) after these calls every member of the system got the information.

In broadcast model M_1 for a vertex $u \in V$ we define $\Omega_1(u)$ to be the set of all broadcast strategies which start from originator u . For any vertex $u \in V$ of the graph $G = (V, E)$ let $t_1^\sigma(u)$ be the broadcast time of u using strategy $\sigma \in \Omega_1(u)$, i.e. $t_1^\sigma(u)$ is the first moment at which every vertex of the scheme got the information by strategy σ . We set $t_1(u) = \max\limits_{\sigma \in \Omega_1(u)} t_1^\sigma(u)$.

Actually $t_1(u)$ is the broadcast time from vertex u in the worst broadcast strategy. Let $t_1(G)$ be the broadcast time of graph G , that is $t_1(G) = \max\limits_{u \in V} t_1(u)$. Similarly for models M_2 , M_3 : $\Omega_2(u)$, $t_2(u)$, $t_2(G)$, $\Omega_3(u)$, $t_3(u)$, and $t_3(G)$ can be defined. From these definitions it follows that

$$\Omega_1(u) \subseteq \Omega_2(u) \subseteq \Omega_3(u). \tag{1.1}$$

For $i = 1, 2, 3$ we define $\tau_i(n) = \min\limits_{\substack{G=(V,E)\\|V|=n}} t_i(G)$.

From (1.1) it follows that $t_1(G) \le t_2(G) \le t_3(G)$ for every connected graph G , and hence $\tau_1(n) \le \tau_2(n) \le \tau_3(n)$ for every positive integer n .

In Section 5 we establish upper bounds on $\tau_2(n)$ and $\tau_3(n)$. Optimal graphs in model M_1 are described in Section 6 and a lower bound for $\tau_3(n)$ is derived in Section 7. For trees we establish even exact results (Section 3 with preparations in Section 2). Here we can algorithmically determine the broadcast times (Section 4).

2. Auxiliary results concerning optimal trees

In addition to the notions presented in the Introduction we need the following concepts.

For model $M_i (i = 1, 2, 3)$ we define $t_i(u, v) = \max\limits_{\sigma \in \Omega_i(u)} t_i^\sigma(u, v)$, where $t_i^\sigma(u, v)$ is the broadcast time when broadcasting according to strategy σ starts from originator u and the information comes to vertex v .

We denote by $\rho(v)$ the local degree of vertex v . Suppose now that we are given a connected tree H . At first we notice that for every vertex u of any tree H the sets of strategies $\Omega_1(u)$ and $\Omega_2(u)$ (but not $\Omega_3(u)$) are the same. Hence $t_1(u) = t_2(u)$ and $t_1(H) = t_2(H)$ for every tree H . In this part we use the abbreviation $t(u)$ for $t_1(u)$ and for $t_2(u)$.

First we consider the following problem. For given broadcast time t construct a tree with root u having maximal number of vertices $g(t)$, for which $t(u) = t$. This tree is called an optimal tree with root u and broadcast time t or in short (OTR, u, t) .

Let for fixed broadcast time $t(u) = t$ an optimal tree T with root u be constructed and let σ_0 be a strategy for which $t(u) = t^{\sigma_0}(u) = \max\limits_\sigma t^\sigma(u)$. Denote by $u_1, u_2, \dots, u_k$ the neighbors of root u . By the tree structure we can assume that under the strategy σ_0 in the unit of time i $(i = 1, \dots, k)$ the vertex u sends information to vertex u_i . After removing (in our minds) from the optimal tree all edges (u, u_i) for $i = 1, \dots, k$ we get trees T_i $(i = 1, \dots, k)$. It is clear that $\max\limits_{1 \le i \le k} t(u_i) = t(u_k)$ (where for $i = 1, \dots, k$ $t(u_i)$ is the broadcast time from u_i in tree T_i), because otherwise, if $\max\limits_{1 \le i \le k} t(u_i) = t(u_j) > t(u_k)$ for some $1 \le j < k$, then by changing the steps j and k in the broadcast strategy σ_0 we would get a strategy σ_0' for which $t^{\sigma_0'}(u) > t^{\sigma_0}(u) = \max\limits_\sigma t^\sigma(u)$. This is a contradiction. It is also clear that for all $i = 1, 2, \dots, k$ the trees T_i are $\left(OTR, u_i, t(u_i)\right)$.

