THE COMPLETE NONTRIVIAL–INTERSECTION THEOREM FOR SYSTEMS OF FINITE SETS

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Abstract

Recently we proved in [4] a complete intersection theorem for systems of finite sets. Now we establish such a result for nontrivial–intersection systems (in the sense of Hilton and Milner [7]).

1. Introduction and result

The Theorem presented and proved in this paper can be viewed as an extension or improvment of our recent Complete Intersection Theorem [4] and may be called Complete Nontrivial–Intersection Theorem. It goes considerably beyond the well–known Hilton/Milner Theorem [7]. We put here the result in proper perspective by a brief sketch of key steps in the development initiated with the pioneering paper [1] by Erdös,Ko, and Rado.

Since we use again our methods from [4], we also keep the notation of this earlier paper as far as possible.

$\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$.

For $k, n \in \mathbb{N}$, $k \leq n$, we set

$$2^{[n]} = \big\{F : F \subset [1, n]\big\},\ \binom{[n]}{k} = \big\{F \in 2^{[n]} : |F| = k\big\}.$$

A system of sets $\mathcal{A} \subset 2^{[n]}$ is called t–intersecting , if

$$|A_1 \cap A_2| \geq t \ \text{ for all } \ A_1, A_2 \in \mathcal{A},$$

and $I(n, t)$ denotes the set of all such systems.

We denote by $I(n, k, t)$ the set of all k–uniform t–intersecting systems (families), that is

$$I(n, k, t) = \left\{\mathcal{A} \in I(n, t) : \mathcal{A} \subset \binom{[n]}{k}\right\}.$$

The investigation of the function

$$M(n, k, t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|,\ 1 \leq t \leq k \leq n,$$

and the structure of maximal systems was initiated by Erdös, Ko, and Rado.

Theorem EKR [1].

For $1 \leq t \leq k$ *and* $n \geq n_0(k, t)$ *(suitable)*

$$M(n, k, t) = \binom{n-t}{k-t}.$$

Clearly, the system

$$\mathcal{A}(n, k, t) = \left\{A \in \binom{[n]}{k} : [1, t] \subset A\right\}$$

is t–intersecting , has cardinality $\binom{n-t}{k-t}$, and is therefore optimal for $n \geq n_0(k, t)$.

The smallest $n_0(k,t)$ has been determined by Frankl [2] for $t \geq 15$ and subsequently by Wilson [3] for all t :

$$n_0(k,t) = (k-t+1)(t+1).$$

In the recent paper [4] we settled all the remaining cases

$$2k - t < n < (k-t+1)(t+1).$$

In particular we proved the long standing so called $4m$–Conjecture (Erdös, Ko, Rado 1938, see also [5] and survey [6]):

$$M(4m, 2m, 2) = \left| \left\{ F \in \binom{[4m]}{2m} : F \cap [1, 2m] \geq m+1 \right\} \right|.$$

We also proved the General Conjecture of Frankl ([2], 1978), that is, for $1 \leq t \leq k \leq n$

$$M(n,k,t) = \max_{0 \leq i \leq \frac{n-t}{2}} |\mathcal{F}_i|,$$

where

$$\mathcal{F}_i = \left\{ F \in \binom{[n]}{k} : |F \cap [1, t+2i]| \geq t+i \right\} \quad \text{for } 0 \leq i \leq \frac{n-t}{2}. \tag{1.1}$$

Theorem AK [4].

For $1 \leq t \leq k \leq n$ *with*

(i) $(k-t+1)\left(2 + \frac{t-1}{r+1}\right) < n < (k-t+1)\left(2 + \frac{t-1}{r}\right)$ *for some* $r \in \mathbb{N} \cup \{0\}$ *we have*

$$M(n,k,t) = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ *is — up to permutations — the unique optimum.*
(By convention $\frac{t-1}{r} = \infty$ *for* $r = 0$ *.)*

(ii) $(k-t+1)\left(2 + \frac{t-1}{r+1}\right) = n$ *for* $r \in \mathbb{N} \cup \{0\}$ *we have*

$$M(n,k,t) = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals — up to permutations — either $\mathcal{F}_r$ *or* $\mathcal{F}r+1$ *.*

An $\mathcal{A} \in I(n,k,t)$ (resp. $\mathcal{A} \in I(n,t)$) is called *non–trivial* if $|\bigcap_{A \in \mathcal{A}} A| < t$, and $\tilde{I}(n,k,t)$ (resp. $\tilde{I}(n,t)$) denotes all non–trivial families from $I(n,k,t)$ (resp. $I(n,t)$). Let

$$\tilde{M}(n,k,t) = \max_{\mathcal{A} \in \tilde{I}(n,k,t)} |\mathcal{A}|, 1 \leq t \leq k \leq n.$$

Hilton and Milner proved in 1967

Theorem HM [7].

$$\tilde{M}(n,k,1) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1, \quad \textit{if} \;\; n > 2k.$$

For $t > 1$ a considerable step was taken in 1978 by Frankl, who determined $\tilde{M}(n,k,t)$, if n is large enough.

Theorem F [8].

