SHADOWS AND ISOPERIMETRY UNDER THE SEQUENCE–SUBSEQUENCE RELATION

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1. Introduction and results

One of the basic results in extremal set theory was discovered in [1] and rediscovered in [2]: For a given number of k–element subsets of an n–set the shadow, that is, the set of $(k-1)$–element subsets contained in at least one of the specified k–element subsets, is minimal, if the k–element subsets are chosen as an initial segment in the squashed order (see [10]; called colex order in [11]), that is, a k–element subset A precedes a k–element subset B, if the largest element in $A \triangle B$ is in B. A closely related result was discovered in [3] and rediscovered in [5]: For a given number $u \in [0, 2^n]$ of arbitrary subsets of an n–set the "Hamming distance 1"–boundary is minimal for the initial segment of size u, also called in short " u–th initial segment", in the H–order (of [3]), that is, if one chooses all subsets of cardinality less than $n-k$ (k suitable) and all remaining subsets of cardinality $n-k$, whose complements are in the initial segment of the squashed order.

In this paper we consider sequences and subsequences rather than sets and subsets.

The basic objects are $\mathcal{X}^n = \prod^n \mathcal{X}$ for $\mathcal{X} = \{0,1\}$ and $n \in \mathbb{N}$, and operations of deletion $\bigtriangledown_i$, $\bigtriangledown$ and of insertion $\triangle_i$, $\triangle$. Here $\bigtriangledown_i$ (resp. $\triangle_i$) means that letter i $(i = 0, 1)$ is deleted (resp. inserted) and $\bigtriangledown$ (resp. $\triangle$) means the deletion (resp. insertion) of any letter.

So for $A \subset \mathcal{X}^n$ we get the *down shadow*

$$\bigtriangledown A = \{x^{n-1} \in \mathcal{X}^{n-1} : x^{n-1} \text{ is subsequence of some } a^n \in A\} \tag{1.1}$$

and the *up shadow*

$$\triangle A = \{x^{n+1} \in \mathcal{X}^{n+1} : \text{ some subsequence of } x^{n+1} \text{ is in } A\}. \tag{1.2}$$

In other words, $\bigtriangledown A$ are all sequences of length $n-1$ obtained by omitting any letter in the sequences of A. Then $\bigtriangledown_i A$ are all those sequences obtained by omitting the letter i. Clearly,

$$\bigtriangledown A = \bigtriangledown_0 A \cup \bigtriangledown_1 A \tag{1.3}$$

and analogously

$$\triangle A = \triangle_0 A \cup \triangle_1 A. \tag{1.4}$$

We describe now our results.

A. Shadows for fixed level and specific letter.

The ℓ–th level is the set of sequences (or words)

$$\mathcal{X}^n_\ell = \left\{ x^n \in \mathcal{X}^n : \sum_{t=1}^n x_t = \ell \right\}. \tag{1.5}$$

We consider sets $B \subset \mathcal{X}^n_{n-k}$ of cardinality v, $0 \le v \le \binom{n}{k}$, and their shadows $\bigtriangledown_0 B$, $\triangle_1 B$ (the other shadows can be estimated similarily by symmetry).

The unique binomial representation of v is

$$v = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s} \tag{1.6}$$

(with $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$).

Whereas Katona used in [2] and also in [5] the function F :

$$F(k,v) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}, \tag{1.7}$$

we introduce and need here the functions $\overset{\triangledown}{F}$ and $\overset{\vartriangle}{F}$, which play the analogue roles for the new shadow problems:

$$\overset{\triangledown}{F}(k,v) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_s - 1}{s-1} \tag{1.8}$$

and

$$\overset{\vartriangle}{F}(k,v) = \binom{a_k + 1}{k} + \binom{a_{k-1} + 1}{k-1} + \cdots + \binom{a_s + 1}{s}. \tag{1.9}$$

Theorem 1. *For all* $B \subset \mathcal{X}^n_{n-k}$ *with* $|B| = v$

(i) $| \triangledown_0 B| \geq \overset{\triangledown}{F}(k,v)$,

(ii) $| \vartriangle_1 B| \geq \overset{\vartriangle}{F}(k,v)$,

and

(iii) *both bounds are optimal.*

B. Shadows of arbitrary sets under deletion of any letter.

For any integer $u \in [0, 2^n]$ we use the unique binomial representation

$$u = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \cdots + \binom{\alpha_t}{t} \tag{1.10}$$

(with $n > \alpha_k > \cdots > \alpha_t \geq t \geq 1$) and observe that for an initial H–order segment S with $|S| = u$

$$\begin{aligned} | \triangledown S| &= \binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_t - 1}{t-1} \\ &= \overset{\triangledown}{G}(n,u), \quad \text{say.} \end{aligned} \tag{1.11}$$

Theorem 2. *For every* $A \subset \mathcal{X}^n$, $|\bigtriangledown A| \geq \overset{\bigtriangledown}{G}(n,|A|)$ *and the bound is achieved by the* $|A|$*–th initial segment in* H*–order .*

This result was first obtained by D.E. Daykin and T.N. Danh [8]. We are grateful to David for his dramatic story about the complexity of their (first) proof. It gave us the impetus to (quickly) find a proof with fairly lengthy calculations with binomial coefficients. Subsequently Daykin/Danh gave also another proof, which can be found in the collection [9]. Then we gave a very "short proof" in [9] based on Lemma 6 of [6] and our inequality (2.5) below. Unfortunately, as was kindly pointed out by David, the original proof of (2.5) has an error in equation (6) of [9].

C. Shadows of arbitrary sets under insertion of any letter.

For u in the representation (1.10) we define

$$\overset{\bigtriangleup}{G}(n,u) = \binom{n+1}{n+1} + \binom{n+1}{n} + \cdots + \binom{n+1}{k+1} + \binom{\alpha_k+1}{k} + \cdots + \binom{\alpha_t+1}{t}. \tag{1.12}$$

Theorem 3. [1] *For every* $A \subset \mathcal{X}^n$, $|\bigtriangleup A| \geq \overset{\bigtriangleup}{G}(n,|A|)$ *, and the bound is achieved by the* $|A|$*–th initial segment in* H*–order .*

Remarks:

1. It must be emphasized that the H–order minimizes *simultaneously both*, the lower and the upper shadows. There is no such phenomenon in the Boolean lattice for "Kruskal/Katona"–type shadows. It has immediate consequences for isoperimetric problems.
2. Theorem 1 can be derived easily from Theorems 2 and 3 like Kruskal–Katona's Theorem from Harper's Theorem.

