A FRACTAL FUNDAMENTAL DOMAIN WITH 12-FOLD SYMMETRY

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ABSTRACT. Square triangle tilings are relevant models for quasicrystals. We introduce a new self-similar tile-substitution which yields the well-known non-periodic square triangle tilings of Schlottmann. It is shown that the new tilings are locally derivable from Schlottmann's, but not vice versa, and that they are mutually locally derivable with the undecorated square triangle tilings. Furthermore, the role of the window (acceptance domain) for these tilings as a fundamental domain of the hexagonal lattice is discussed.

1. SQUARE TRIANGLE SUBSTITUTIONS

Nonperiodic tilings, like Penrose tilings, are important models for physical quasicrystals. Besides the three 3-dimensional tilings with icosahedral symmetry (see for instance [1] and references therein) the most relevant models for physical quasicrystals are those planar tilings with 5-fold (resp. 10-fold), 8-fold and 12-fold symmetry. The celebrated Penrose tilings show statistical 10-fold symmetry (compare [4]), and there are two particular Penrose tilings showing global 5-fold dihedral symmetry [6]. The most prominent tilings with 8-fold symmetry are certainly the Ammann-Beenker tilings, see Ammann's P4 in [6]. These two (families of) tilings use two different building blocks (*prototiles*) only, and both of them can be generated by a tile-substitution. A tile-substitution is given by a set of prototiles $T_1, \ldots, T_m$, a substitution factor λ, and a rule how to replace the enlarged prototiles λT_i with congruent copies of the prototiles. For an example, see Figure 1; see also [5] for further details, precise definitions and a wealth of examples. A tile-substitution is

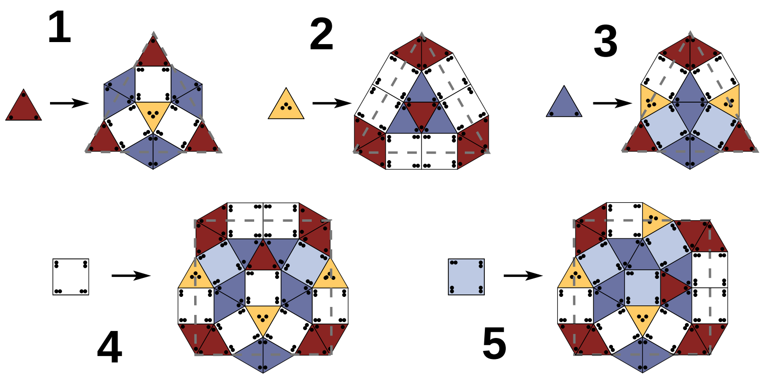

FIGURE 1. A substitution rule by Martin Schlottmann for a square triangle tiling.

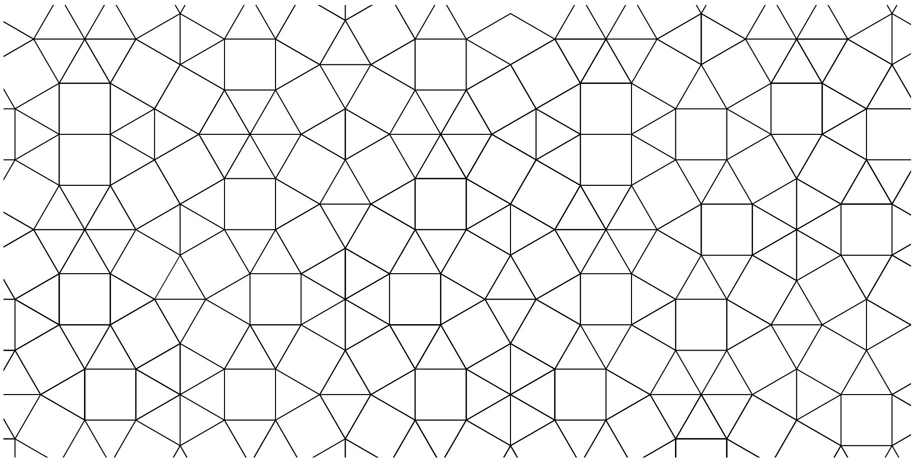

FIGURE 2. A patch of a square-triangle tiling, generated by the tile-substitution shown in Figure 1 (after deleting colour and decorations).