For $1 \le t \le k \le n$ *and* $n > n_1(k,t)$ *(suitable) in the cases*

(a) $t+1 \le k \le 2t+1$ *:* $\tilde{M}(n,k,t) = |\mathcal{V}_1(n,k,t)|$ *, where*

$$\mathcal{V}_1(n,k,t) = \left\{ V \in \binom{[n]}{k} : |[1,t+2] \cap V| \ge t+1 \right\} \tag{1.2}$$

(b) $k > 2t+1$ *:* $\tilde{M}(n,k,t) = |\mathcal{V}_2(n,k,t)|$ *, where*

$$\mathcal{V}_2(n,k,t) = \left\{ V \in \binom{[n]}{k} : [1,t] \subset V, V \cap [1+t,k+1]| \ne \varnothing \right\} \cup \big\{ [1,k+1] \smallsetminus \{i\} : i \in [1,t] \big\}. \tag{1.3}$$

Moreover, for every $\mathcal{V} \in \tilde{I}(n,k,t)$ *with* $|\mathcal{V}| = \tilde{M}(n,k,t)$ *one has* $\mathcal{V} \equiv \mathcal{V}_1(n,k,t)$ *in the case* $k \le 2t+1$ *and* $\mathcal{V} \equiv \mathcal{V}_2(n,k,t)$ *in the case* $k > 2t+1$ *, provided that* $n > n_1(k,t)$ *.*

We note that $\mathcal{V}_1(n,k,t) = \mathcal{F}_1$ ($\mathcal{F}_i$–s are defined in (1.1)). The natural questions are: "What is the value of $n_1(k,t)$? (see [9], [10]). What is the value of $\tilde{M}(n,k,t)$, if $n < n_1(k,t)$?" In [10] it was asked whether $n_1(k,t) \sim c \cdot kt$.

In the present paper, we answer all these questions by determining $\tilde{M}(n,k,t)$ for all n,k,t . Our main result is the following

Theorem.

(a) $2k - t < n \le (t+1)(k-t+1)$ *:*

$$\tilde{M}(n,k,t) = M(n,k,t)$$

and the value of $M(n,k,t)$ *is specified in Theorem AK.*

(b) $(t+1)(k-t+1) < n$ *and* $k \le 2t+1$ *:*

$$\tilde{M}(n,k,t) = |\mathcal{F}_1| = |\mathcal{V}_1|$$

and $\mathcal{F}_1$ *is — up to permutations — the unique optimum.*

(c) $(t+1)(k-t+1) < n$ *and* $k > 2t+1$ *:*

$$\tilde{M}(n,k,t) = \max\big\{ |\mathcal{V}_1|, |\mathcal{V}_2| \big\},$$

and — up to permutations — $\mathcal{V}_1$ *or* $\mathcal{V}_2$ *are the only solutions.*

2. Left compressed sets

We recall first well–known notions, which we need.

Definition 2.1: For $A_1 = \{i_1, i_2, \dots, i_s\} \in \binom{[n]}{s}$, $i_1 < i_2 < \dots < i_s$ and $A_2 = \{j_1, j_2, \dots, j_s\} \in \binom{[n]}{s}$, $j_1 < j_2 < \dots < j_s$, we write $A_1 \prec A_2$ iff $i_\ell \le j_\ell$ for all $1 \le \ell \le s$, that is, A_1 can be obtained from A_2 by *left–pushing.* Furthermore let $\mathcal{L}(A_2)$ be the set of all sets obtained this way from A_2 .

Also set $\mathcal{L}(\mathcal{A}) = \bigcup_{A\in\mathcal{A}} \mathcal{L}(A)$ for any $\mathcal{A} \subset 2^{[n]}$.

Definition 2.2: $\mathcal{A} \subset 2^{[n]}$ is said to be *left compressed* or stable iff $\mathcal{A} = \mathcal{L}(\mathcal{A})$.

Definition 2.3: We denote by $LI(n,k,t) \subset I(n,k,t)$ (resp. $L\tilde{I}(n,k,t) \subset \tilde{I}(n,k,t)$) the set of all stable systems from $I(n,k,t)$ (resp. from $\tilde{I}(n,k,t)$). It is well known and easily follows with the shifting technique of [1] that

$$M(n,k,t) = \max_{\mathcal{A}\in I(n,k,t)} |\mathcal{A}| = \max_{\mathcal{A}\in LI(n,k,t)} |\mathcal{A}|.$$

It is also known (c.f. Frankl [11]), that for $t = 1$ we have analogously

$$\tilde{M}(n,k,1) = \max_{\mathcal{A}\in \tilde{I}(n,k,1)} |\mathcal{A}| = \max_{\mathcal{A}\in L\tilde{I}(n,k,1)} |\mathcal{A}|.$$

Using the approach in [11] one can extend this to every t and this is presented as (i) in the Proposition below.

Let now $\mathcal{A} \in I(n,k,t)$ be such that $\left|\bigcap_{A\in\mathcal{A}} A\right| = 0$, let $I_0(n,k,t)$ denote all such families from $I(n,k,t)$, and let $M_0(n,k,t) = \max_{\mathcal{A}\in I_0(n,k,t)} |\mathcal{A}|$.

Obviously, $I_0(n,k,t) \subset \tilde{I}(n,k,t) \subset I(n,k,t)$ and $M_0(n,k,t) \le \tilde{M}(n,k,t) \le M(n,k,t)$.

We gain more insight from an interesting identity with a simple proof. We state it as (ii) in the Proposition eventhough it is not used in this paper. To the contrary, it follows from the Theorem, which says that all optimal families in $\tilde{I}(n,k,t)$ belong to $I_0(n,k,t)$.

Proposition.

(i)

$$\tilde{M}(n,k,t) = \max_{\mathcal{A}\in \tilde{I}(n,k,t)} |\mathcal{A}| = \max_{\mathcal{A}\in L\tilde{I}(n,k,t)} |\mathcal{A}|. \tag{2.1}$$

(ii) $\tilde{M}(n,k,t) = M_0(n,k,t)$ *for all* $n, k,$ *and* t .

