Examples of pp-definable subgroups in the homotopy category of a gentle algebra.

This talk is:

- based on arXiv preprint 1911.07691
- the first of two talks, the second based also on 2004.06854

Plan for the talk today:

- motivate via examples from the module category
- discuss model theory of modules
- recall compactly generated triangulated categories]
- discuss 'many sorted' model theory for these
- recall 'string complexes' and 'homotopy words'
- give some examples of the groups in the title

In what remains we consider the gentle algebra

$\Lambda = KQ/I$,

where
$$\begin{cases} Q = \begin{tikzcd} & & 4 & \\ 1 \arrow[loop left, "\delta"] \arrow[r, "\alpha"] & 2 \arrow[ur, "\eta"] \arrow[r, "\beta"] & 3 \arrow[loop right, "\varepsilon"] \arrow[ll, bend left, "\gamma"] \end{tikzcd} \\ K \text{ is a field} \\ I = \langle \gamma\beta, \beta\alpha, \alpha\gamma, \delta^2, \varepsilon^2 \rangle \end{cases}$$

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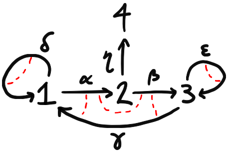
§1: pp-definable subgroups of modules.

Consider the word $w = \beta^{-1}\varepsilon^{-1}\gamma^{-1}\delta\gamma\varepsilon^{-1}$

$2 \to 3 \to 3 \to 1 \leftarrow 1 \leftarrow 3 \to 3$

For any Λ-module M let



$$wM = \left\{ m_0 \in e_2 M \;\middle|\; \exists\, m_1, \dots, m_6 \in M \text{ such that } \left(m_i \in \begin{cases} e_1 M & (i = 3,4) \\ e_3 M & (\text{otherwise}) \end{cases} \right) \text{ and } \left(\begin{matrix} \beta m_0 = m_1, \varepsilon m_1 = m_2, \gamma m_2 = m_3 \\ m_3 = \delta m_4, m_4 = \gamma m_5, \varepsilon m_5 = m_6 \end{matrix} \right) \right\}$$

The m_i's are described as 'filling in' the schema

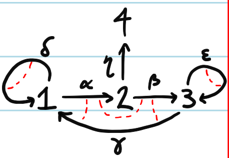
$$m_0 \xrightarrow{\beta} m_1 \xrightarrow{\varepsilon} m_2 \xrightarrow{\gamma} m_3 \xleftarrow{\delta} m_4 \xleftarrow{\gamma} m_5 \xrightarrow{\varepsilon} m_6$$

Remarks:

- The subspaces wM arise when one uses 'functorial filtrations' to classify f.d modules [Butler, Ringel, '87]
- if $\theta : M \to N$ is a homomorphism of Λ-modules then $\theta(wM) \subseteq wN$, hence $w : \Lambda\text{-Mod} \to \mathrm{Vect}_k$ is a functor

We can realise elements of wM by means of Λ-module homomorphisms $\theta : S \to M$ where S is a string module.

To see this, let $S = \bigoplus_{n=0}^{11} k b_i$ and define the action of Λ using the diagram



$$b_0 \xleftarrow{\eta} b_1 \xrightarrow{\beta} b_2 \xrightarrow{\varepsilon} b_3 \xrightarrow{\gamma} b_4 \xleftarrow{\delta} b_5 \xleftarrow{\gamma} b_6 \xrightarrow{\varepsilon} b_7 \xrightarrow{\gamma} b_8 \xrightarrow{\delta} b_9 \xrightarrow{\alpha} b_{10} \xrightarrow{\eta} b_{11}$$

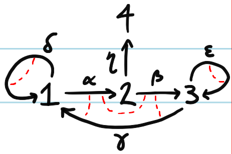
$$m_0 \xrightarrow{\beta} m_1 \xrightarrow{\varepsilon} m_2 \xrightarrow{\gamma} m_3 \xleftarrow{\delta} m_4 \xleftarrow{\gamma} m_5 \xrightarrow{\varepsilon} m_6$$