self-similar, if the enlarged prototiles are not only replaced by copies of $T_1, \dots, T_m$, but if the enlarged tiles λT_i can be dissected into copies of $T_1, \dots, T_m$. Figure 4 shows an example of a self-similar tile-substitution.

Whereas the 10-fold and 8-fold symmetric examples above can be generated by a substitution rule with two prototiles, the situation is more different for 12-fold symmetric tilings. There are nice examples of substitution tilings with three prototiles which are 12-fold symmetric, for instance Socolar's square rhomb hexagon tilings, or Gähler's shield tilings (see for instance [5]). On the other hand, there are several 12-fold symmetric tilings by squares and triangles, but they cannot be generated by a simple substitution rule. The first substitution rule yielding square triangle tilings with 12-fold symmetry known to the author aware of was found by Schlottmann and is published for instance in [7]. This tile-substitution is shown in Figure 1. Note that it is not self-similar: for instance, a triangle is not replaced by a triangular patch of tiles (a *patch* is any finite collection of tiles), but by a slightly larger patch. Nevertheless, the construction ensures that the overlapping parts coincide exactly. Schlottmann's substitution does not work with one kind of triangle and one kind of square. He introduces three different kinds of triangles, and two different kinds of squares, which are distinguished by colour and markings in Figure 1. Thus in fact we have five prototiles in this case.

Iterating the substitution rule for the marked tiles, one can produce arbitrary large tilings by marked squares and triangles. After deleting the markings one obtains a plain square triangle tiling as in Figure 2. In this way, one obtains a quasiperiodic tiling with statistical 12-fold symmetry. (There is no precise definition of 'quasiperiodic' agreed on today. For a discussion of possible definitions of quasiperiodicity, compare [1].) The unmarked square triangle tiling and the square triangle tiling with markings are *mutually locally derivable* (mld), which means: one can be obtained from the other by local rules. In particular, the following construction can serve as such a rule: In the square triangle without markings, the 'supertiles' can be found in a unique way, see Figure 3. Comparison with the rule in Figure 1 then

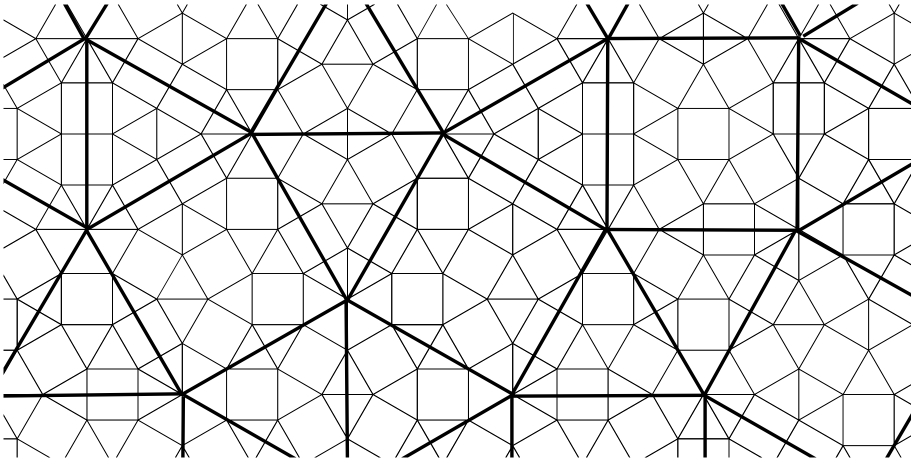

FIGURE 3. The same patch as in Figure 2, with the 'supertiles' indicated.

yields the markings of each individual tile, depending on the type of supertile which contains it. These considerations yield the following result.