On the other hand, since the tree T is assumed to be optimal, necessarily

$$t(u_1) = t(u_2) = \dots = t(u_k) = t - k. \tag{2.1}$$

Indeed, if otherwise for some $j \in \{1, \ldots, k\}$ $t(u_j) < t(u_k)$, then by taking subtree T_k instead of subtree T_j we will get a tree T' with $t(T') = t(T)$ and number of vertices $|T'| > |T|$, which is a contradiction. Hence

$$g(t) = \max_k k\, g(t-k) + 1. \tag{2.2}$$

The first values of the function g are

$$g(1) = 2,\ g(2) = 3,\ g(3) = 5,\ g(4) = 7,\ g(5) = 11,\ g(6) = 16,\ g(7) = 23. \tag{2.3}$$

It can be shown that for $t \geq 8$

$$g(t) = 3 \cdot g(t-3) + 1. \tag{2.4}$$

Therefore, using the initial values in (2.3), we have

Lemma 1. *(Models M_1 and M_2)*

For given broadcast time $t \geq 7$ the optimal tree with root u , for which $t(u) = t$, has $g(t)$ vertices, where

$$g(t) = \begin{cases} \frac{11 \cdot 3^{\frac{t-3}{3}} - 1}{2} & \textit{for } t \equiv 0 \mod 3 \\ \frac{47 \cdot 3^{\frac{t-7}{3}} - 1}{2} & \textit{for } t \equiv 1 \mod 3 \\ \frac{23 \cdot 3^{\frac{t-5}{3}} - 1}{2} & \textit{for } t \equiv 2 \mod 3. \end{cases} \tag{2.5}$$

Lemma 2.

For any vertices $v, a \in V$ of the tree $T = (V, E)$ $t(a, v) \leq t(v) - \rho(v) + 1$.

Moreover, for any $v \in V$ there exists an $a_0 \in V$ with $t(a_0, v) = t(v) - \rho(v) + 1$.

Proof: For any $v, a \in V$ we consider the unique path $v \to w_1 \to w_2 \to \ldots \to w_s \to a$ between v and a .

It is clear that $t(v, a) = \rho(v) + \sum_{i=1}^{s} \rho(w_i) - s$, $t(a, v) = \rho(a) + \sum_{i=1}^{s} \rho(w_i) - s$, and hence

$$t(a, v) = t(v, a) - \rho(v) + \rho(a). \tag{2.6}$$

From the definition of $t(v)$ it follows that

$$t(v) \geq t(v, a) + \rho(a) - 1. \tag{2.7}$$

Therefore $t(a, v) \leq t(v) - \rho(v) + 1$, as claimed. Moreover, since $t(v)$ is the broadcast time of v , there exist a $u_0 \in V$ and a strategy σ_0 such that

$t(v) = \max_{u \in V} t(v, u) = t(v, u_0)$. Obviously $\rho(u_0) = 1$. Taking $a = u_0$ in (2.6) we get

$$t(u_0, v) = t(v, u_0) - \rho(v) + \rho(u_0) = t(v) - \rho(v) + 1.$$

3. Construction of optimal trees

a) Models M_1 and M_2 .

Again we use the abbreviation $t(u)$ for $t_1(u)$ and $t_2(u)$.

For given t_0 we consider the set $\mathcal{T}(t_0)$ of all connected trees having broadcast time t_0 . We define $f(t_0) = \max_{T \in \mathcal{T}(t_0)} |T|$, where $|T|$ is number of vertices in tree T . We call the tree T t_0–optimal , if $t(T) = t_0$ and $|T| = f(t_0)$, and present now our main tool for determining the quantity $f(t_0)$.

Lemma 3. *For every $t_0 \geq 2$ there exists an t_0–optimal tree T having a center of symmetry, that is, there is a vertex v_0 in T such that after removal of v_0 the tree T is decomposed into trees $H_1, \ldots, H_s$ with equal cardinalities $|H_1| = |H_2| = \cdots = |H_s|$ and $t(w_1) = t(w_2) = \cdots = t(w_s)$. Here w_i $(i = 1, \ldots, s)$ are neighbors of v_0 and $t(w_i)$ is the broadcast time of H_i , when broadcasting starts from root w_i . Moreover, if $t_o \geq 5$, then* every *optimal tree has a center of symmetry.*

Proof: Suppose T is t_0–optimal , that is $t(T) = t_0$ and $|T| = f(t_0)$. Let v be any vertex of T with $\rho(v) \geq 2$.