Moreover, for every $\mathcal{A} \in \tilde{I}(n,k,t)$ *with* $|\mathcal{A}| = \tilde{M}(n,k,t)$ *one has* $\mathcal{A} \in I_0(n,k,t)$ *as well.*

Proof: (i) For integers $1 \le i < j \le n$ and a family $\mathcal{F} \subset 2^{[n]}$ let us define the well–known (i,j)–shift S_{ij} as follows:

$$S_{ij}(F) = \begin{cases} \big(F \smallsetminus \{j\}\big) \cup \{i\} & \text{if } i \notin F, j \in F, \big((F \smallsetminus \{j\}) \cup \{i\}\big) \notin \mathcal{F} \\ F & \text{otherwise} \end{cases},$$

$$S_{ij}(\mathcal{F}) = \big\{S_{ij}(F) : F \in \mathcal{F}\big\}.$$

It is well–known and easy to show (see e.g. Proposition 2.1 [11]), that $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$, $\mathcal{F} \subset \binom{[n]}{k}$ implies $S_{ij}(\mathcal{F}) \subset \binom{[n]}{k}$, and that for $\mathcal{F} \in I(n,k,t)$ also $S_{ij}(\mathcal{F}) \in I(n,k,t)$.

Let $\mathcal{A} \in \tilde{I}(n,k,t)$ be a family with $|\mathcal{A}| = \tilde{M}(n,k,t)$. We apply the (i,j)–shift to $\mathcal{A}$. Then either $S_{ij}(\mathcal{A}) \in \tilde{I}(n,k,t)$ or $S_{ij}(\mathcal{A}) \in I(n,k,t) \smallsetminus \tilde{I}(n,k,t)$. In the first case we continue the shifting until we obtain a stable family. Suppose then that the second possibility occurs. In this case necessarily $\Big|\bigcap\limits_{A \in \mathcal{A}} A\Big| = t-1$ and $\Big|\bigcap\limits_{A \in S_{ij}(\mathcal{A})} A\Big| = t$. Without loss of generality we can assume that $\bigcap\limits_{A \in \mathcal{A}} A = \{1,2,\dots,t{-}1\}$, $i = t, j = t{+}1$ and that $\bigcap\limits_{A \in S_{tt+1}(\mathcal{A})} A = \{1,2,\dots,t\}$, which immediately implies $|A \cap \{t,t+1\}| \ge 1$ for all $A \in \mathcal{A}$.

Since $\mathcal{A}$ is of maximal size, necessarily

$$\left\{G : [1,t+1] \subset G \in \binom{[n]}{k}\right\} \subset \mathcal{A}. \tag{2.2}$$

As $\bigcap\limits_{A \in \mathcal{A}} A = \{1,2,\dots,t-1\}$ and $\bigcap\limits_{A \in S_{tt+1}(\mathcal{A})} = \{1,2,\dots,t\}$, there are $A_1, A_2 \in \mathcal{A}$ with

$$A_1 \cap [1,t+1] = \{1,2,\dots,t\} \text{ and } A_2 \cap [1,t+1] = \{1,2,\dots,t-1,t+1\}. \tag{2.3}$$

Now, instead of S_{tt+1} we keep applying the (i,j)–shift for $1 \le i < j \le n$ with $i,j \notin \{t,t+1\}$. Then (2.2) and (2.3) imply that $\bigcap\limits_{A \in S_{ij}(\mathcal{A})} A = \{1,2,\dots,t-1\}$, i.e. $S_{ij}(\mathcal{A}) \in \tilde{I}(n,k,t)$ for all $1 \le i < j \le n$, $i,j \notin \{t,t+1\}$. We note that $S_{ij}(\mathcal{A}) = \mathcal{A}$ for all i,j with $i \le t-1$, since $\bigcap\limits_{A \in \mathcal{A}} A = \{1,2,\dots,t-1\}$.

Hence (to avoid a new notation), we may assume that $S_{ij}(\mathcal{A}) = \mathcal{A}$ for all $1 \le i < j \le n$, $i,j \notin \{t,t+1\}$ and that

$$A_1 = \{1,2,\dots,t,t+2,\dots,k+1\}, A_2 = \{1,2,\dots,t-1,t+1,\dots,k+1\}.$$

Together with (2.2) this yields

$$\mathcal{B} = \big\{[1,t-1] \cup B : B \subset [t,k+1], |B| = k-t+1\big\} \subset \mathcal{A}.$$

Now we can apply an arbitrary (i,j)–shift , $1 \le i < j \le n$ and $\mathcal{B}$ will not change. Therefore $\Big|\bigcap\limits_{A \in \mathcal{A}} A\Big| < t$ will be maintained.

(ii) Suppose to the opposite that there exists an $\mathcal{A} \in \tilde{I}(n,k,t) \setminus I_0(n,k,t)$, i.e. $1 \leq \left| \bigcap\limits_{A \in \mathcal{A}} A \right| < t$, with $|\mathcal{A}| = \tilde{M}(n,k,t)$. Without loss of generality we can assume $1 \in A$ for all $A \in \mathcal{A}$.

