A SPLITTING PROPERTY OF MAXIMAL ANTICHAINS

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Abstract

In every *dense* poset $\mathcal{P}$ every maximal antichain S may be partitioned into disjoint subsets S_1 and S_2, such that the union of the upset of S_1 with the downset of S_2 yields the entire poset: $\mathcal{U}(S_1) \cup \mathcal{D}(S_2) = \mathcal{P}$. To find a similar splitting of maximal antichains in posets is NP–hard in general.

1. Introduction

Let $\mathcal{P} = (P, <p)$ be a finite poset and let H be a subset of P. The *downset* $\mathcal{D}(H)$ of the subset H is

$$\mathcal{D}(H) := \{x \in P : \exists s \in H(x \leq s)\}.$$

The *upset* of H is

$$\mathcal{U}(H) = \{x \in P : \exists s \in H(s \leq x)\}.$$

We introduce also the sets

$$\mathcal{D}^*(H) = \{x \in P : \exists s \in H(x < s)\}$$

and

$$\mathcal{U}^*(H) = \{x \in P : \exists s \in H(s < x)\}.$$

A subset $S \subset P$ is called *antichain* or *Sperner system* if no two elements of S are comparable. An antichain S is *maximal* if for every antichain $S' \subset P$, $S \subset S'$ implies $S = S'$. It is easy to see that an antichain S is maximal iff

$$\mathcal{U}(S) \cup \mathcal{D}(S) = P. \tag{1}$$

Let X be an n–element set and let S be the set of its k–element subsets. Furthermore let $S_1 \subset S$ be the family of all k–element subsets containing a fixed element of X, and let $S_2 = S = S \setminus S_1$. Then it is easy to check that $\mathcal{D}(S_1) \cup \mathcal{U}(S_2) = 2^X$. In fact a similar *splitting* can be done for every maximal antichain of 2^X. The purpose of this note is to show this claim and to generalize it to a wide class of posets. The structure of this paper is as follows: Section 2 describes maximal antichains in general posets which satisfy the splitting property. Section 3 determines a large class of posets where every maximal antichain possesses such splittings. Section 4 gives examples for posets which satisfy the splitting property, and some examples which don't. Finally Section 5 shows that to find a suitable splitting of a maximal antichain in posets is NP–hard in general.

It is very convenient to use the following notation:

For $H, G \subset P$ we write $H >|< G$ iff for all $h \in H$ and all $s \in G$ h and s are incomparable. For $s, s' \in \mathcal{P}$ and $G \subset P$ we also write $s >|< s'$ instead of $\{s\} >|< \{s'\}$ and $s >|< G$ instead of $\{s\} >|< G$.

2. Dense sets in posets

In this Section we describe a class of maximal antichains which satisfy the property described above.

Definition. Let $\mathcal{P} = (P, < p)$ be a finite partially ordered set (poset). The subset $H \subset P$ is called *dense* in the poset $\mathcal{P}$ if any non–empty *open interval* $\langle x, y\rangle = \{z \in P : x <_p z <_p y\}$ which contains at least one element of H also contains at least one other element of it.

Definition. We say that the maximal antichain $S \subset P$ satisfies the *splitting property* if there exists a partition $\{S_1, S_2\}$ of S such that

$$\mathcal{U}(S_1) \cup \mathcal{D}(S_2) = P \tag{2}$$

holds. We also say that the poset $\mathcal{P}$ satisfies the splitting property if every one of its maximal antichains satisfies it.

It is easy to find a poset $\mathcal{P}$ and a maximal antichain S in $\mathcal{P}$ such that S has no splitting. Indeed if $\mathcal{P}$ has a (non–maximal and non–minimal) element s which is comparable to any other element of P then $S = \{s\}$ is a maximal antichain and doesn't satisfy the splitting property. On the other hand we give two different proofs for the following statement:

Theorem 2.1. *Let S be a maximal, dense antichain in the finite poset $\mathcal{P}$. Then S satisfies the splitting property.*

First proof. We proceed by induction on the cardinality of S.