Let $v_1, v_2, \ldots, v_k$ be the neighbors of v . If we remove (in mind) the vertex v , then the tree T decomposes into trees $T_1 = (V_1, E_1)$, $T_2 = (V_2, E_2), \ldots, T_k = (V_k, E_k)$ with roots $v_1, \ldots, v_k$. Let the labelling be such that $t(v_1) \leq t(v_2) \leq \cdots \leq t(v_k)$, where $t(v_i)$ is the broadcast time of T_i when broadcasting starts from vertex v_i .

Now let us estimate the quantity $t(a, b)$ for $a \in T_i$ and $b \in T_j$ $(i \neq j)$. We see by Lemma 2 that

$$t(a, b) \leq t(v_i) + k + t(v_j)$$

and there exist $a' \in T_i$, $b' \in T_j$ for which $t(a', b') = t(v_i) + k + t(v_j)$.

Since $t(v_1) \leq t(v_2) \leq \cdots \leq t(v_k)$ we obtain

$$t(T) = \max\left\{ t(v_{k-1}) + k + t(v_k);\ \max_{a,b \in T_k} t(a, b) \right\}.$$

Now we show that

$t(v_1) = t(v_2) = \cdots = t(v_{k-1})$ and that $|T_1| = |T_2| = \cdots = |T_{k-1}| = |T_0|$,

where T_0 is the tree with root v_{k-1} and $t(T_0) = t(v_{k-1})$, having a maximal number of vertices. According to Lemma 1 $|T_0| = g\big(t(v_{k-1})\big)$. Indeed, if it is not the case we can change every tree $T_i (i = 1, \ldots, k-1)$ to T_0 and get the tree T' with $|T'| > |T|$.

But it is easy to verify that $t(T') = t(T)$, which contradicts the optimality of tree T .

Now if $t(v_k) = t(v_{k-1})$, then $|T_k| \leq |T_0|$ and we can change also T_k to T_0 to get tree T'' , for which $|T''| \geq |T|$, $t(T'') = t(T)$, and v is the center of symmetry of T'' . Suppose that $t(v_k) > t(v_{k-1})$ and consider the neighbors of vertex v_k : $u_1, u_2, \dots, u_{r-1}, v$. If we remove (in mind) the vertex v_k , then the tree T is decomposed into trees $L_1, \dots, L_{r-1}, L(v)$ with roots $u_1, \dots, u_{r-1}, v$. Let $t(u_1) \leq t(u_2) \leq \dots \leq t(u_{r-1})$, where $t(u_i)$ $(i = 1, \dots, r-1)$ is the broadcast time of L_i when broadcasting starts from vertex u_i . Clearly $t(v) = k - 1 + t(v_{k-1})$, where $t(v)$ is the broadcast time of $L(v)$, when broadcasting starts from vertex v .

We have to consider two cases: (i) $t(v) \leq t(u_{r-1})$ and (ii) $t(v) > t(u_{r-1})$.

If we are in case (i), then it can be shown as above that $t(u_1) = t(u_2) = \dots = t(u_{r-2}) = t(v)$, $|L_1| = |L_2| = \dots = |L_{r-2}| = |L(v)|$, and if $t(u_{r-1}) = t(u_1) = \dots = t(u_{r-2}) = t(v)$, then v_k is the center of the tree T . Otherwise we will continue our procedure by considering the neighbors of u_{r-1} . Hence the principle case is (ii): $t(v) = k - 1 + t(v_{k-1}) > t(u_{r-1})$.

In this case we have already shown that $t(u_1) = t(u_2) = \dots = t(u_{r-1})$; $|L_1| = \dots = |L_{r-1}| = |L_0|$ where L_0 is the tree with root u_{r-1} , $t(L_0, u_{r-1}) = t(u_{r-1})$, and having maximal number of vertices equal to $g\big(t(u_{r-1})\big)$ (see Lemma 1). Hence $t(v_k) = r - 1 + t(u_{r-1})$ and by our assumption $r - 1 + t(u_{r-1}) > t(v_{k-1})$.

It is easy to verify that in this case (ii) we have $t(T) = t(v_{k-1}) + k + r - 1 + t(u_{r-1})$.