$w = \beta^{-1}\varepsilon^{-1}\gamma^{-1}\delta\gamma\varepsilon^{-1}$

So $b_0 = \eta b_1$, $\beta b_1 = b_2$, and so on. When μ is an arrow such that b_i is not the tail of μ, $\mu b_i = 0$. So, for example, $\varepsilon b_7 = 0$. There are also relations such as $e_4 b_0 = b_0$, $e_2 b_1 = b_1$, and so on. Note this gives $\varepsilon^2 b_6 = \varepsilon b_7 = 0, \dots$ We now claim

$$wM = \{ \theta(b_1) : \theta \in \mathrm{Hom}_{\Lambda\text{-Mod}}(S, M) \}$$

For $\subseteq$, given $m_0 \in wM$ choose $m_1, \ldots, m_6$ as before.
Then define θ by $\quad m_0 \xrightarrow{\beta} m_1 \xrightarrow{\varepsilon} m_2 \xrightarrow{\gamma} m_3 \xleftarrow{\delta} m_4 \xleftarrow{\gamma} m_5 \xrightarrow{\varepsilon} m_6$



$$\begin{tikzcd}[column sep=small]
b_0 \arrow[d] & b_1 \arrow[l, "\eta"'] \arrow[r, "\beta"] \arrow[d] & b_2 \arrow[r, "\varepsilon"] \arrow[d] & b_3 \arrow[r, "\gamma"] \arrow[d] & b_4 \arrow[d] & b_5 \arrow[l, "\delta"'] \arrow[d] & b_6 \arrow[l, "\gamma"'] \arrow[r, "\varepsilon"] \arrow[d] & b_7 \arrow[r, "\gamma"] \arrow[d] & b_8 \arrow[r, "\delta"] \arrow[d] & b_9 \arrow[r, "\alpha"] \arrow[d] & b_{10} \arrow[r, "\eta"] \arrow[d] & b_{11} \arrow[d, "\theta"] \\
\eta m_0 & m_0 \arrow[l, "\eta"'] \arrow[r, "\beta"] & m_1 \arrow[r, "\varepsilon"] & m_2 \arrow[r, "\gamma"] & m_3 & m_4 \arrow[l, "\delta"'] & m_5 \arrow[l, "\gamma"'] \arrow[r, "\varepsilon"] & m_6 \arrow[r, "\gamma"] & \gamma m_6 \arrow[r, "\delta"] & \delta\gamma m_6 \arrow[r, "\alpha"] & \alpha\delta\gamma m_6 \arrow[r, "\eta"] & \eta\alpha\delta\gamma m_6
\end{tikzcd}$$

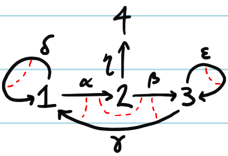
The inclusion $\supseteq$ is easier.

We now interpret wM using model theory (logic).
We claim there exists $A \in \mathrm{Mat}_{13\times 7}(\Lambda)$ such that

$$wM = \left\{ m_0 \in M : \exists\, m_1, \ldots, m_6 \in M \text{ with } A \begin{pmatrix} m_0 \\ \vdots \\ m_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Proof:

$w = \beta^{-1}\varepsilon^{-1}\gamma^{-1}\delta\gamma\varepsilon^{-1}$



Choose

$$A = \begin{array}{c} \left(\begin{array}{ccccccc}
1-e_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-e_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-e_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-e_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-e_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-e_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-e_3 \\
\hline
\beta & -e_3 & 0 & 0 & 0 & 0 & 0 \\
0 & \varepsilon & -e_3 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma & -e_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -e_1 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & -e_1 & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon & -e_3
\end{array}\right) \end{array}
\begin{array}{l}
e_2 m_0 = m_0 \\
e_3 m_1 = m_1 \\
\vdots \\
\\
\\
\vdots \\
e_3 m_6 = m_6 \\
\beta m_0 = m_1 \\
\varepsilon m_1 = m_2 \\
\vdots \\
\\
\vdots \\
\varepsilon m_5 = m_6
\end{array}$$

$m_0 \xrightarrow{\beta} m_1 \xrightarrow{\varepsilon} m_2 \xrightarrow{\gamma} m_3 \xleftarrow{\delta} m_4 \xleftarrow{\gamma} m_5 \xrightarrow{\varepsilon} m_6$

Note that we could replace A with a smaller matrix, but for this talk we only care that some A works.