D. Two isoperimetric inequalities.

It has been emphasized in [7] that isoperimetric inequalities in discrete metric spaces are fundamental principles in combinatorics. The goal is to minimize the union of a specified number of spheres of constant radius. We speak of an isoperimetric inequality, if this minimum is assumed for a set of sphere–centers, which themselves form a sphere (or quasi–sphere, if numbers do not permit a sphere).

For any $A \subset \mathcal{X}^* = \bigcup\limits_{n=1}^{\infty} \mathcal{X}^n$ and any distance d we define (the union of spheres of radius r)

$$\Gamma_d^r(A) = \big\{x^{n'} \in \mathcal{X}^* : d(x^{n'}, a^n) \leq r \text{ for some } a^n \in A\big\}. \tag{1.13}$$

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[1] A referee kindly pointed out to us that the equivalence of Theorem 2 and Theorem 3 can be derived with a theorem in "Variational principle in discrete extremal problems" by Bezrukov (Reihe Informatik Bericht tr–ri–94–152, Universität–GH–Paderborn).

A prototype of a discrete isoperimetric inequality is the one discovered in [3], rediscovered in [5], and proved again in [6]. Here d equals the Hamming distance d_H and is defined on $\mathcal{X}^n \times \mathcal{X}^n$.

We recall the result. For

$$G(n,u) = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k} + \binom{\alpha_k}{k-1} + \cdots + \binom{\alpha_t}{t-1} \tag{1.14}$$

and any $A \subset \mathcal{X}^n$

$$|\Gamma^r_{d_H}(A)| \geq G(n,|A|) \tag{1.15}$$

and the bound is achieved by the $|A|$–th initial segment in H–order (this is a sphere of radius k , if $|A| = \sum\limits_{j=0}^{k} \binom{n}{j}$).

We define now two distances, θ and δ , in $\mathcal{X}^*$. For $x^m, y^{m'} \in \mathcal{X}^*$ $\theta(x^m, y^{m'})$ counts the minimal number of insertions and deletions which transform one word into the other. $\delta(x^m, y^{m'})$ counts the minimal number of operations, if also exchanges of letters are allowed. Thus $\delta(x^m, y^{m'}) \leq \theta(x^m, y^{m'})$.

Now observe that from (1.15) and our Theorems 2 and 3, we get immediately two inequalities.

Corollary 1. *For* $A \subset \mathcal{X}^n$

(i) $\Gamma^1_\theta(A) \leq \overset{\triangledown}{G}(n,|A|) + \overset{\vartriangle}{G}(n,|A|)$,

(ii) $\Gamma^1_\delta(A) \leq \overset{\triangledown}{G}(n,|A|) + \overset{\vartriangle}{G}(n,|A|) + G(n,|A|)$,

and both bounds are achieved by the $|A|$*–th initial segment in* H*–order .*

Moreover, in *Theorem 4 of Section 6* we have established those inequalities for *every radius* r . The exact formulation and the proof require a technical setup.

2. Auxiliary results

A. Numerical inequalities.

While working on [7] Gyula Katona drew attention to the approach of Eckhoff/Wegner [4] to prove Kruskal/Katona via the following inequality for F , defined in (1.7).

Lemma 1 (see [4]): For $k > 1$, $v \leq v_0 + v_1$,

$$F(k,v) \leq \max\bigl(v_0, F(k,v_1)\bigr) + F(k-1,v_0). \tag{2.1}$$

In fact, he used this idea also in his proof of the isoperimetric inequality for the Hamming space. He just had to establish the corresponding inequality for G , defined in (1.14).

Lemma 2 (Lemma 6 of [6]): If $0 \leq u_1 \leq u_0$ and $u \leq u_0 + u_1$, then

$$G(n,u) \leq \max\big(u_0, G(n-1,u_1)\big) + G(n-1,u_0). \tag{2.2}$$

The discoveries in the present paper are similar inequalities for $\overset{\triangledown}{F}$, $\overset{\vartriangle}{F}$, $\overset{\triangledown}{G}$, and $\overset{\vartriangle}{G}$ (defined in (1.8), (1.9), (1.11), and (1.12)), which for cardinalities of shadows resp. boundaries considered describe their values for segments in the H–order .

We state first the inequalities for F . They are proved in the same way as those for G below.

$\overset{\triangledown}{F}$**–inequality:** For $k > 1$, if $v \leq v_0 + v_1$ and $v_0 < \overset{\triangledown}{F}(k,v)$, then

$$\overset{\triangledown}{F}(k,v) \leq \overset{\triangledown}{F}(k,v_1) + \overset{\triangledown}{F}(k-1,v_0). \tag{2.3}$$

$\overset{\vartriangle}{F}$**–inequality:** For $k > 1$, if $v \leq v_0 + v_1$, then

$$\overset{\vartriangle}{F}(k,v) \leq \max\big(v_0+v_1, \overset{\vartriangle}{F}(k,v_1)\big) + \overset{\vartriangle}{F}(k-1,v_0). \tag{2.4}$$

Next we derive the inequalities for G .

$\overset{\triangledown}{G}$**–inequality:** If $w_1 \leq w_0 < \overset{\triangledown}{G}(n,w)$ and $w \leq w_0 + w_1$, then

$$\overset{\triangledown}{G}(n,w) \leq \overset{\triangledown}{G}(n-1,w_0) + \overset{\triangledown}{G}(n-1,w_1). \tag{2.5}$$

$\overset{\vartriangle}{G}$**–inequality:** If $0 \leq u_1 \leq u_0$, $u \leq u_0 + u_1$, then

$$\overset{\vartriangle}{G}(n,u) \leq \max\big(u_0+u_1, \overset{\vartriangle}{G}(n-1,u_1)\big) + \overset{\vartriangle}{G}(n-1,u_0). \tag{2.6}$$

Proofs: From the definitions of the numerical functions we have

$$G(n,u) + u = \overset{\vartriangle}{G}(n,u) \quad \text{for } u \text{ as in (1.10)}$$

and the equivalence of (2.2) and (2.6) immediately follows.

Next we show (2.5). For u as in (1.10) denote by $\ell_n(u)$ and $r_n(u)$ the smallest j with $\alpha_j > j$ and the number of i's with $\alpha_i = i$, respectively.