Using the shifting technique described in the proof of (i) we will get an $\mathcal{A}'$ with $\mathcal{A}' \subset L\tilde{I}(n,k,t)$, $|\mathcal{A}'| = |\mathcal{A}| = \tilde{M}(n,k,t)$ and still $1 \in A'$ for all $A' \in \mathcal{A}'$. Now we consider $A'' = \{2,3,\dots,k+1\}$ and show that $|A'' \cap A'| \geq t$ for all $A' \in \mathcal{A}'$, i.e. $\{A''\} \cup \mathcal{A}' \in \tilde{I}(n,k,t)$, which will lead to a contradiction with the maximality of $\mathcal{A}$.

Let us assume that there exists an $A' \in \mathcal{A}'$ for which

$$|A' \cap A''| = |A' \cap [2,k+1]| \leq t-1.$$

Since $\mathcal{A}'$ is stable, we can assume $A' = \{1,2,\dots,t,k+2,\dots,2k-t+1\}$. Moreover, since $\mathcal{A}'$ is stable and $\mathcal{A}' \in \tilde{I}(n,k,t)$, also $A''' = \{1,2,\dots,t-1,t+1,\dots,k+1\} \in \mathcal{A}'$. Now $|A' \cap A'''| = t-1$ contradicts $\mathcal{A}' \in I(n,k,t)$.

3. Generating sets

In this Section we repeat concepts from Section 2 of [4] and restate the simple, but basic properties, expressed in Lemmas 1–5 of [4], again in Lemmas 1–5. Only Lemma 1 has been slightly modified.

Definition 3.1: For any $B \in 2^{[n]}$ we define the upset $\mathcal{U}(B) = \{B' \in 2^{[n]} : B \subset B'\}$. More generally for $\mathcal{B} \subset 2^{[n]}$ we define

$$\mathcal{U}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{U}(B).$$

Definition 3.2: For any $\mathcal{B} \subset \binom{[n]}{k}$ a set $g(\mathcal{B}) \subset \bigcup\limits_{i \leq k} \binom{[n]}{i}$ is called a generating set of $\mathcal{B}$, if $\mathcal{U}\left(g(\mathcal{B}) \cap \binom{[n]}{k}\right) = \mathcal{B}$. Furthermore, $G(\mathcal{B})$ is the set of all generating sets of $\mathcal{B}$. $\big(G(\mathcal{B}) \neq \varnothing$, because $\mathcal{B} \in G(\mathcal{B})\big)$.

Lemma 1. *Let* $\mathcal{A} \in \binom{[n]}{k}$ *and* $n > 2k-t$.

Then $\mathcal{A} \in I(n,k,t)$ *(resp.* $\mathcal{A} \in \tilde{I}(n,k,t)$ *) if and only if* $g(\mathcal{A}) \in I(n,t)$ *(resp.* $g(\mathcal{A}) \in \tilde{I}(n,t)$ *) for every* $g(\mathcal{A}) \in G(\mathcal{A})$.

Next we introduce further basic concepts.

Definition 3.3: For $B = \{b_1, b_2, \dots, b_{|B|}\} \subset [1,n], b_1 < b_2 < \dots < b_{|B|}$, write the biggest element $b_{|B|}$ as $s^+(B)$. Also for $\mathcal{B} \subset 2^{[n]}$ set

$$s^+(\mathcal{B}) = \max_{B \in \mathcal{B}} s^+(B).$$

Definition 3.4: Let $\mathcal{A} \subset \binom{[n]}{k}$ be left compressed, i.e. $\mathcal{A} = \mathcal{L}(\mathcal{A})$. For any generating set $g(\mathcal{A}) \in G(\mathcal{A})$ consider $\mathcal{L}\big(g(\mathcal{A})\big)$ and introduce its set of minimal (in the sense of set–theoretical inclusion) elements $\mathcal{L}_*\big(g(\mathcal{A})\big)$.

Furthermore, define $G_*(\mathcal{A}) = \big\{g(\mathcal{A}) \in G(\mathcal{A}) : \mathcal{L}_*\big(g(\mathcal{A})\big) = g(\mathcal{A})\big\}$. (Notice that $\mathcal{A} \in G_*(\mathcal{A})$).

We continue with simple properties.

Lemma 2. *For a left compressed* $\mathcal{A} \subset \binom{[n]}{k}$ *and any* $g(\mathcal{A}) \in G(\mathcal{A})$

(i) $\mathcal{L}_*\big(g(\mathcal{A})\big) \in G(\mathcal{A})$

(ii) $s^+\big(\mathcal{L}_*\big(g(\mathcal{A})\big) \le s^+\big(g(\mathcal{A})\big)\big)$

(iii) *for* $A \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ *and* $B \prec A$ *we have either* $B \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ *or there exists a* $B' \in \mathcal{L}_*\big(g(\mathcal{A})\big)$ *with* $B' \subset B$.

The next important properties immediately follow from the definition of $G_*(\mathcal{A})$ and the left–compressedness of $\mathcal{A}$.