Base case. The maximal antichain $S = \{s\}$ satisfies $s < x$ or $x < s$ for all $x \in P \setminus S$. It cannot occur that $x < s < y$ for any $x, y \in P$, because otherwise by the denseness of S $|\langle x, y\rangle \cap S| \geq 2$ in contradiction to $|S| = 1$. Therefore we have either a maximal element 1 or a minimal element 0 equal to s and the splitting property holds.

General case. Let $\mathcal{P} = (P, < p)$ be a finite poset with a maximal dense antichain S and assume that for every finite poset and for every maximal antichain with fewer elements than S we already know the statement.

Assume first that for some $s \in S$ neither

$$\mathcal{D}^*(s) \subset \mathcal{D}^*\big(S \setminus \{s\}\big) \tag{3}$$

nor

$$\mathcal{U}^*(s) \subset \mathcal{U}^*\big(S \setminus \{s\}\big) \tag{4}$$

holds, that is, for some $x \in \mathcal{D}^*(s)$ and some $y \in \mathcal{U}^*(s)$

$$\{x, y\} >\!\!|\!< S \setminus \{s\}, \tag{5}$$

because S is antichain. Since $s \in \langle x, y\rangle$, by the denseness of S there is an $s' \in \langle x, y\rangle \cap \bigl(S \smallsetminus \{s\}\bigr)$ and this contradicts (5).

We know therefore that for every $s \in S$ (3) or (4) holds. By symmerty we can assume that (3) holds for an arbitrary, but fixed, s. Then

Claim 2.2. The antichain $S' = S \smallsetminus \{s\}$ is maximal and dense in the poset $\mathcal{P}' = \bigl(P \smallsetminus \mathcal{U}(s), <_p\bigr)$.

Proof. By (3) and the maximality of S every element $x \in P' = P \smallsetminus \mathcal{U}(s)$ is comparable with some element of S'. So S' is maximal in $\mathcal{P}'$. Furthermore, for any $x, y \in P'$ in obvious notation

$$\langle x, y\rangle_{\mathcal{P}'} = \langle x, y\rangle_{\mathcal{P}} , \tag{6}$$

because otherwise there exists a $z \in \langle x, y\rangle_{\mathcal{P}} \cap \mathcal{U}(s)$ and $s \le z \le y$, $y \in \mathcal{U}(s)$ contradicts $y \in P'$. Equation (6) implies that S' is dense in the poset $\mathcal{P}'$.

By our induction hypothesis we have a partition $\{S'_1, S'_2\}$ of S' with $\mathcal{D}(S'_1) \cup \mathcal{U}(S'_2) = P'$ and hence

$$\mathcal{D}(S'_1) \cup \mathcal{U}\bigl(S'_2 \cup \{s\}\bigr) = P.$$

Second proof. Let S be a dense maximal antichain in the poset $\mathcal{P}$ and let $<_{\mathrm{ord}}$ be an arbitrary linear ordering of S. For every element $x \in P \setminus S$, $x \in \mathcal{D}^*(S)$ let $f(x)$ be the greatest element $s \in S$ (with respect to the ordering $<_{\mathrm{ord}}$) such that $x <_p s$. If $x \in \mathcal{U}^*(S)$ then let $f(x)$ be the smallest element $s \in S$ (with respect to the linear ordering $<_{\mathrm{ord}}$) such that $s <_p x$. Finally, define

$$S_1 = \bigl\{f(x) : x \in \mathcal{D}^*(S)\bigr\} \quad \text{and} \quad S_2 = \bigl\{f(x) : x \in \mathcal{U}^*(S)\bigr\}.$$

Clearly $\bigl\{S_1, S_2 \cup (S \smallsetminus S_1)\bigr\}$ satisfies the splitting condition (2), if $S_1 \cap S_2 = \varnothing$.