Let

$$\overline{u}(n-1) \triangleq u - \overset{\triangledown}{G}(n,u)$$
$$= \binom{n-1}{n-1} + \cdots + \binom{n-1}{k+1} + \binom{\alpha_k - 1}{k} + \cdots + \binom{\alpha_{\ell_n(u)} - 1}{\ell_n(u)}. \tag{2.7}$$

By (1.11) and (1.14)

$$\overset{\triangledown}{G}(n,u) = \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_{\ell_n(u)} - 1}{\ell_n(u) - 1} + r_n(u)$$
$$= G\big(n-1, \overline{u}(n-1)\big) + r_n(u). \tag{2.8}$$

Moreover by the binomial coefficient representation

$$u+1 = \begin{cases} \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \cdots + \binom{\ell_n(u)}{\ell_n(u)-1} & \text{if } \ \alpha_t = t = 1 \\ \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \cdots + \binom{\alpha_t}{t} + \binom{t-1}{t-1} & \text{otherwise} \end{cases} \tag{2.9}$$

(1.11) implies

$$\overset{\triangledown}{G}(n,u+1) = \begin{cases} \overset{\triangledown}{G}(n,u) & \text{if } \ \alpha_t = t = 1 \\ \overset{\triangledown}{G}(n,u) + 1 & \text{otherwise.} \end{cases} \tag{2.10}$$

By the definition of binomial coefficient representation, $\overline{u}(n-1)$ in (2.7) is non–decreasing in u for fixed n (c.f. (2.9)).

For w_1 , w_0 and w with

$$w_1 \le w_0 < \overset{\triangledown}{G}(n,u) \tag{2.11}$$

and

$$w \le w_0 + w_1, \tag{2.12}$$

we let $w^* = w$, if $r_n(w) = 0$, and otherwise

$$w^* = \binom{n}{n} + \cdots + \binom{n}{k'+1} + \binom{\beta_{k'}}{k'} + \cdots + \binom{\beta_{\ell_n(w)}}{\ell_n(w)} + \binom{\ell_n(w)}{\ell_n(w)-1}$$
$$= w - r_n(w) + \ell_n(w), \tag{2.13}$$

if the representation of w is

$$w = \binom{n}{n} + \cdots + \binom{n}{k'+1} + \binom{\beta_{k'}}{k'} + \cdots + \binom{\beta_s}{s}$$
$$= \binom{n}{n} + \cdots + \binom{n}{k'+1} + \binom{\beta_{k'}}{k'} + \cdots + \binom{\beta_{\ell_n(w)}}{\ell_n(w)} + \binom{\ell_n(w)-1}{\ell_n(w)-1} + \cdots + \binom{\ell_n(w)-r_n(w)}{\ell_n(w)-r_n(w)}.$$

Write

$$w_0^* = w_0 + (w^* - w) \quad \text{and} \quad w_1^* = w_1. \tag{2.14}$$

Then by the definitions of w^* , w_0^* , w_1^* , and (2.10) (used repeatedly),

$$r_n(w^*) = 0, \tag{2.15}$$

$$\overset{\triangledown}{G}(n, w^*) = \overset{\triangledown}{G}(n, w) + (w^* - w) - 1, \quad \text{if} \quad w^* \neq w, \tag{2.16}$$

and

$$\overset{\triangledown}{G}(n-1, w_0^*) \leq \overset{\triangledown}{G}(n-1, w_0) + (w^* - w) - \tau(w_0^*), \tag{2.17}$$

where $\tau(w_0^*) = 1$, if $r_{n-1}(w_0^*) = 0$ and $w_0^* \neq w_0$, and $\tau(w_0^*) = 0$ otherwise. So, by (2.11), (2.14), and (2.16)

$$w_1^* \leq w_0^* \leq \overset{\triangledown}{G}(n, w^*), \tag{2.18}$$

which with (2.7), (2.8) and (2.15) yields

$$\begin{aligned} \overline{w}_0^*(n-2) + G\big(n-2, \overline{w}_0^*(n-2)\big) &= w_0^* - \overset{\triangledown}{G}(n-1, w_0^*) + G\big(n-2, \overline{w}_0^*(n-2)\big) \\ &= w_0^* - r_{n-1}(w_0^*) \leq w_0^* \leq \overset{\triangledown}{G}(n, w^*) \\ &= G\big(n-1, \overline{w}^*(n-1)\big). \end{aligned} \tag{2.19}$$

Moreover, by the first inequality in (2.18) and the monotonicity of $\overline{u}(n-1)$ (as a function of u),

$$\overline{w}_1^*(n-2) \leq \overline{w}_0^*(n-2). \tag{2.20}$$

Now we assume that (2.5) does not hold and derive a contradiction. With (2.12) we obtain

$$w - \overset{\triangledown}{G}(n, w) < w_0 - \overset{\triangledown}{G}(n-1, w_0) + w_1 - \overset{\triangledown}{G}(n-1, w_1). \tag{2.21}$$

When $w^* \neq w$ then by (2.7) and (2.16) the LHS of (2.22) is $w - \overset{\triangledown}{G}(n, w^*) + (w^* - w) - 1 = \overline{w}^*(n-1) - 1$ and by (2.7), (2.14) and (2.17) the RHS of (2.22) is not bigger than $w_0 - \overset{\triangledown}{G}(n-1, w_0^*) + (w^* - w) - \tau(w_0^*) + \overline{w}_1^*(n-2) \leq \overline{w}_0^*(n-2) + \overline{w}_1^*(n-2)$.

Thus we have

$$\overline{w}^*(n-1) \leq \overline{w}_0^*(n-2) + \overline{w}_1^*(n-2). \tag{2.22}$$

By our notation in (2.7), (2.22) certainly implies (2.23), when $w^* = w$ (so $w_0^* = w_0$).

Finally, with (2.19), (2.20), and (2.23) we obtain from Lemma 2,

$$G\big(n-1, \overline{w}^*(n-1)\big) \leq G\big(n-2, \overline{w}_0^*(n-2)\big) + G\big(n-2, \overline{w}_1^*(n-2)\big). \tag{2.23}$$

This implies (2.5) (a contradiction to our assumption), because by (2.8), (2.15), and (2.16) the LHS of (2.23) is

$$\overset{\triangledown}{G}(n,w^*) = \begin{cases} \overset{\triangledown}{G}(n,w) + w^* - w - 1 & \text{if } \ w \neq w^*, \\ \overset{\triangledown}{G}(n,w) & \text{if } \ w = w^* \ (\text{note } r_n(w) = 0) \end{cases}$$

and by (2.8), (2.14), and (2.17) the RHS of (2.23) is

$$\begin{aligned} &\overset{\triangledown}{G}(n-1,w_0^*) + \overset{\triangledown}{G}(n-1,w_1^*) - \big(r_{n-1}(w_0^*) + r_{n-1}(w_1^*)\big) \\ &\leq \overset{\triangledown}{G}(n-1,w_0) + \overset{\triangledown}{G}(n-1,w_1) + w^* - w - \tau(w_0^*) - \big(r_{n-1}(w_0^*) + r_{n-1}(w_1^*)\big) \\ &\leq \overset{\triangledown}{G}(n-1,w_0) + \overset{\triangledown}{G}(n-1,w_1) + \begin{cases} w^* - w - 1 & \text{if } \ w \neq w^* \\ 0 & \text{if } \ w = w^*, \end{cases} \end{aligned}$$

and otherwise (2.5) trivially follows from (2.8), (2.23), and $r_n(w_0) = 0$.