Lemma 3. *For a left compressed* $\mathcal{A} \subset \binom{[n]}{k}$ *and* $g(\mathcal{A}) \in G_*(\mathcal{A})$ $\mathcal{A}$ *is a disjoint union*

$$\mathcal{A} = \dot{\bigcup_{E \in g(\mathcal{A})}} \mathcal{D}(E),$$

where

$$\mathcal{D}(E) = \left\{ B \in \binom{[n]}{k} : B = E \,\dot{\cup}\, B_1, B_1 \subset \big[s^+(E)+1, n\big], |B_1| = k - |E| \right\}.$$

Lemma 4. *For a left compressed* $\mathcal{A} \subset \binom{[n]}{k}$ *and* $g(\mathcal{A}) \in G_*(\mathcal{A})$ *choose* $E \in g(\mathcal{A})$ *such that* $s^+(E) = s^+\big(g(\mathcal{A})\big)$ *and consider the set of elements of* $\mathcal{A}$ *, which are only generated by* E *, that is,*

$$\mathcal{A}_E = \big(\mathcal{U}(E) \smallsetminus \mathcal{U}\big(g(\mathcal{A}) \smallsetminus \{E\}\big)\big) \cap \binom{[n]}{k}.$$

Then

$$\mathcal{A}_E = \mathcal{D}(E) \quad \textit{and} \quad |\mathcal{A}_E| = \binom{n - s^+(E)}{k - |E|}.$$

Lemma 5. *Let* $\mathcal{A} \in LI(n,k,t)$ *,* $g(\mathcal{A}) \in G_*(\mathcal{A})$ *, and let* $E_1, E_2 \in g(\mathcal{A})$ *have the properties*

$$i \notin E_1 \cup E_2,\ j \in E_1 \cap E_2$$

for some $i, j \in [1,n]$ *with* $i < j$ *. Then*

$$|E_1 \cap E_2| \ge t+1.$$

Finally, we use the following convention.

Definition 3.5: For $\mathcal{A} \in LI(n,k,t)$ we set

$$s_{\min}\big(G(\mathcal{A})\big) = \min_{g(\mathcal{A}) \in G(\mathcal{A})} s^+\big(g(\mathcal{A})\big).$$

4. The main auxiliary results and proof of the Theorem

For given n, k, t , $n > k > t \geq 1$, and every i , $2 \leq i \leq k - t + 1$, we consider the set

$$\mathcal{H}_i = \left\{ H \in \binom{[t+i]}{t+1} : [1,t] \subset H \right\} \cup \left\{ H \in \binom{[t+i]}{t+i-1} : [t+1, t+i] \subset H \right\}.$$

We have

$$\left| \mathcal{U}(\mathcal{H}_i) \cap \binom{[n]}{k} \right| = \binom{n-t}{k-t} - \binom{n-t-i}{k-t} + t \binom{n-t-i}{k-t-i+1}.$$

Let us note that

$$\mathcal{U}(\mathcal{H}_2) \cap \binom{[n]}{k} = \mathcal{V}_1 = \mathcal{F}_1 \qquad \text{(see (1.1), (1.2))}$$

and

$$\mathcal{U}(\mathcal{H}_{k-t+1}) \cap \binom{[n]}{k} = \mathcal{V}_2 \quad \text{(see (1.3))}.$$

Our main auxiliary result, which essentially proves the Theorem, is

Lemma 6. *Let* $n > (t+1)(k-t+1)$, $\mathcal{A} \in L\tilde{I}(n,k,t)$ *with* $|\mathcal{A}| = \tilde{M}(n,k,t)$, *and let* $g(\mathcal{A}) \in G(\mathcal{A})$ *satisfy* $s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big)$, *then*

$$g(\mathcal{A}) = \mathcal{H}_i \quad \textit{for some} \quad i, 2 \leq i \leq k-t+1.$$

Proof: By Lemma 2 we have for some $g(\mathcal{A}) \in G_*(\mathcal{A})$ $\quad s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big) = \ell$, say, and we let $g(\mathcal{A}) = g_0(\mathcal{A}) \,\dot{\cup}\, g_1(\mathcal{A})$, where

$$g_0(\mathcal{A}) = \big\{ B \in g(\mathcal{A}) : s^+(B) = \ell \big\}, g_1(\mathcal{A}) = g(\mathcal{A}) \smallsetminus g_0(\mathcal{A}).$$

It is easy to verify that

$$\ell \geq t+2. \tag{4.1}$$

The elements in $g_0(\mathcal{A})$ have an important property, which follows immediately from Lemma 5:

(P) for any $E_1, E_2 \in g_0(\mathcal{A})$ with $|E_1 \cap E_2| = t$ necessarily $|E_1| + |E_2| = \ell + t$.

Now we consider the cardinality of the intersection of the elements of $g_1(\mathcal{A})$: let

$$\left| \bigcap_{B \in g_1(\mathcal{A})} B \right| = \tau.$$

We distinguish the two cases $\tau < t$ and $\tau \geq t$.

Case $\tau < t$:

In this case we almost repeat the proof of Lemma 6 [4], only some parameters are changed.

We partition $g_0(\mathcal{A})$ according to the cardinalities of its members

$$g_0(\mathcal{A}) = \bigcup_{t<i<\ell} \mathcal{R}_i,\ \mathcal{R}_i = g_0(\mathcal{A}) \cap \binom{[n]}{i}.$$

Of course, some of the $\mathcal{R}_i$–s can be empty.

Let $\mathcal{R}_i' = \big\{E \subset [1, \ell-1] : E \cup \{\ell\} \in \mathcal{R}_i\big\}$.

So $|\mathcal{R}_i| = |\mathcal{R}_i'|$ and for $E' \in \mathcal{R}_i'$ $|E'| = i - 1$.

From property (P) we know that for any $E_i' \in \mathcal{R}_i'$, $E_j' \in \mathcal{R}_j'$ with $i + j \neq \ell + t$

$$|E_1' \cap E_2'| \geq t.$$

We shall prove (under the present conditions $n > (t+1)(k-t+1)$ and $\tau < t$) that all $\mathcal{R}_i$–s are empty, i.e. the case $\tau < t$ is impossible.