The last remark above says that wM is the set of solutions to a positive primitive (=pp) formula.

In general: a pp-formula (in one free variable x_0) has the form

ignore when $c=1$

$$\varphi(x_0) = \left(\exists x_1, \dots, x_{c-1} : \bigwedge_{i=1}^{r} \sum_{j=0}^{c-1} a_{ij} x_j = 0\right)$$

where $(a_{ij})_{i,j} \in \mathrm{Mat}_{r \times c}(\Lambda)$ $(r, c \geq 1)$.

This is a formula in the language of Λ-modules.

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§2: model theory of modules

The language of modules over Λ is defined and denoted

$$\mathcal{L}_\Lambda = \langle \mathrm{pred}_\Lambda, \mathrm{func}_\Lambda, \mathrm{ar}_\Lambda \rangle$$

where

$$\mathrm{pred}_\Lambda = \{0\}, \ \mathrm{func}_\Lambda = \{+\} \cup \{a- : a \in \Lambda\}$$

are the predicate and function symbols.

The arity function $\mathrm{ar}_\Lambda : \mathrm{pred}_\Lambda \sqcup \mathrm{func}_\Lambda \to \mathbb{N}$ is given by

$$0 \mapsto 0, \ + \mapsto 2, \ a- \mapsto 1$$

By introducing a countable set of variables x_i $(i \in \mathbb{N})$, the terms inside $\mathcal{L}_\Lambda$ are defined by saying:
- each x_i is a term
- if t and t' are terms then $t+t'$ is a term
- if t is a term and $a \in \Lambda$ then at is a term

The atomic formulas are those of the form

$$t = t' \quad \text{or} \quad t = 0$$

where t and t' are terms. The formulas in $\mathcal{L}_\Lambda$ are built from: the variables x_i, atomic formulas, binary connectives $\wedge$, $\vee$, and $\Rightarrow$, negation $\neg$, and the quantifiers $\forall$ and $\exists$.

A positive primitive formula $\varphi(x_0, \dots, x_{n-1})$ in $n \geq 1$ free variables has the form

$$\varphi(x_0, \dots, x_{n-1}) = \left(\exists y_n, \dots, y_m \bigwedge_{i=1}^{r} \psi_i(x_0, \dots, x_{n-1}, y_n, \dots, y_m)\right)$$

where each $\psi_i(x_0, \dots, x_{n-1}, y_n, \dots, y_m)$ is atomic.

The $\mathcal{L}_\Lambda$-structures are given by a set M together with an interpretation $0_M \in M$ of the predicate symbol, and of the functions $+_M : M^2 \to M$ and $a_M - : M \to M$ $(a \in \Lambda)$.

The theory Th_Λ of Λ-modules is given by the usual module axioms, for example distributivity is given by the formula

$$\forall x_0, x_1, \quad a(x_0 + x_1) = a(x_0) + a(x_1) \qquad (a \in \Lambda)$$

A model for Th_Λ is an $\mathcal{L}_\Lambda$-structure satisfying the module axioms: hence is the same thing as a Λ-module. There is also a notion of a homomorphism of $\mathcal{L}_\Lambda$-structures, which corresponds to a homomorphism of Λ-modules.

Positive primitive formulas are important in the model theory of modules due to the following.

Theorem (Baur, '76) Any formula in $\mathcal{L}_\Lambda$ is equivalent (relative to Th_Λ) to a boolean combination $(\wedge, \vee, \neg)$ of $\forall\exists$ sentences and positive primitive formulas.

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§3: compactly generated Δ-ated categories.

Let $\mathcal{T}$ be a triangulated category with suspension functor $\Sigma: \mathcal{T} \to \mathcal{T}$. The only example we are interested in today is

$$\mathcal{T} = \underbrace{K}_{\text{homotopy category}}(\underbrace{\Lambda\text{-Proj}}_{\text{complexes of* projective modules.}})$$

* possibly inf. dim.

We assume also $\mathcal{T}$ has small coproducts.
So if I is any set and X, Y_i $(i \in I)$ are objects in $\mathcal{T}$, the universal properties give a morphism

$$\coprod_i \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \to \mathrm{Hom}_{\mathcal{T}}(X, \coprod_i Y_i)$$

We call the object X compact provided, for any such collection $(Y_i : i \in I)$, the map above is bijective.