Let us prove that $t(v_{k-1}) = t(u_{r-1})$ or equivalently that $|T_0| = |L_0|$. Suppose that $t(v_{k-1}) > t(u_{r-1})$ (or equivalently that $|T_0| > |L_0|$). Then in tree T we remove the edge (v_k, u_1) with the rooted subtree (L_0, u_1) and add the new edge (v, v') with the rooted subtree (T_0, v')

Using the restriction $r - 1 + t(u_{r-1}) > t(v_{k-1})$ it is easy to verify that for the obtained tree T' we have $t(T') = t(T)$. However this contradicts the optimality of T , since $|T'| > |T|$. Similarly it can be proved that $t(v_{k-1}) < t(u_{r-1})$ is impossible. Hence $t(v_{k-1}) = t(u_{r-1}) = t_1$, $|T_0| = |L_0|$ and $t(T) = 2t_1 + k + r - 1$, $|T| = (k + r - 2) \cdot |T_0| + 2$.

Now we can transform our tree T into the new one T^* as follows: we remove vertex v_k with edges (v_k, v) , (v_k, u_i) for $i = 1, \dots, r-1$, we add edges (v, u_i) for $i = 1, \dots, r-1$ and add a new vertex v' with rooted subtree (T_0, v') and edge (v, v') .

We verify that $t(T^*) = t(T) = 2t_1 + k + r - 1$ and

$$|T^*| = (k + r - 1)|T_0| + 1 \geq |T|. \tag{3.1}$$

However, since T is optimal, we should have equality in (3.1), which occurs only when $|T_0| = 1$ (or equivalently when $t_1 = 0$), i.e. all vertices v_i, u_j $(i = 1, \dots, k-1; j = 1, \dots, r-1)$ are terminal vertices in T . Hence, if $|T_0| = 1$, we have $|T| = k + r$ and $t_0 = t(T) = k + r - 1 = |T| - 1$. However, it is very easy to construct for every $t_0 \geq 5$ a tree (not necessary optimal) having more than $t_0 + 1$ vertices.

Therefore, if $t_0 \geq 5$, the assumption (ii) $t(v) > t(u_{r-1})$ is impossible and hence for $t_0 \geq 5$ every optimal tree has a center of symmetry.

We verify that for $t_0 = 3$ $|T| = 4$, that for $t_0 = 4$ $|T| = 5$, and that all connected trees on 4 or 5 vertices are optimal. Among these optimal trees there are stars (which have center of symmetry) on 4 or 5 vertices, and this fact completes the proof.

Now let v be the center of symmetry of an t_0–optimal tree T , $t_0 \geq 2$. That is, removing vertex v from T the tree T will be decomposed into s subtrees $T_1, \ldots, T_s$ with roots $v_1, \ldots, v_s$; $t(T_1, v_1) = t(T_2, v_2) = \cdots = t(T_s, v_s) = t_1$ and $|T_1| = |T_2| = \cdots = |T_s| = g(t_1)$, where $v_1, \ldots, v_s$ are the neighbors of v and $g(t_1)$ is described in Lemma 1.

We verify that $t(T) = t_0 = 2t_1 + s$ and

$$|T| = s \cdot g(t_1) + 1 = (t_0 - 2t_1)g(t_1) + 1. \tag{3.2}$$

Therefore, by optimality of T t_1 maximizes the quantity

$$\max_{0 \leq x < \frac{t_0}{2}} (t_0 - 2x)g(x) = (t_0 - 2t_1)g(t_1).$$

Using (2.3) and (2.5) it is not difficult to find (details are omitted) an appropriate t_1 (and hence value s) for every fixed broadcast time $t_0 \geq 2$ we have

Theorem 1. *(Models M_1 and M_2)*

Let T be an optimal tree for which $t(T) = t_0$ and $t_0 \geq 2$. Then

$$f(t_0) = |T| = \begin{cases} 3 & \textit{for } t_0 = 2 \\ 4 & \textit{for } t_0 = 3 \\ 5 & \textit{for } t_0 = 4 \\ 7 & \textit{for } t_0 = 5 \\ 9 & \textit{for } t_0 = 6 \end{cases}$$

and for $t_0 \geq 7$

$$f(t_0) = |T| = \begin{cases} 5 \cdot g\left(\frac{t_0-5}{2}\right) + 1, & \textit{if } t_0 \equiv 1 \mod 2 \\ 6 \cdot g\left(\frac{t_0-6}{2}\right) + 1, & \textit{if } t_0 \equiv 0 \mod 2. \end{cases}$$

Using Theorem 1 and (2.5) the following result can be proved (details are omitted).