□

B. A calculus of iterative applications for $\overset{\triangledown}{G}$, $\overset{\vartriangle}{G}$, and G .

We present here a rather technical result (Lemma 4 below), which is needed only for the proof of Theorem 4. Recall that for u , $1 \leq u \leq 2^n$,

$$u = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{\alpha_k}{k} + \binom{\alpha_{k-1}}{k-1} + \cdots + \binom{\alpha_t}{t},$$

$$\overset{\triangledown}{G}(n,u) = \binom{n-1}{n-1} + \cdots + \binom{n-1}{k+1} + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_t - 1}{t-1},$$

$$G(n,u) = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{n}{k} + \binom{\alpha_k}{k-1} + \cdots + \binom{\alpha_t}{t-1},$$

and

$$\overset{\vartriangle}{G}(n,u) = \binom{n+1}{n+1} + \cdots + \binom{n+1}{k+1} + \binom{\alpha_k + 1}{k} + \cdots + \binom{\alpha_t + 1}{t}.$$

All these functions are increasing in u and they transform binomial representations into binomial representations. This makes it easy to apply them repeatedly.

We notice that the representation of $\overset{\triangledown}{G}(n,u)$ may be not unique, due to the appearance of the term $\binom{0}{0}$. However, it causes no difficulties to apply the functions, because both representations (if they exist) always give the same result, when $\overset{\triangledown}{G}$, G or $\overset{\vartriangle}{G}$ are applied.

More specifically, the non–uniqueness happens only when $\alpha_t = t = 1$ in (1.10), and with the notation $\ell_n(u) = \ell$ (say) in the proof of (2.5),

$$\begin{aligned}\overset{\triangledown}{G}(n,u) &= \binom{n-1}{n-1} + \cdots + \binom{n-1}{k+1} + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_\ell - 1}{\ell - 1} \\ &\quad + \binom{\ell-2}{\ell-2} + \cdots + \binom{1}{1} + \binom{0}{0} \\ &= \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{\alpha_k - 1}{k-1} + \cdots + \binom{\alpha_\ell - 1}{\ell - 1} + \binom{\ell-1}{\ell-2} \triangleq v \text{ say.}\end{aligned}$$

For the first representation of v

$$\overset{\triangledown}{G}(n-1,v) = \binom{n-2}{n-2} + \cdots + \binom{n-2}{k-1} + \binom{\alpha_k - 2}{k-2} + \cdots + \binom{\alpha_\ell - 2}{\ell - 2} + \binom{\ell-3}{\ell-3} + \cdots + \binom{0}{0},$$

$$G(n-1,v) = \binom{n-1}{n-1} + \cdots + \binom{n-1}{k-1} + \binom{\alpha_k - 1}{k-2} + \cdots + \binom{\alpha_\ell - 1}{\ell - 2} + \binom{\ell-2}{\ell-3} + \cdots + \binom{1}{0},$$

and

$$\overset{\triangle}{G}(n-1,v) = \binom{n}{n} + \cdots + \binom{n}{k} + \binom{\alpha_k}{k-1} + \cdots + \binom{\alpha_\ell}{\ell - 1} + \binom{\ell-1}{\ell-2} + \cdots + \binom{2}{1} + \binom{1}{0},$$

and for the second representation of v ,

$$\overset{\triangledown}{G}(n-1,v) = \binom{n-2}{n-2} + \cdots + \binom{n-2}{k-1} + \binom{\alpha_k - 2}{k-2} + \cdots + \binom{\alpha_\ell - 2}{\ell - 2} + \binom{\ell-2}{\ell-3},$$

$$G(n-1,v) = \binom{n-1}{n-1} + \cdots + \binom{n-1}{k-1} + \binom{\alpha_k - 1}{k-2} + \cdots + \binom{\alpha_\ell - 1}{\ell - 2} + \binom{\ell-1}{\ell-3},$$

and

$$\overset{\triangle}{G}(n-1,v) = \binom{n}{n} + \cdots + \binom{n}{k} + \binom{\alpha_k}{k-1} + \cdots + \binom{\alpha_\ell}{\ell - 1} + \binom{\ell}{\ell-2}.$$

They really have the same values.

For two functions ϕ , $\psi : \mathbb{N} \to \mathbb{N}$ we write $\psi\big(\phi(\cdot)\big)$ as $\psi \circ \phi(\cdot)$ and thus we can define

$$\overset{\triangledown}{G}{}^{\circ p}(n,\cdot) = \overset{\triangledown}{G}(n-p+1,\cdot) \circ \overset{\triangledown}{G}(n-p+2,\cdot) \circ \cdots \circ \overset{\triangledown}{G}(n,\cdot), \tag{2.24}$$

$$G^{\circ q}(n,\cdot) = G(n,\cdot) \circ G(n,\cdot) \circ \cdots \circ G(n,\cdot), \tag{2.25}$$

and

$$\overset{\triangle}{G}{}^{\circ s}(n,\cdot) = \overset{\triangle}{G}(n+s-1,\cdot) \circ \overset{\triangle}{G}(n+s-2,\cdot) \circ \cdots \circ \overset{\triangle}{G}(n,\cdot) \tag{2.26}$$

with p , q , and s factors, respectively.

We can also define $\overset{\triangledown}{G}{}^{\circ p}(n+s,\cdot) \circ \overset{\triangle}{G}{}^{\circ s}(n,\cdot)$, $G^{\circ q} \circ \overset{\triangle}{G}{}^{\circ p}$ etc.

Directly from the definitions the functions in (2.25) – (2.27) can be calculated.