Theorem 1. *Schlottmann's square triangle tilings are mld with the plain square triangle tilings in Figure 2.*

It is easy to see that there can't be a square triangle tiling with perfect 12-fold symmetry: there is no local constellation with 12-fold symmetry. The best one can hope for is a tiling with 12-fold symmetry apart from some finite part in its centre. However, by a general result in [4], all tilings considered in the present paper show statistical 12-fold symmetry. Roughly spoken, this means that each orientation occurs with the same frequency throughout the tiling.

There are some reasons for requiring the substitution rule to be self-similar. For instance, the frequencies and the areas of the prototiles can be read off from the substitution matrix in this case. (In fact, to obtain the frequencies, one needs only that the rule is non-overlapping.) Here we present for the first time a substitution rule producing Schlottmann's square triangle tilings which is self-similar. This rule is shown in Figure 4.

In order to achieve self-similarity, one has to divide each equilateral triangle in Schlottmann's rule into two right-angled triangles, and each square in Schlottmann's rule into four smaller squares. Triangles of type 1 (resp. 2) in Figure 1 are divided into two triangles of type 1 (resp. 2) in Figure 4. Triangles of type 3 in Figure 1 can be divided into two right-angled triangles in two different ways: Either into two triangles of type 3 in Figure 4, or into one triangle of type 4 and one of type 7 in Figure 4 (regarding whether it is divided along its vertical mirror axis, or along another axis).

Squares of type 4 in Figure 1 are divided into two squares of type 5 and two squares of type 6 in Figure 4. Squares of type 5 in Figure 1 are divided into one square of type 5, one of type 6, one of type 8 and one of type 9 in Figure 4. (Figure 5 gives an idea how the smaller tiles fit together to yield the larger squares and triangles.)

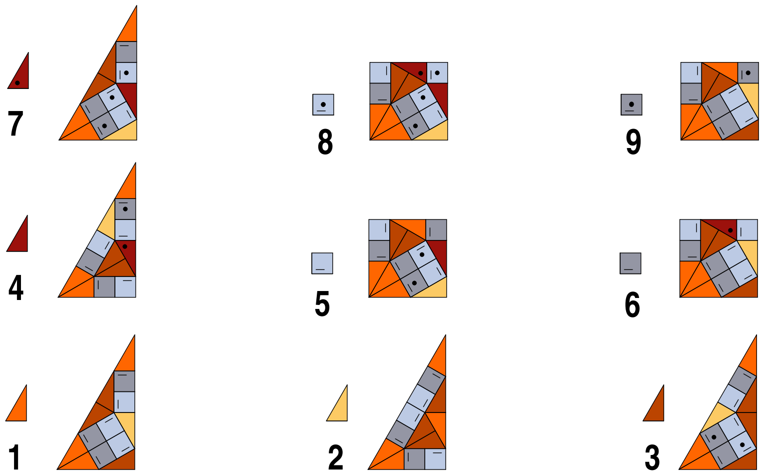

FIGURE 4. A self-similar tile-substitution rule for a square triangle tiling.

Theorem 2. *The self-similar substitution in Figure 4 yields the same plain square triangle tilings as Schlottmann's substitution in Figure 1.*

This is achieved just by gluing four pairwise adjacent squares into one square, gluing together two right-angled triangles along the longer cathetus, and delete all decorations (colours and points).
Nevertheless, let us emphasise that Schlottmann's square triangle tilings and our square triangle tilings are not mld.

Theorem 3. *Schlottmann's square triangle tilings are locally derivable from our square triangle tilings, but not vice versa.*

By Theorem 1, Theorem 3 implies Theorem 2. One part — Schlottmann's tilings are locally derivable from our's — is clear from the construction.