Corollary 1. *For large t_0*

$$t_0 = \frac{6}{\log_2 3} \cdot \log_2 |T| + 0(1) \sim 3.785 \log_2 |T|.$$

b) Model M_3 .

Since the optimal trees in models M_2 and M_3 are similar (but not the same!) we represent only the results.

We have to calculate now the quantity $g'(t_0)$, which as in case of model M_2 is defined to be the cardinality of optimal tree H with root u , that is $t_3(u,H) = t_0$ and for any tree H' with $t_3(u,H') = t_0$ it follows that $|H| \geq |H'|$. The initial values of $g'(t_0)$ are $g'(1) = 2$, $g'(2) = 3$, $g'(3) = 4$, $g'(4) = 5$, $g'(5) = 7$, $g'(6) = 10$, $g'(7) = 13$, $g'(8) = 17$, $g'(9) = 22$, $g'(10) = 31$, $g'(11) = 41$, $g'(12) = 53$, $g'(13) = 69$, $g'(14) = 94$, $g'(15) = 125$, $g'(16) = 165$, $g'(17) = 213$, $g'(18) = 283$.

Lemma 1'. *(Model M_3) For $t_0 \geq 18$ we have*

$$g'(t_0) = \begin{cases} \frac{94\cdot 4^{\frac{t_0-10}{5}}-1}{3}, & \text{if } t_0 \equiv 0 \mod 5 \\ \frac{31\cdot 4^{\frac{t_0-6}{5}}-1}{3}, & \text{if } t_0 \equiv 1 \mod 5 \\ \frac{10\cdot 4^{\frac{t_0-2}{5}}-1}{3}, & \text{if } t_0 \equiv 2 \mod 5 \\ \frac{850\cdot 4^{\frac{t_0-18}{5}}-1}{3}, & \text{if } t_0 \equiv 3 \mod 5 \\ \frac{283\cdot 4^{\frac{t_0-14}{5}}-1}{3}, & \text{if } t_0 \equiv 4 \mod 5 \ . \end{cases}$$

Lemma 3'. *(Model M_3) For every $t_0 \geq 2$ every optimal tree has a center of symmetry.*

Remark: The difference between Lemma 3 and 3' is the following: in the model M_2 for $t_0 = 3$ and $t_0 = 4$ there are trees which are optimal but do not have a center of symmetry, in model M_3 there are no such exceptions.

Lemma 2 can be repeated for the model M_3 .

Theorem 1'. *(Model M_3) Let H be a t_0–optimal tree and $t_0 \geq 18$. Then*

$$|H| = \begin{cases} 8 \cdot g'\left(\frac{t_0-11}{2}\right) + 1, & \text{if } t_0 \equiv 1 \mod 2 \\ 7 \cdot g'\left(\frac{t_0-10}{2}\right) + 1, & \text{if } t_0 \equiv 0 \mod 2, \end{cases}$$

where g' is the quantity described in Lemma 1'.

Corollary 1'. *(Model M_3) For large t_0* $t_0 \sim 5 \cdot \log_2 |H|$.

At the end of this paragraph we discuss the structures of optimal trees in models M_2 and M_3 .

Let T and H be optimal trees in models M_2 and M_3 , respectively, and let $t_2(T) = t_3(H) = t_0$ and let t_0 be large. From Lemmas 3 and 3' it follows:

In T and H there are centers of symmetry $v \in T$ and $u \in H$. Now for $t_0 \equiv 1 \mod 2$ we have $\rho(v) = 5$, $\rho(u) = 8$ and for $t_0 \equiv 0 \mod 2$ we have $\rho(v) = 6$, $\rho(u) = 7$. The distance from v to every terminal point in the tree T is of order $\frac{t_0}{6}$ and the distance from u to every terminal point in the tree H is of order $\frac{t_0}{10}$.