Suppose to the opposite that $s \in S_1 \cap S_2$. Then for some $x \in \mathcal{D}^*(S)$, $y \in \mathcal{U}^*(S)$ $f(x) = f(x) = s$. In this case $s \in \langle x, y\rangle$ and by the denseness of S there is an $s' \in S \cap \langle x, y\rangle$ with $s' \ne s$. The linear ordering gives us an order between s and s'. If, e.g., $s <_{\mathrm{ord}} s'$ then by our definition of f $f(x)$ cannot be s, a contradiction. The other case $s' <_{\mathrm{ord}} s$ leads to a contradiction with $f(y) = s$.

3. Dense posets

In this Section a wide class of posets will be described where every maximal antichain possesses the splitting property.

Definition. The poset $\mathcal{P}$ is called *(weakly) dense* if every non–empty *open intervall* $\langle x, y\rangle = \{x <_p z <_p y\}$ contains at least two elements.

Definition. The poset $\mathcal{P}$ is called *(strongly) dense* if for every non–empty intervall $\langle x, y\rangle$ and for every element z of this intervall there exists another element z' of the intervall such that $z >\!\!|\!\!< z'$.

It is easy to show that for finite posets the two notions of denseness are equivalent. However, for infinite posets the two notions are not necessarily equal. For example the totally ordered set of the rational numbers is weakly dense but not strongly dense. When dealing with finite posets it is therefore sufficient to speak about *dense* posets.

Theorem 3.1. *For a finite and dense poset $\mathcal{P}$ every maximal antichain S has the splitting property.*

We give again two proofs. The first one is a slightly changed version of the first proof of Theorem 2.1. The second one is based on a reduction to Theorem 2.1.

First proof. We proceed by induction on the cardinality of S.

Base case. The maximal antichain $S = \{s\}$ satisfies $s < x$ or $x < s$ for all $x \in P \setminus S$. It cannot occur that $x < s < y$ for any $x, y \in P$ because otherwise by (strong) denseness there is a $z \in \langle x, y \rangle$ with $z >\!\!|\!\!< s$ and this contradicts the maximality of S. Therefore we have either a maximal element 1 or a minimal element 0 equal to s and the splitting property holds.

General case. Let $\mathcal{P} = (P, <_p)$ be a finite dense poset and let S be a maximal antichain in $\mathcal{P}$. Assume that for every finite poset and for every maximal antichain with fewer elements than S we already know the statement.

Asssume first (again) that for some $s \in S$ (5) holds. Since $s \in \langle x, y \rangle$, by the (strong) denseness of $\mathcal{P}$ there is a $z \in \langle x, y \rangle$ with $z >\!\!|\!\!< s$. However, even $z >\!\!|\!\!< s$, because $z > s'$ (or $z < s'$) for some $s' \in S$ would imply $s' < y$ (or $x < s'$) in contradiction to (5). Finally, $z >\!\!|\!\!< s$ contradicts the maximality of S.

We are (again) left with the case where by symmetry (3) holds for an arbitrary, but fixed, s.

Claim 3.2. The poset $\mathcal{P}' = (P \setminus \mathcal{U}(s), <_p)$ is dense and the antichain $S' = S \setminus \{s\}$ is maximal in it.

Proof. By (3) and the maximality of S every element $x \in P' = P \setminus \mathcal{U}(s)$ is comparable with some element of S'. So S' is maximal in $\mathcal{P}'$.

Furthermore, for any $x, y \in P'$ (again) (6) holds and this implies that $\mathcal{P}'$ is dense, because $\mathcal{P}$ is dense.

By our induction hypothesis we have a partition $\{S_1', S_2'\}$ of S' with $\mathcal{D}(S_1') \cup \mathcal{U}(S_2') = P'$ and hence

$$\mathcal{D}(S_1') \cup \mathcal{U}\big(S_2' \cup \{s\}\big) = P.$$

Second proof. We are going to construct a subposet $\mathcal{P}'$ in $\mathcal{P}$ where the antichain S is dense and we show that any good splitting of S in $\mathcal{P}'$ is also suitable for the original poset $\mathcal{P}$.