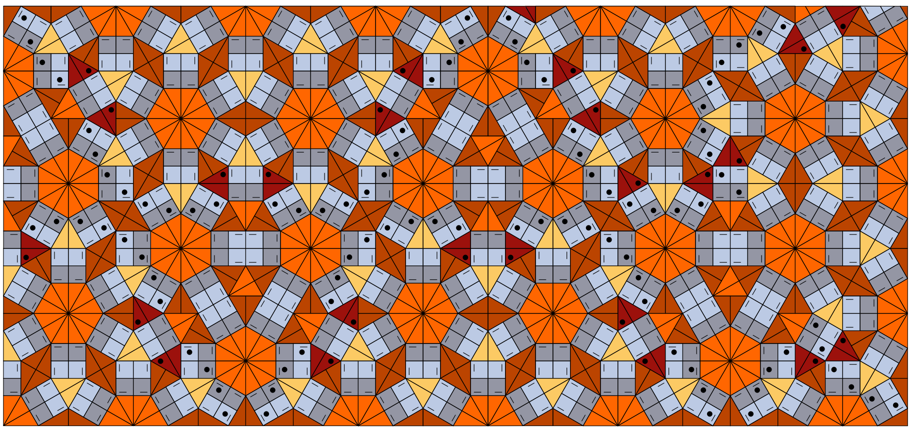

FIGURE 5. A part of a tiling generated by the rule in Figure 4.

Let us phrase the other directions as follows: Our square triangle tilings just fail slightly to be mutually locally derivable with Schlottmann's square triangle tilings. In most cases it is clear how to divide Schlottmann's tiles into our prototiles. The only ambiguity is caused by Schlottmann's prototile 3. Depending on the surrounding, it can be divided into two of our prototiles of type 3, or into one of type 4 and one of type 7. Usually, the next order supertile tells us in which one out of the two ways to divide. The only situation where this fails is if the considered Schlottmann prototile of type 3 is contained in a supertile of type 3, which again is contained in a second order supertile of type 3, which again is contained in a third order supertile of type 3, and so on. Altogether, many of our square triangle tilings are mld with the corresponding ones of Schlottmann, but not all of them.

A Fractal with 12-fold Symmetry

Many interesting non-periodic tilings can be obtained by projection from high dimensional lattices, via a so-called cut-and-project scheme (CPS), see for instance [1] and references therein. A CPS for our square triangle tilings is given as follows. In the following diagram, let Λ be a lattice of full rank in $\mathbb{R}^4$, let π_1, π_2 projections such that $\pi_1|_\Lambda$ is injective, and $\pi_2(\Lambda)$ is dense in $\mathbb{R}^2$. Let W be a compact set of positive Lebesgue measure — the so-called *window* — such that the Lebesgue measure of its boundary is zero.

$$
\begin{array}{ccccc}
\mathbb{R}^2 & \xleftarrow{\pi_1} & \mathbb{R}^4 & \xrightarrow{\pi_2} & \mathbb{R}^2 \\
\cup & & \cup & & \cup \\
V & & \Lambda & & W
\end{array}
\tag{1}
$$

Then the set $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is called a *model set* (or a *cut-and-project set*). To be precise, this defines a *regular* model set. Without the requirement that the boundary of W has Lebesgue measure zero, V is still called a model set, but not a regular one.

Obviously, V is a set of points in $\mathbb{R}^2$, not a tiling. We can translate V into a tiling by joining points in V with a certain appropriate distance by an edge. Any tiling obtained in such a way is called a *cut-and-project tiling.*

We ask whether our plain square triangle tiling is a cut-and-project tiling. So we go the opposite direction: Our plain square triangle tiling yields a point set V by considering its vertex set. (Then we obtain the tiling back from V by joining all vertex pairs with distance 1.) There is a standard method to obtain Λ from V, see for instance [3]. With V and Λ we know also π_1. Then π_2 can be chosen orthogonally to π_1. Taking a large finite part of V then yields an approximation of W, just by mapping each point x of (the finite part of) V to $\pi_2(\pi_1^{-1}(x))$. (Recall that π_1 is injective.) Such an approximation of W is shown in Figure 6. This approximation used the fifth iteration of our substitution rule on prototile 5, which yields 131.044 vertices of the plain square triangle tiling. Let us mention that this window has been computed before, see for instance [2].