It can be shown, that every vertex $v' \in T$ with $d(v, v') < \frac{t_0}{6} - 3$ ($d(v, v')$ means distance between v and v'), has local degree $\rho(v') = 4$, and for every $u' \in H$ with $d(u, u') < \frac{t_0}{10} - 6$, $\rho(u') = 5$.

4. An algorithm for determining the broadcast time of a tree

In this section we present an algorithm for determination of the broadcast time of any given tree.

a) Models M_1 and M_2 .

Let us have to find the broadcast time $t(u)$ of vertex u in tree $T = (V, E)$. Suppose vertex u has neighbors $u_1, \ldots, u_k$, which have the broadcast times $t(u_1), \ldots, t(u_k)$ in trees $T_i = (V_i, E_i)$ with roots $u_i (i = 1, \ldots, k)$, respectively.

It is clear that the broadcast time of vertex u is $t(u) = \max_{1 \le i \le k} t(u_i) + k$.

Our algorithm is based on this fact.

The algorithm:

Step 1: Label the terminal vertices of tree T with 0 , that is, if $\rho(v) = 1$, then $\ell(v) = 0$.

Step 2: For all vertices v (v has no label), if $\rho(v) = k$ and all $k-1$ neighbors $v_1, \ldots, v_{k-1}$ of v except v_k are labeled, then we label the vertex v with $\ell(v) = \max_{1 \le i \le k-1} \ell(v_i) + k - 1$.

Step 3: If all neighbors $v_1, \ldots, v_k$ of the vertex v are labeled (v has no label), then we label vertex v with $\ell(v) = \max_{1 \le i \le k} \ell(v_i) + k$.

Step 4: The broadcast time of vertex v (which got the label in step 3) equals its label: $t(v) = \ell(v)$.

Step 5: If every $v \in T$ has $\ell(v)$ go to step 7. If v' is neighbor of v and the broadcast time $t(v)$ of vertex v is known, but $t(v')$ is not known, then cancel the labels $\ell(v) = t(v)$ and $\ell(v')$.

Step 6: If $\rho(v) = k$ and v has neighbors $v_1, v_2, \ldots, v_{k-1}, v'$, then we label vertex v with $\ell(v) = \max_{1 \le i \le k-1} \ell(v_i) + k - 1$. Go to step 3.

Step 7: Stop.

It can be verified (details are omitted) that this algorithm assigns to every vertex its broadcast time.

b) Model M_3 **.**

A similar algorithm can be designed and we leave it to the reader.

5. An upper bound for $\tau_2(n)$ and $\tau_3(n)$

Lemma 4. *For any connected graph* $G = (V, E)$ *with diameter* d *and* $\rho(G) \leq k$ *we have*

(a) $t_2(G) \leq d(k-1)+1$ (b) $t_3(G) \leq dk$.

Proof:

(a) We have to prove that in model M_2 $t(v,u) \leq d(k-1)+1$ for any $v, u \in V$.

Let $v \to w_1 \to w_2 \to \ldots \to w_{s-1} \to u$ be the shortest path from v to u . Since $\rho(v) \leq k$, $\rho(w_i) \leq k$ for $i = 1, \ldots, s-1$, after at most k units of time the vertex w_1 will be informed, after at most $2k-1$ units of time the information comes to w_2 etc. and after at most $s(k-1)+1$ units of time the information comes to vertex u .

Since the graph G has diameter d we have $s \leq d$. Therefore $t(v,u) \leq d(k-1)+1$ for any $v, u \in V$

(b) The proof is similar.

We need the following result due to Bollobás and de la Vega [9].

Theorem [9]. *Suppose* $\varepsilon > 0$ *and* $k \geq 3$ *are fixed. Then if* d *is sufficiently large there exists a graph* $G = (V, E)$ *with diameter* d *and* $\rho(G) \leq k$ *for which*

$$|V| = n \geq \frac{1-\varepsilon}{2kd\log_2(k-1)} \cdot (k-1)^{d-1}.$$

Actually for every $k \geq 3$ and large d we have

$$\log_2 n \geq (d-1)\log_2(k-1) - 0(\log d).$$

Using (a) of Lemma 4 and Theorem [9] we conclude that for any fixed $k \geq 3$ and sufficiently large d there exists a graph $G = (V, E)$, $\rho(G) \leq k$, with diameter d , $|V| = n$, and broadcast time

$$t_2(G) \leq \frac{k-1}{\log_2(k-1)} \cdot \log_2 n.$$

We verify that $\min\limits_{k\geq 3} \frac{k-1}{\log_2(k-1)} = \frac{3}{\log_2 3} \approx 1.89$ and that the minimum is assumed for $k = 4$.