Lemma 3. *With the convention* $\binom{k}{\ell} = 0$ *for* $\ell < 0$

$$\overset{\triangledown}{G}{}^{\circ p}(n,u) = \binom{n-p}{n-p} + \cdots + \binom{n-p}{k+1-p} + \binom{\alpha_k - p}{k-p} + \cdots + \binom{\alpha_t - p}{t-p}, \tag{2.27}$$

$$G^{\circ q}(n,u) = \binom{n}{n} + \cdots + \binom{n}{k+1-q} + \binom{\alpha_k}{k-q} + \cdots + \binom{\alpha_t}{t-q}, \tag{2.28}$$

and

$$\overset{\triangle}{G}{}^{\circ s}(n,u) = \binom{n+s}{n+s} + \cdots + \binom{n+s}{k+1} + \binom{\alpha_k + s}{k} + \cdots + \binom{\alpha_t + s}{t}. \tag{2.29}$$

Here (2.29) is well–known from the isoperimetric theorem in the Hamming space.
Another important property of G–type functions is the commutativity of the $\circ$–operation :

$$\overset{\triangledown}{G}\circ G(n,u) = G\circ\overset{\triangledown}{G}(n,u) = \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{n-1}{k-1} + \binom{\alpha_k - 1}{k-2} + \cdots + \binom{\alpha_t - 1}{t-2}, \tag{2.30}$$

$$\overset{\triangledown}{G}\circ\overset{\triangle}{G}(n,u) = \overset{\triangle}{G}\circ\overset{\triangledown}{G}(n,u) = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{n}{k} + \binom{\alpha_k}{k-1} + \cdots + \binom{\alpha_t - 1}{t-1}, \tag{2.31}$$

and

$$G\circ\overset{\triangle}{G}(n,u) = \overset{\triangle}{G}\circ G(n,u) = \binom{n+1}{n+1} + \cdots + \binom{n+1}{k+1} + \binom{n+1}{k} + \binom{\alpha_k + 1}{k-1} + \cdots + \binom{\alpha_t - 1}{t-1}. \tag{2.32}$$

Applying (2.28) – (2.30) and (2.31) – (2.33) repeatedly or by calculation we establish general rules.

Lemma 4. *We have*

$$\begin{aligned}
&\overset{\triangledown}{G}{}^{\circ p} \circ G^{\circ q} \circ \overset{\triangle}{G}{}^{\circ s}(n,u) = \overset{\triangledown}{G}{}^{\circ p} \circ \overset{\triangle}{G}{}^{\circ s} \circ G^{\circ q}(n,u) \\
&= G^{\circ q} \circ \overset{\triangledown}{G}{}^{\circ p} \circ \overset{\triangle}{G}{}^{\circ s}(n,u) = G^{\circ q} \circ \overset{\triangle}{G}{}^{\circ s} \circ \overset{\triangledown}{G}{}^{\circ p}(n,u) \\
&= \overset{\triangle}{G}{}^{\circ s} \circ \overset{\triangledown}{G}{}^{\circ p} \circ G^{\circ q}(n,u) = \overset{\triangle}{G}{}^{\circ s} \circ G^{\circ q} \circ \overset{\triangledown}{G}{}^{\circ r}(n,u) \\
&= \binom{n+s-p}{n+s-p} + \binom{n+s-p}{n+s-p-1} + \cdots + \binom{n+s-p}{k+1-p-q} \\
&\quad + \binom{\alpha_k + s - p}{k-p-q} + \cdots + \binom{\alpha_t + s - p}{t-p-q}
\end{aligned} \tag{2.33}$$

for u *as in (1.10),* $0 \le p, q, s$ *.*

3. Proof of Theorem 1

Denote an initial segment in squashed order (see [10]) over $\mathcal{X}_k^n$ by S and write $\overline{S}$ for the set of complements of the members of S. Thus $\overline{S} \subset \mathcal{X}_{n-k}^n$ and $|\overline{S}| = |S| = v$, say. We speak here about the complementary squashed order or in short about the CS–order.

We consider first $\nabla_0 \overline{S}$ and $\nabla_1 \overline{S}$.

Lemma 5. *For the initial segment* $\overline{S}$ *defined above*

(i) $\nabla_0 \overline{S}$ *is the* $\overset{\triangle}{F}(k,v)$*–th initial segment in the CS–order on* $\mathcal{X}_{n-k}^{n-1}$

and

(ii) $\triangle_1 \overline{S}$ *is the* $\overset{\triangle}{F}(k,v)$*–th initial segment in the CS–order on* $\mathcal{X}_{n+1-k}^{n+1}$.

Proof: (i) We use the expansion (1.6) for v and look at any $s^n \in \overline{S}$:

$$s_{t_i} = 0 \text{ for } i = 1,2,\dots,k \text{ and } 1 \le t_1 < t_2 < \dots < t_k \le n.$$

By the definition of the CS–order there must be a j such that for all $i \in (j,k]$ $t_i = a_i + 1$ and for all $i \le j$ $t_i \le a_j$. Now suppose that we delete for some index ℓ s_{t_ℓ}. We can assume that $s_{t_\ell - 1} = 1$, because otherwise we can delete $s_{t_{\ell-1}}$ and get the same subsequence. Let $s'^{\,n-1}$ be the resulting subsequence, $t'_i = t_i$ for $i < \ell$ and $t'_{i-1} = t_i$ for $i > \ell$.

Choose now $j' = \max(\ell, j)$ and notice that for $i \le j' - 1$, $t'_i \le a_{j'} - 1$, for $i > j' - 1$ $t'_{i-1} = a_i = (a_i - 1) + 1$, and for all i $s'_{t'_i} = 0$.

Therefore the resulting subsequence $s'^{\,n-1}$ falls into the $\overset{\triangledown}{F}(k,v)$–th initial segment in CS–order.

Conversely, given a sequence $s'^{\,n-1}$ in $\mathcal{X}_{n-k}^{n-1}$ and in the $\overset{\triangledown}{F}(k,v)$–th initial segment the forgoing argument provides a way to find an s^n in the v–th initial segment from which $s'^{\,n-1}$ is obtainable by deleting a 0.

(ii) Use again the s^n described above and let $s''^{\,n+1}$ be obtained by inserting a 1 before $s_{t_{\ell''}}$, $t''_i = t_i$ for $i < \ell''$ and $t''_i = t_i + 1$ for $i \ge \ell''$.

Then $s''_{t''_i} = 0$ for all i and for $i \le j''$ $t''_i \le a_i + 1$; for $i > j''$ $t''_i = a_i + 2 = (a_i + 1) + 1$, if we choose $j'' = \max(j, \ell - 1)$.