The statistical 12-fold symmetry of the tilings imply the perfect 12-fold symmetry of the window. This is visible in the figure. Moreover, the image implies that the window is a fractal set. This is plausible, since the expanding self-similarity of the tiling results in a contracting self-similarity of the window, in other words: There is an iterated function system of which the window is the unique compact solution. In general, such solutions tend to be of fractal appearance.

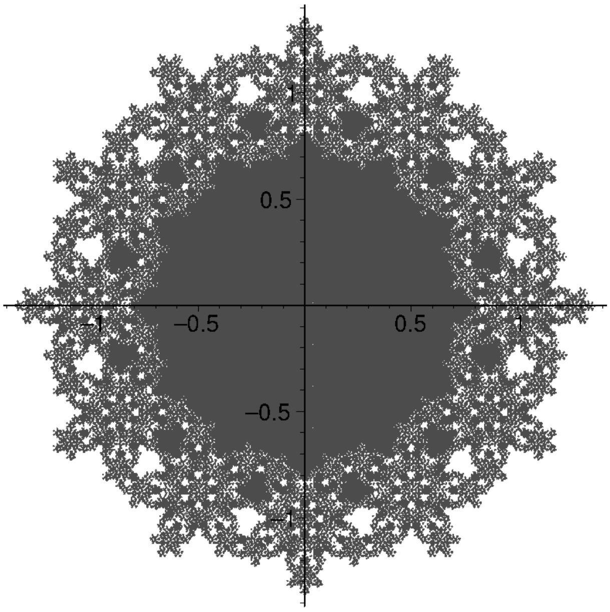

FIGURE 6. The window of the square triangle tiling in Figure 2.

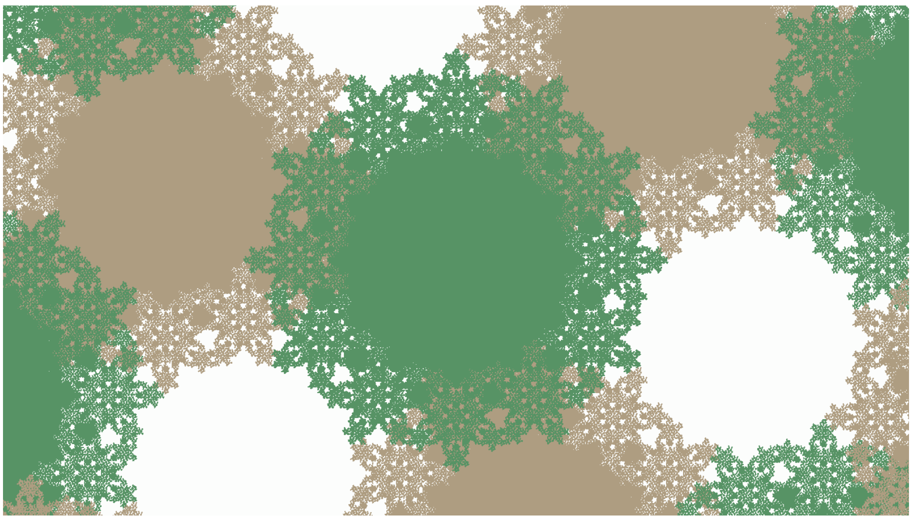

FIGURE 7. The window is possibly a fundamental domain for the hexagonal lattice.

The window has a further very interesting property: It is probably a fundamental domain of the hexagonal lattice, even though it has a larger (12-fold) symmetry than the point group of the hexagonal lattice (6-fold). For details, we refer to future work.

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