If for all i with $\mathcal{R}_i \neq \varnothing$ one has $\mathcal{R}_{\ell+t-i} = \varnothing$, then (by property (P))

$$g' = \big(g(\mathcal{A}) \smallsetminus g_0(\mathcal{A})\big) \cup \left(\bigcup_{t<i<\ell} \mathcal{R}_i' \right) \in \tilde{I}(n,t),$$

$$\left| \mathcal{U}(g') \cap \binom{[n]}{k} \right| \geq |\mathcal{A}| \ \text{ and } \ s^+(g') < s^+\big(g(\mathcal{A})\big),$$

which is a contradiction.

Suppose then that for some i we have $\mathcal{R}_i \neq \varnothing$ and $\mathcal{R}_{\ell+t-i} \neq \varnothing$.

If a) $i \neq \ell + t - i$, *or, equivalently,* $i \neq \frac{\ell+t}{2}$, then we consider the sets

$$\begin{aligned} f_1 &= g_1(\mathcal{A}) \cup \big(g_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_i' \\ f_2 &= g_1(\mathcal{A}) \cup \big(g_0(\mathcal{A}) \smallsetminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_{\ell+t-i}'. \end{aligned}$$

We know (see property (P)) that $f_1, f_2 \in I(n,k)$ and that we are in the case $\tau < t$, that is, $\left| \bigcap_{\mathcal{B} \in g_1(\mathcal{A})} B \right| < t$ and so $f_1, f_2 \in \tilde{I}(n,k)$ as well.

Hence

$$\mathcal{B}_i = \mathcal{U}(f_i) \cap \binom{[n]}{k} \in \tilde{I}(n,k,t) \ \text{ for } \ i = 1, 2.$$

The desired contradiction shall take the form

$$\max_{i=1,2} |\mathcal{B}_i| > |\mathcal{A}|. \tag{4.2}$$

We consider the set $\mathcal{A} \smallsetminus \mathcal{B}_1$.

From the construction of f_1 and the $\mathcal{R}_i'$–s it follows that $\mathcal{A} \smallsetminus \mathcal{B}_1$ consists of those elements of $\binom{[n]}{k}$, which are extensions only of the elements from $\mathcal{R}_{\ell+t-i}$. We determine their number (using Lemma 4)

$$|\mathcal{A} \smallsetminus \mathcal{B}_1| = |\mathcal{R}_{\ell+t-i}| \cdot \binom{n-\ell}{k-\ell-t+i}. \tag{4.3}$$

By similar arguments one gets

$$|\mathcal{B}_1 \smallsetminus \mathcal{A}| \geq |\mathcal{R}_i| \cdot \binom{n-\ell}{k-i+1}. \tag{4.4}$$

Analogously, we have

$$|\mathcal{A} \smallsetminus \mathcal{B}_2| = |\mathcal{R}_i| \cdot \binom{n-\ell}{k-i} \quad \text{and} \tag{4.5}$$

$$|\mathcal{B}_2 \smallsetminus \mathcal{A}| \geq |\mathcal{R}_{\ell+t-i}| \cdot \binom{n-\ell}{k-\ell-t+i+1}. \tag{4.6}$$

Now (4.3) – (4.6) enable us to state the negation of (4.2) in the form

$$\left.\begin{aligned} |\mathcal{R}_i|\binom{n-\ell}{k-i+1} &\leq |\mathcal{R}_{\ell+t-i}|\binom{n-\ell}{k-\ell-t+i}, \\ |\mathcal{R}_{\ell+t-i}|\binom{n-\ell}{k-\ell-t+i+1} &\leq |\mathcal{R}_i|\binom{n-\ell}{k-i}. \end{aligned}\right\} \tag{4.7}$$

Since $|\mathcal{R}_i| \neq \varnothing$, $|\mathcal{R}_{\ell+t-i}| \neq \varnothing$, from (4.7) one has

$$(n-\ell-k+i)(n-k+t-i) \leq (k-i+1)(k-\ell-t+i+1).$$

However this is false, because $n \geq (t+1)(k-t+1)+1 \geq 2k-t+2$ and consequently $n-k+t-i > k-i+1$ as well as $n-\ell-k+i > k-\ell-t+i+1$.

Hence (4.2) holds in contradiction to the optimality of $\mathcal{A}$. Therefore, in the case $\tau < t$ necessarily $\mathcal{R}_i = \varnothing$ for all $i \neq \frac{\ell+t}{2}$.

If b) $i = \frac{\ell+t}{2}$, then we consider $\mathcal{R}'_{\frac{t+\ell}{2}}$ and recall that for $B \in \mathcal{R}'_{\frac{t+\ell}{2}}$

$$|B| = \frac{t+\ell}{2} - 1 \quad \text{and} \quad B \subset [1, \ell-1].$$

By the pigeon–hole principle there exists a $j \in [1, \ell-1]$ and a $\mathcal{T} \subset \mathcal{R}'_{\frac{t+\ell}{2}}$ such that $j \notin B$ for all $B \in \mathcal{T}$ and

$$|\mathcal{T}| \geq \frac{\frac{\ell-t}{2}}{\ell-1}\left|\mathcal{R}'_{\frac{t+\ell}{2}}\right|. \tag{4.8}$$

By Lemma 5 we have $|B_1 \cap B_2| \geq t$ for all $B_1, B_2 \in \mathcal{T}$ and since we are in the case $\tau < t$, that is $\left| \bigcap_{B \in g_1(\mathcal{A})} B \right| < t$, then

$$f' = \left(g(\mathcal{A}) \setminus \mathcal{R}_{\frac{t+\ell}{2}} \right) \cup \mathcal{T} \in \tilde{I}(n,t).$$