Clearly, such an $s''^{\,n-1}$ is in the $\overset{\triangle}{F}(k,v)$–th initial segment in the CS–order. The same argument gives also the reverse implication.

$\square$

Proof of Theorem 1 (i) and (ii) by induction on n:

The cases $n = 1,2$ are done by simple inspection. For any ℓ, m, j, $C \subset \mathcal{X}^\ell$, $D \subset \mathcal{X}^m$, and $E \subset \mathcal{X}^j$ let

$$C_i = \{(c_1, \dots, c_\ell) : (c_1, \dots, c_\ell, i) \in C\}(\subset \mathcal{X}^{\ell-1}), \tag{3.1}$$

$$D * i = \{(d_1, \dots, d_m, i) : (d_1, \dots, d_m) \in D\}(\subset \mathcal{X}^{m+1}), \tag{3.2}$$

and

$$\hat{E}_i = \{(e_1, \dots, e_j) : e_j = i \text{ and } (e_1, \dots, e_j) \in E\}(\subset \mathcal{X}^j) \tag{3.3}$$

for $i = 0, 1$.

(i) for $n > 2$ **:**

Since $B_0 \subset \nabla_0 B, (\nabla_0 B_i) * i \subset \nabla_0 B (i = 0, 1)$ and $(\nabla_0 B_0) * 0 \cap (\nabla_0 B_1) * 1 = \varnothing$, either $|\nabla_0 B| \geq |B_0| \geq \overset{\nabla}{F}(k, |B|)$ or by (2.3) and induction hypothesis (IH) $|\nabla_0 B| \geq |\nabla_0 B_0| + |\nabla_0 B_1| \overset{(*)}{\geq} \overset{\nabla}{F}(k-1, |B_0|) + \overset{\nabla}{F}(k, |B_1|) \geq \overset{\nabla}{F}(k, |B|)$, where $(*)$ is justified by $B_0 \subset \mathcal{X}^{n-1}_{n-k}$, and $B_1 \subset \mathcal{X}^{n-1}_{n-k-1}$.

(ii) for $n > 2$ **:**

Recall the definition of the operator " $\wedge$ " in (3.3).

Considering $\triangle_1 B = (\widehat{\triangle_1 B_1})_1 \cup (\widehat{\triangle_1 B})_0$, $(\widehat{\triangle_1 B})_0 = (\triangle_1 B_0) * 0$, $B * 1 \subset (\widehat{\triangle_1 B})_1$ and $(\triangle_1 B_1) * 1 \subset (\widehat{\triangle_1 B})_1$, by (2.4) and IH,

$$|\triangle_1 B| \geq \max(|B|, |\triangle_1 B_1|) + |\triangle_1 B_0| \geq \max\big(|B|, \overset{\triangle}{F}(k, |B_1|)\big) + \overset{\triangle}{F}(k-1, |B_0|) \geq \overset{\triangle}{F}(k, |B|).$$

(iii) follows by Lemma 5.

□

4. Proof of Theorem 2

Lemma 6. *For the initial segment* S *in the* H*–order* $\triangle S$ *equals the* $\overset{\triangle}{G}(n, |S|)$*–th initial segment in the* H*–order , and* ∇S *equals the* $\overset{\nabla}{G}(n, |S|)$*–th initial segment in* H*–order .*

Proof: By the definitions of the two orders and direct inspection, we first get, that for some k and m , and the m–th initial segment S' (of level $n-k$) in the CS–order

$$S = \left(\bigcup_{\ell=0}^{n-(k+1)} \mathcal{X}^n_\ell \right) \cup S' , \tag{5.1}$$

$$\triangle S = \left(\bigcup_{\ell=0}^{n-k} \mathcal{X}^{n+1}_\ell \right) \cup \triangle_1 S' , \tag{5.2}$$

and

$$\bigtriangledown S = \left(\bigcup_{\ell=0}^{n-(k+1)} \mathcal{X}^{n-1} \right) \cup \bigtriangledown_0 S' . \tag{5.3}$$

The rest of the proof follows from Lemma 5.

Proof of Theorem 2 by induction on n **:** For $n = 2$ the statement is readily verified. From the IH for $n-1$ we proceed to n .

Next observe that, by convention (3.1) and (3.2), $\bigcup\limits_{i=0}^{1} (\bigtriangledown A_i) * i \subset \bigtriangledown A$, $\bigcap\limits_{i=0}^{1} (\bigtriangledown A_i) * i = \varnothing$ and that therefore
$|\bigtriangledown A| \geq \sum\limits_{i=0}^{1} |\bigtriangledown A_i| \geq \sum\limits_{i=0}^{1} \overset{\bigtriangledown}{G}(n-1, |A_i|)$ (by the IH).

According to the $\bigtriangledown$–inequality this can be lower bounded with the desired $\overset{\bigtriangledown}{G}(n, |A|)$, if $|A_0|, |A_1| < \overset{\bigtriangledown}{G}(n, |A|)$. Otherwise we have for some i $|A_i| = \max(|A_0|, |A_1|) \geq \overset{\bigtriangledown}{G}(n, |A|)$ and we are done again, because $\bigtriangledown A \supset A_i$.

The achievability follows from Lemma 6.

□

5. Proof of Theorem 3

The proof goes in exactly the same way as the proof of Theorem 1, (ii) (and the " $\triangle_1$ " part of (iii)), except that here we use (2.6), Lemma 6 and the observations: $\triangle A = (\widehat{\triangle A})_1 \cup (\widehat{\triangle A})_0$, $(\triangle A_i) * i \subset (\widehat{\triangle A})_i$ and $A * i \subset (\widehat{\triangle A})_i$ (for $i = 0, 1$).

6. General isoperimetric theorems

We use now the calculus of iterative applications of $\overset{\bigtriangledown}{G}$, $\overset{\triangle}{G}$, and G described in Section 2 B.

Fortunately our Theorems 2, 3 and Theorem H ([3]) establish the *Inheritance property* for the operations $\bigtriangledown$, $\triangle$, and $\Gamma^1_{d_H}$ (recall definition (1.13)). In the sequel, we abbreviate $\Gamma^1_{d_H}$ as Γ_{d_H} and as Γ . If S is an initial segment in H–order , then so are $\bigtriangledown S$, $\triangle S$, and $\Gamma_{d_H} S$. This enables us to apply these theorems repeatedly. Formally, we introduce

$$\bigtriangledown^\ell A = \bigtriangledown\big(\bigtriangledown \dots \bigtriangledown (\bigtriangledown A) \dots\big), \tag{6.1}$$

$$\triangle^\ell A = \triangle\big(\triangle \dots \triangle (\triangle A) \dots\big), \tag{6.2}$$

and

$$\Gamma^\ell_{d_H} A = \Gamma\big(\Gamma \dots \Gamma(\Gamma A)\dots\big) = \big\{x^n \in \mathcal{X}^n : d_H(x^n, a^n) \le \ell \ \text{ for some } \ a^n \in A\big\} \tag{6.3}$$

and state the results.