Similarly, using Lemma 4 (b) and again Theorem [9] we have that $t_3(G) \leq \frac{k}{\log_2(k-1)} \log_2 n$, $\min \frac{k}{\log_2(k-1)} = 2,5$, and that the minimum is assumed for $k = 5$.

We summarize our findings.

Theorem 2. *For sufficiently large* n

(a) $\tau_2(n) \le 1.89 \log_2 n$ (b) $\tau_3(n) \le 2,5 \cdot \log_2 n$.

6. Some optimal graphs (Model M_1)

In this Section we present for broadcast model M_1 some graphs on n vertices, where $n \le 10$ and $n = 14$, with minimum possible broadcast time, that is, for these graphs

$$\tau_1(n) = t_1(G),\ G = (V, E),\ |V| = n.$$

For $4 \le n \le 8$ the optimal graphs are cycles C_n .

Denote their vertex set by $V_n^* = \{0, 1, \dots, n-1\}$ and their edge set by E_n^* . For $n = 10$ $\tau_1(10) = 4$ and the optimal graph is the well-known Peterson graph $G = \big(\{0, 1, \dots, 4\} \cup \{0', 1', \dots, 4'\}\ ,\ E_5^* \cup E_5' \cup \big\{\{i, i'\} : i = 0, 1, \dots, 4\}\big\}\big)$, where $E_5' = \big\{\{0', 2'\}, \{2', 4'\}, \{4', 1'\}, \{1', 3'\}, \{3', 0'\}\big\}$.

For $n = 9$ $\tau_1(9) = 4$ and the optimal graph is obtained from Peterson's graph by removing one vertex with its edges.

For $n = 14$ $\tau_1(14) = 5$ and the optimal graph is

$$G = \big(V_{14}^*, E_{14}^* \cup \big\{\{0, 5\}, \{1, 10\}, \{2, 7\}, \{3, 12\}, \{4, 9\}, \{6, 11\}, \{8, 13\}\big\}\big).$$

It is necessary to note that these graphs — except for the graphs on 9 and 10 vertices — are optimal even for broadcast model M_2 .

7. A lower bound for $\tau_3(n)$

Let $G = (V, E)$ be a connected graph for which $t_3(G) = t_0$, that is $t_0 = t_3(G) = \max\limits_{u \in V} \max\limits_{\sigma \in \Omega_3(u)} t_3^{\sigma}(u)$. We take an arbitrary originator $v \in V$ and consider the following strategy $\sigma_0 \in \Omega(v)$.

In any unit of time t' , $t' \in \{1, 2, \dots, t_0 - 1\}$ let $N(t') = N_1(t') \cup N_2(t')$ be the set of informed vertices after t' units of time, where N_2 is the set of "new" informed vertices, that is N_2 is the set of those vertices of N which were not informed after $t' - 1$ units of time. It means that every vertex $u_i \in N_2$ $(i = 1, \dots, |N_2|)$ in the t'–th moment received the information from some subsets $V_i \subset N_1$ $(i = 1, \dots, |N_2|)$, $V_i \cap V_j = \varnothing$. Then the strategy σ_0 is the following: in the $(t'+1)$–th unit of time every $u_i \in N_2$, $(i = 1, \dots, |N_2|)$ sends the information back to anyone vertex from subset V_i , $(i = 1, \dots, |N_2|)$.

Hence, using the strategy σ_0 , after $t' + 1$ units of time the cardinality of the set of informed vertices could increase at most by $|N_1|$. So, if we denote by $n(k)$ $(k =$

$2, \ldots, t)$ the cardinality of the set of informed vertices in the k–th unit of time, then we have

$$|V| \leq n(k) \leq n(k-1) + n(k-2).$$

From here for t_0 we have

$$|V| \leq n(t_0) \leq c \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{t_0} \quad \text{or} \quad t_0 \geq \frac{1}{\log_2 \frac{1+\sqrt{5}}{2}} \log_2 |V| \sim 1.44 \log |V|.$$

Theorem 3. *(Model M_3)* $\tau_3(n) \geq 1.44 \cdot \log_2 n$.

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