We show now that under the condition $n > (t+1)(n-k+1)$ one has

$$\left| \mathcal{U}(f') \cap \binom{[n]}{k} \right| > |\mathcal{A}|. \tag{4.9}$$

Indeed, let us write

$$\mathcal{A} = \mathcal{U}\left(g(\mathcal{A}) \cap \binom{[n]}{k} \right) = \mathcal{D}_1 \dot{\cup} \mathcal{D}_2, \quad \text{where}$$

$$\mathcal{D}_1 = \mathcal{U}\left(g(\mathcal{A}) \setminus \mathcal{R}_{\frac{t+\ell}{2}} \right) \cap \binom{[n]}{k},$$

$$\mathcal{D}_2 = \left(\mathcal{U}\left(\mathcal{R}_{\frac{t+\ell}{2}} \right) \setminus \mathcal{U}\left(g(\mathcal{A}) \setminus \mathcal{R}_{\frac{t+\ell}{2}} \right) \right) \cap \binom{[n]}{k},$$

and

$$\mathcal{U}(f') \cap \binom{[n]}{k} = \mathcal{D}_1 \dot{\cup} \mathcal{D}_3, \quad \text{where}$$

$$\mathcal{D}_3 = \left(\mathcal{U}(\mathcal{T}) \setminus \mathcal{U}\left(g(\mathcal{A}) \setminus \mathcal{R}_{\frac{t+\ell}{2}} \right) \right) \cap \binom{[n]}{k}.$$

In this terminology, equivalent to (4.9) is

$$|\mathcal{D}_3| > |\mathcal{D}_2|. \tag{4.10}$$

We know (see Lemma 4) that

$$|\mathcal{D}_2| = \left| \mathcal{R}_{\frac{t+\ell}{2}} \right| \cdot \binom{n-\ell}{k - \frac{t+\ell}{2}}, \tag{4.11}$$

and it is easy to show that

$$|\mathcal{D}_3| \geq |\mathcal{T}| \binom{n-\ell+1}{k - \frac{t+\ell}{2} + 1}. \tag{4.12}$$

In the light of (4.8), (4.10) – (4.12) sufficient for (4.9) is

$$\frac{\ell - t}{2(\ell-1)} \binom{n-\ell+1}{k - \frac{t+\ell}{2} + 1} > \binom{n-\ell}{k - \frac{t+\ell}{2}}$$

or equivalently

$$(\ell - t)(n - \ell + 1) > 2(\ell - 1)\left(k - \frac{t+\ell}{2} + 1\right)$$

or ($\ell > t+1$ by (4.1))

$$n > \frac{2(\ell-1)(k-t+1)}{\ell - t}.$$

This is true, since $n > (t+1)(k-t+1)$ and

$$(t+1)(k-t+1) \geq \frac{2(\ell-1)(k-t+1)}{\ell-t} \quad \text{for every} \quad \ell \geq t+2.$$

Case $\tau \geq t$:

Since $\mathcal{A}$ is stable, clearly

$$\bigcap_{B \in g_1(\mathcal{A})} B = [1, \tau].$$

We also have

$$\ell = s^+\big(g(\mathcal{A})\big) > \tau.$$

Next we recall the definitions of $g_0(\mathcal{A})$ and $g_1(\mathcal{A})$:

$$g_0(\mathcal{A}) = \big\{B \in g(\mathcal{A}) : s^+(B) = \ell\big\}, g_1(\mathcal{A}) = g(\mathcal{A}) \smallsetminus g_0(\mathcal{A})$$

and observe the following important properties of the elements of $g_0(\mathcal{A})$, which immediately follow from left–compressedness arguments:

for all $B \in g_0(\mathcal{A})$

(P') $|B \cap [1, \tau]| \geq \tau - 1$,

(P") if $|B \cap [1,\tau]| = \tau - 1$, then $[\tau + 1, \ell] \subset B$.

At first let us show that $\tau < t+2$.

Indeed, in the case $\tau \geq t+2$, using Property (P') we have

$$|B_1 \cap B_2 \cap [1, \tau]| \geq \tau - 2 \geq t \quad \text{for all} \quad B_1, B_2 \in g(\mathcal{A})$$

and hence removing from every member of $g_0(\mathcal{A})$ the element ℓ , i.e. obtaining $g_0'(\mathcal{A}) = \big\{B \subset [1, \ell-1] : B \cup \{\ell\} \in g_0(\mathcal{A})\big\}$, we arrive at the generating set

$$f'' = \big(g(\mathcal{A}) \smallsetminus g_0(\mathcal{A})\big) \cup g_0'(\mathcal{A}), \tag{4.13}$$

for which we have

$$f'' \in \tilde{I}(n,k), \left|\mathcal{U}(f'') \cap \binom{[n]}{k}\right| \geq |\mathcal{A}|, \quad \text{but} \quad s^+(f'') < \ell = s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big),$$

a contradiction.

Therefore we have only two possibilities for τ : $\tau = t$ and $\tau = t+1$.

Subcase $\tau = t+1$:

We must have $\ell = t+2$, because otherwise, if $\ell > t+2$, then as in the case directly above we remove the element ℓ from every member of $g_0(\mathcal{A})$ and get the generating set f'' (see (4.13)) for which we know

$$\left|\mathcal{U}(f'') \cap \binom{[n]}{k}\right| \geq |\mathcal{A}|, s^+(f'') < \ell = s^+\big(g(\mathcal{A})\big) = s_{\min}\big(G(\mathcal{A})\big). \tag{4.14}$$

However, using the properties (P') and (P") it is easy to verify, that still $f'' \in \tilde{I}(n,k)$, which contradicts (4.14).