Proposition 1. For every $A \subset \mathcal{X}^n$, $|A| = u$

(i) $|\bigtriangledown^\ell A| \ge \overset{\bigtriangledown}{G}{}^{\circ\ell}(n,u)$

(ii) $|\bigtriangleup^\ell A| \ge \overset{\bigtriangleup}{G}{}^{\circ\ell}(n,u)$

(iii) $|\Gamma^\ell_{d_H} A| \ge G^{\circ\ell}(n,u)$

and all these bounds are achieved by the u–th initial segment in H–order .

Now we turn to the distances θ and δ in order to generalize Corollary 1. Here operations are combined and the commutative law for the numerical functions (Lemma 4 in Section 2) is needed.

Fortunately this commutative law holds also for the operations $\bigtriangledown$, $\bigtriangleup$, and Γ ! Indeed, using the short notation

$$\bigtriangledown\{x^n\} = \bigtriangledown x^n,\ \bigtriangleup\{x^n\} = \bigtriangleup x^n,\ \Gamma\{x^n\} = \Gamma x^n,$$

we see that

$$\bigtriangledown\{\bigtriangleup x^n\} = \bigtriangleup\{\bigtriangledown x^n\},\ \Gamma\{\bigtriangleup x^n\} = \bigtriangleup\{\Gamma x^n\},\ \bigtriangledown\{\Gamma x^n\} = \Gamma\{\bigtriangleup x^n\}. \tag{6.4}$$

Therefore the commutative law holds for every $A \subset \mathcal{X}^n$:

$$\bigtriangledown(\bigtriangleup A) = \bigtriangleup(\bigtriangledown A),\ \Gamma(\bigtriangleup A) = \bigtriangleup(\Gamma A),\ \bigtriangledown(\Gamma A) = \Gamma(\bigtriangledown A). \tag{6.5}$$

Moreover, it is clear that for every $A \subset \mathcal{X}^n$

$$\Gamma^\ell A \subset \bigtriangledown^\ell(\bigtriangleup^\ell A) = \bigtriangleup^\ell(\bigtriangledown^\ell A) \ \text{ for } \ \ell \le n. \tag{6.6}$$

Here strict inclusion can occur:

$$\Gamma(1,0) = \big\{(0,0),(1,0),(1,1)\big\} \ne \mathcal{X}^2 = \bigtriangledown\big(\bigtriangleup(1,0)\big). \tag{6.7}$$

However, strict inclusion does not occur, if S is an initial segment in H–order .

Proposition 2. If S is an initial segment in H–order , $|S| = u$, then

(i) $|\bigtriangleup^\ell(\bigtriangledown^\ell S)| = |\bigtriangledown^\ell(\bigtriangleup^\ell S)| = \overset{\bigtriangleup}{G}{}^{\circ\ell} \circ \overset{\bigtriangledown}{G}{}^{\circ\ell}(n,u) = G^{\circ\ell}(n,u) = |\Gamma^\ell S|$

and

(ii) $\triangle^\ell(\triangledown^\ell S) = \triangledown^\ell(\triangle^\ell S) = \Gamma^\ell S$.

Proof: For (i) the first equalities are justified by (6.6) and Proposition 1 and the last equality is (the easy) part of Theorem H. The remaining equality follows from Lemma 4 with the choices $p = s = \ell$, $q = 0$ and $p = s = 0$, $q = \ell$, respectively: both quantities equal $\binom{n}{n} + \cdots + \binom{n}{k+1-\ell} + \binom{\alpha_k}{k-\ell} + \cdots + \binom{\alpha_t}{t-\ell}$. Notice that (i) and (6.6) imply (ii).

$\square$

Now we consider arbitrary sets $A \subset \mathcal{X}^n$ and the distances θ, δ .

Proposition 3. For any $A \subset \mathcal{X}^n$, $r > 0$ and any ℓ_i, ℓ_i' $(i = 1, 2)$ with $\ell_2 - \ell_1 = \ell_2' - \ell_1'$ and $\ell_2 < \ell_2'$

(i) $\triangledown^{\ell_2}(\triangle^{\ell_1} A) \subset \triangledown^{\ell_2'}(\triangle^{\ell_1'} A)$

and

(ii) $\Gamma_\theta^r A = \bigcup_{\ell=-r}^r \triangledown^{\lfloor (r+\ell)/2 \rfloor}(\triangle^{\lfloor (r-\ell)/2 \rfloor} A) = \bigcup\limits_{\ell=0}^{r-1} \big(\triangledown^\ell(\triangle^{r-\ell} A)\big) \cup \big(\triangledown^\ell(\triangle^{r-1-\ell} A)\big) \cup \triangledown^r A,$

where by convention $\triangle^0 A = \triangledown^0 A = A$.

Proof: Obviously, for all ℓ ,

$$A \subset \triangledown^\ell(\triangle^\ell A) \tag{6.8}$$

and therefore by the commutative law (6.5)

$$\triangledown^{\ell_2}(\triangle^{\ell_1} A) \subset \triangledown^{\ell_2}\big(\triangle^{\ell_1}\big(\triangledown^{\ell_2'-\ell_2}(\triangle^{\ell_2'-\ell_2} A)\big)\big) = \triangledown^{\ell_2}\big(\triangle^{\ell_1}\big(\triangledown^{\ell_2'-\ell_2}(\triangle^{\ell_1'-\ell_1} A)\big) = \triangledown^{\ell_2'}(\triangle^{\ell_1'} A)\big),$$

and thus (i) is verified.