Let D contain the maximum elements of $\mathcal{D}^*(S)$ in $\mathcal{P}$ and U contain the minimum elements of $\mathcal{U}^*(S)$ in $\mathcal{P}$. Take the subposet $\mathcal{P}'$ where $P' = S \cup D \cup U$ and $<_{\mathcal{P}'}$ is the induced order. It is obvious that a suitable splitting of S in $\mathcal{P}'$ supplies a good splitting in $\mathcal{P}$ as well.

Claim 3.3. The antichain S is maximal and dense in poset $\mathcal{P}'$.

Proof. The maximality of S is obvious. For an intervall $\langle x,y\rangle_{\mathcal{P}'}$ in $\mathcal{P}'$ with $\langle x,y\rangle_{\mathcal{P}'}\cap S\neq\varnothing$ nesessarily $x\in D$ and $y\in U$. But then by the definitions of D and U we have actually $\langle x,y\rangle_{\mathcal{P}}\subset S$. Since $S\subset P'$ we even have $\langle x,y\rangle_{\mathcal{P}'}=\langle x,y\rangle_{\mathcal{P}}\subset S$. By the denseness of $\mathcal{P}$ for $s\in\langle x,y\rangle_{\mathcal{P}'}\cap S$ there exists a $z\neq s$ with $z\in\langle x,y\rangle_{\mathcal{P}'}\cap S$. This shows the denseness of S in $\mathcal{P}'$.

The application of Theorem 2.1 completes the proof.

4. Examples and a conjecture

In this section we furnish some posets which satisfy the splitting property and some which do not.

At first we remark that Theorem 2.1 and Theorem 3.1 are different. On the ond hand there exists a dense poset $\mathcal{P}$ with a maximal antichain S , which is not dense there. To show this, let $P=\{a,b,c,d,e,f,g\}$ and let the covering relations be $a<c<f$, $a<d<f$, $b<d<g$ and finally $b<e<g$. This poset is dense, but the maximal antichain $\{a,e\}$ is not dense in $\mathcal{P}$. (Observe that $S_1=\{e\}$ and $S_2=\{a\}$ is a good splitting.) On the other hand choose $P=S\cup C\subset 2^X$ with the induced order, where S is the set of all k–element subsets of X and C is any chain of length three disjoint from S . Clearly, S is a dense antichain and P is not dense.

There are a lot of well–known posets which satisfy the splitting property. For example:

Corollary 4.1. *Every finite geometric lattice (specially every Boolean lattice) satisfies the splitting property.*

Proof. Because every geometric lattice is relatively complemented, it is dense as well (see e.g. [1] Section II.3). Theorem 3.1 gives the result.

Furthermore we remark that the notion of 'denseness' is not necessary for the splitting property to hold. The following lattice is not dense, no maximal antichain is dense in the poset, either, yet it satisfies the splitting property. The symbol $<$ means (again) 'covers' in the poset:

Example 4.2. *[Faigle, [2]] The poset* $\{a<b<c<d<e; a<f<d; b<g<e\}$ *is not dense, and no (non–trivial) maximal antichain is dense in the poset, however, the poset satisfies the splitting property.*

On the other hand, there are distributive lattices which do not satisfy the splitting property. For example:

Example 4.3. *In the distributive lattice given by Aigner [1, p. 34 Figure 2.2 L], the maximal antichain formed by the second level–elements does not satisfy the splitting property.*

It is interesting to remark, that the splitting property of an antichain bears no relationship to the LYM property or even to the Sperner property of the poset. Indeed, in the disjoint union of some r–chains , which is a LYM poset, no maximal Sperner family has good splitting. On the other hand, if the ranked posets $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the splitting property, but the length of the posets are different, then their disjoint union also satisfies the splitting property, but it is not a Sperner poset.

5. Splitting in general is NP–hard

References

1. M. Aigner, “Combinatorial Theory”, Springer Verlag, New York, 1979.
2. U. Faigle, personal communication, 1993.