Hence in the case $\tau = t+1$ one has $\ell = t+2$ and clearly $g(\mathcal{A}) = \mathcal{H}_2$.

Subcase $\tau = t$:

Let $g_0'(\mathcal{A}) = \big\{B \in g_0(\mathcal{A}) : |B \cap [1,t]| = t-1\big\}$.

Clearly $g_0'(\mathcal{A}) \neq \varnothing$, because otherwise $g(\mathcal{A}) \notin \tilde{I}(n,t)$.

Since for every $B \in g_0'(\mathcal{A})$ we have $[t+1,\ell] \subset B$ (see property (P")), we conclude (see also (4.1)) that $t+2 \leq \ell \leq k+1$, and that

$$g_0'(\mathcal{A}) \subset \big\{B \subset [1,\ell] : |B \cap [1,t]| = t-1, [t+1,\ell] \subset B\big\}. \tag{4.15}$$

Also, for every $C \in g(\mathcal{A}) \smallsetminus g_0'(\mathcal{A})$ one has $[1,t] \subset C$ and $|C \cap [t+1,\ell]| \geq 1$, and this together with (4.15) shows that

$$\mathcal{U}\big(g(\mathcal{A})\big) \cap \binom{[n]}{k} = \mathcal{A} \subset \mathcal{U}(\mathcal{H}_{\ell-t}) \cap \binom{[n]}{k}.$$

Since $\mathcal{A}$ is maximal, necessarily $g(\mathcal{A}) = \mathcal{H}_{\ell-t}$. The lemma is proved.

Finally we use the abbreviation

$$S_i \triangleq \left|\mathcal{U}(\mathcal{H}_i) \cap \binom{n}{k}\right| \quad \text{for} \quad 2 \leq i \leq k-t+1$$

and establish the following numerical result.

Lemma 7.

(i) $\max\limits_{2\le i\le k-t+1} S_i = \max\{S_2, S_{k-t+1}\}$

(ii) *if* $k \le 2t+1$, *then* $\max\limits_{2\le i\le k-t+1} S_i = S_2$.

Proof: (i) Let us show that $S_i < S_{i+1}$ implies $S_{i+1} < S_{i+2}$. This yields (i). We have to show that

$$S_i = \binom{n-t}{k-t} - \binom{n-t-i}{k-t} + t\binom{n-t-i}{k-t-i+1}$$
$$< \binom{n-t}{k-t} - \binom{n-t-i-1}{k-t} + t\binom{n-t-i-1}{k-t-i} = S_{i+1}$$

implies

$$S_{i+1} < \binom{n-t}{k-t} - \binom{n-t-i-2}{k-t} + t\binom{n-t-i-2}{k-t-i-1} = S_{i+2}$$

or that

$$t\left(\binom{n-t-i}{k-t-i+1} - \binom{n-t-i-1}{k-t-i}\right) < \binom{n-t-i}{k-t} - \binom{n-t-i-1}{k-t}$$

implies

$$t\left(\binom{n-t-i-1}{k-t-i} - \binom{n-t-i-2}{k-t-i-1}\right) < \binom{n-t-i-1}{k-t} - \binom{n-t-i-2}{k-t}$$

or that (by Pascal's identity)

$$t\binom{n-t-i-1}{k-t-i+1} < \binom{n-t-i-1}{k-t-1}$$

implies

$$t\binom{n-t-i-2}{k-t-i} < \binom{n-t-i-2}{k-t-1}$$

or finally that

$$\frac{t\binom{n-t-i-2}{k-t-i}}{k-t-i+1} < \frac{\binom{n-t-i-2}{k-t-1}}{n-k-i}$$

implies

$$t\binom{n-t-i-2}{k-t-i} < \binom{n-t-i-2}{k-t-1},$$

which is true, because $n \ge 2k-t+1$ and consequently

$$\frac{k-t-i+1}{n-k-i} \le 1.$$

(ii) In the light of (i) it is sufficient to show that for $k \le 2t+1$

$$S_2 = \binom{n-t}{k-t} - \binom{n-t-2}{k-t} + t\binom{n-t-2}{k-t-1} > \binom{n-t}{k-t} - \binom{n-k-1}{k-t} + t = S_{k-t+1}. \tag{4.16}$$

Equivalent to (4.16) is

$$t\binom{n-t-2}{k-t-1} > \binom{n-t-2}{k-t} - \binom{n-k-1}{k-t} + t$$

or

$$t \sum_{j=0}^{k-t-1} \binom{k-t-1}{j}\binom{n-k-1}{k-t-1-j} > \sum_{i=1}^{k-t-1} \binom{k-t-1}{i}\binom{n-k-1}{k-t-i} + t$$

or

$$\sum_{j=0}^{k-t-2} \left(t\binom{k-t-1}{j} - \binom{k-t-1}{j+1}\right)\binom{n-k-1}{k-t-1-j} > 0$$

and this is true, because $k \leq 2t+1$ and consequently

$$t\binom{k-t-1}{j} > \binom{k-t-1}{j+1} \quad \text{for every } j > 0, \quad \text{and}$$

$$t = t\binom{k-t-1}{0} \geq \binom{k-t-1}{1} \quad \text{for } j = 0.$$

Proof of the Theorem:

The claimed statement follows from Theorem AK and Lemmas 6, 7.

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