Again by (6.5) and the definition of distance θ

$$\Gamma_\theta^r A = \bigcup_{r_1 + r_2 \le r} \big(\triangledown^{r_2}(\triangle^{r_1} A)\big). \tag{6.9}$$

Thus by (i) and (6.9)

$$\Gamma_\theta^r = \bigcup_{\ell=-r}^{r} \bigcup_{\substack{r_1 + r_2 \le r \\ r_2 - r_1 = \ell}} \big(\triangledown^{r_2}(\triangle^{r_1} A)\big) = \bigcup_{\ell=-r}^{r} \big(\triangledown^{\lfloor (r+\ell)/2 \rfloor}(\triangle^{\lfloor (r-\ell)/2 \rfloor} A)\big)$$

$$= \bigcup_{\ell=0}^{r-1} \big(\triangledown^\ell(\triangle^{r-\ell} A)\big) \cup \big(\triangledown^\ell(\triangle^{r-1-\ell} A)\big) \cup \triangledown^r A.$$

$\square$

We are now ready to state and prove the main result.

Theorem 4. *For all* $A \subset \mathcal{X}^n$ *and* $r \geq 0$

(i) $|\Gamma_\theta^r A| \geq \sum_{\ell=-r}^{r} \overset{\bigtriangledown}{G}{}^{\circ\lfloor\frac{r+\ell}{2}\rfloor} \circ \overset{\bigtriangleup}{G}{}^{\lfloor\frac{r-\ell}{2}\rfloor}(n,|A|)$

and

(ii) $|\Gamma_\delta^r A| \geq \sum_{\ell=1}^{r} \overset{\bigtriangledown}{G}{}^{\circ\ell} \circ G^{\circ(r-\ell)}(n,|A|) + \overset{\bigtriangleup}{G}{}^{\circ\ell} \circ G^{\circ(r-\ell)}(n,|A|) + G^{\circ r}(n,|A|)$,

where $G^{\circ 0}(n,u) = u$ *, and both bounds are achieved by the* $|A|$*–th initial segment in* H*–order .*

Proof: By our definitions for $0 \leq \ell_i \quad (i=1,2)$ and $n - \ell_2 + \ell_1 \geq 0$

$$\bigtriangledown^{\ell_2}\big(\bigtriangleup^{\ell_1}(\Gamma^{\ell_0} A)\big) \subset \mathcal{X}^{n-\ell_2+\ell_1}. \tag{6.10}$$

(Here Γ^{ℓ_0} is only used for proving (ii).)

Therefore also

$$\bigtriangledown^{\ell_2}\big(\bigtriangleup^{\ell_1}(\Gamma^{\ell_0} A)\big) \cap \bigtriangledown^{\ell_2'}\big(\bigtriangleup^{\ell_1'}(\Gamma^{\ell_0'} A)\big) = \varnothing, \quad \text{if } \ \ell_1 - \ell_2 \neq \ell_1' - \ell_2' \tag{6.11}$$

and (i) as well as its optimality immediately follows from Proposition 3, (6.11) and Proposition 1 (applied twice).

(ii) Similarly to (6.9), we have also

$$\Gamma_\delta^r A = \bigcup_{r_1+r_2+r_0 \leq r} \big(\bigtriangledown^{r_2}\big(\bigtriangleup^{r_1}(\Gamma_{d_H}^{r_0} A)\big)\big) \tag{6.12}$$

and therefore

$$\Gamma_\delta^r A \supset \bigcup_{\ell=1}^{r} \big(\bigtriangledown^{\ell}(\Gamma_{d_H}^{r-\ell} A)\big) \cup \big(\bigtriangleup^{\ell}(\Gamma_{d_H}^{r-\ell} A)\big) \cup \Gamma_{d_H}^{r} A. \tag{6.13}$$

Hence (ii) follows from (6.11), (6.13) and Proposition 1 (applied twice).

Finally, we have to show that the $|A|$–th initial segment in H–order S achieves equality.

By Proposition 2 (ii), Proposition 3 (i), (6.12) and (6.11), and by the monotonicity of $\bigtriangleup^t$, $\bigtriangledown^t$, $\Gamma_{d_H}^t$ in the sets it suffices to show that for all parameters $-r \leq \ell \leq r$, $\ell_1 + \ell_2 + \ell_0 = r$, $\ell_2 - \ell_1 = \ell$, and $\ell_i \geq 0$ for $i = 0,1,2$

$$\bigtriangledown^{\ell_2}\big(\bigtriangleup^{\ell_1}(\Gamma^{\ell_0} S)\big) \subset \begin{cases} \bigtriangledown^{\ell}(\Gamma^{r-\ell} S), & \text{if } \ \ell_2 > \ell_1 \\ \bigtriangleup^{|\ell|}(\Gamma^{r-|\ell|} S), & \text{if } \ \ell_2 < \ell_1 \\ \Gamma^r S, & \text{if } \ \ell_2 = \ell_1. \end{cases}$$

Let us abbreviate $\bigtriangledown^{\ell_2}\big(\bigtriangleup^{\ell_1}(\Gamma^{\ell_0} S)\big) = L$.

Using Proposition 2 (ii) and Proposition 3 (i) we show the desired inclusions.

Case $\ell_2 > \ell_1$ **:**

$$\begin{aligned} L &= \bigtriangledown^{\ell_2-\ell_1}\big(\bigtriangledown^{\ell_1}\big(\triangle^{\ell_1}(\Gamma^{\ell_0}S)\big)\big) \\ &= \bigtriangledown^{\ell}\big(\Gamma^{\ell_1}(\Gamma^{\ell_0}S)\big) = \bigtriangledown^{\ell}(\Gamma^{\ell_1+\ell_0}S) = \bigtriangledown^{\ell}(\Gamma^{r-\ell_2}S) \\ &\subset \bigtriangledown^{\ell}(\Gamma^{r-\ell}S) \quad (\text{as} \quad \ell_2 > \ell_1 \geq 0, r-\ell_2 \leq r-\ell). \end{aligned}$$

Case $\ell_2 < \ell_1$ **:**

$$\begin{aligned} L &= \triangle^{\ell_1-\ell_2}\big(\triangle^{\ell_2}\big(\bigtriangledown^{\ell_2}(\Gamma^{\ell_0}S)\big)\big) \\ &= \triangle^{|\ell|}(\Gamma^{\ell_0+\ell_2}S) = \triangle^{|\ell|}(\Gamma^{r-\ell_1}S) \\ &\subset \triangle^{|\ell|}(\Gamma^{r-|\ell|}S) \quad (\text{as} \quad 0 \leq \ell_2 < \ell_1, r-\ell_1 \leq r+\ell = r-|\ell|), \end{aligned}$$

Case $\ell_2 = \ell_1$ **:**

$$L = \bigtriangledown^{\ell_1}\big(\triangle^{\ell_1}(\Gamma^{\ell_0}S)\big) = \Gamma^{\ell_0+\ell_1}S \subset \Gamma^r S.$$

□

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