We say $\mathcal{T}$ is compactly generated by a set $\mathcal{G}$ of objects in $\mathcal{T}$ provided:
- each $X \in \mathcal{G}$ is a compact object in $\mathcal{T}$,
- if $X \in \mathcal{G}$ then $\Sigma X \in \mathcal{G}$, and
- if M is an object in $\mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(X, M) = 0$ for all $X \in \mathcal{G}$, then we must have $M = 0$.

generalised by Neeman '08

Theorem (Jørgensen '05) Since Λ is sufficiently nice, the category $K(\Lambda\text{-Proj})$ is compactly generated.

Later we will combine details from this theorem with results from [BT, Thesis], to describe a set of isoclasses of indecomposable compact objects in $K(\Lambda\text{-Proj})$.

Assumption: in what follows we assume that $\mathcal{T}$ is compactly generated, (that $\mathcal{T}$ has small coproducts) and that the full subcategory $\mathcal{T}^c$ of $\mathcal{T}$ consisting of compact objects is skeletally small.

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§4: The canonical many-sorted language.

Remarks:
- in the second talk we'll look at notions of purity, adapted from Λ-Mod to $\mathcal{T}$ by Krause
- for today we replace $\mathcal{L}_\Lambda$ by the following

Definition (Garkusha, Prest, '05) Let $\mathcal{T}$ be a triangulated category satisfying the above Assumption. Fix a chosen set $\mathcal{S}$ of isoclasses inside $\mathcal{T}^c$.

The canonical $\mathcal{S}$-sorted language for $\mathcal{T}$ is defined and denoted

$$\mathcal{L}_{\mathcal{T}}^{\mathcal{S}} = \langle \mathrm{pred}_{\mathcal{T}}^{\mathcal{S}}, \mathrm{func}_{\mathcal{T}}, \mathrm{ar}_{\mathcal{T}}^{\mathcal{S}}, \mathrm{sort}_{\mathcal{T}}^{\mathcal{S}} \rangle$$

where

- $\mathrm{pred}_{\mathcal{T}}^{\mathcal{S}} = \{ 0_G : G \in \mathcal{S} \}$,
- $\mathrm{func}_{\mathcal{T}}^{\mathcal{S}} = \{ +_G : G \in \mathcal{S} \} \cup \{ \circ a : a \in \mathrm{Hom}_{\mathcal{T}}(G,H), G, H \in \mathcal{S} \}$
- $\mathrm{ar}_{\mathcal{T}}^{\mathcal{S}}(0_G) = 0$, $\mathrm{ar}_{\mathcal{T}}^{\mathcal{S}}(+_G) = 2$, $\mathrm{ar}_{\mathcal{T}}^{\mathcal{S}}(\circ a) = 1$
- $\mathrm{sort}_{\mathcal{T}}^{\mathcal{S}} : \mathrm{pred}_{\mathcal{T}}^{\mathcal{S}} \cup \mathrm{func}_{\mathcal{T}}^{\mathcal{S}} \to \bigsqcup_{n \in \mathbb{N}} \mathcal{S}^n$ is a function given by

$$0_G \mapsto G \in \mathcal{S}^1, \quad +_G \mapsto (G,G,G) \in \mathcal{S}^3, \quad \circ a \mapsto (H,G) \in \mathcal{S}^2$$

As we did with the language $\mathcal{L}_\Lambda$ of Λ-modules, one can define sorted terms, atomic formulas, formulas and pp-formulas in $\mathcal{L}_{\mathcal{T}}^{\mathcal{S}}$. For example one begins by fixing a countable set $\mathcal{V}_G$ of variables for each sort $G \in \mathcal{S}$, each term has a sort, ...

To understand the above... Given any object M in $\mathcal{T}$, we define an $\mathcal{L}_{\mathcal{T}}^{\mathcal{S}}$-structure $\underline{M}$ as follows.

Let $\underline{M} = \{ \mathrm{Hom}_{\mathcal{T}}(G,M) : G \in \mathcal{S} \}$, where

- the symbol 0_G is interpreted as $0 \in \mathrm{Hom}_{\mathcal{T}}(G,M)$
- the symbol $+_G$ —— " —— the addition here
- each symbol $\circ a$ —— " —— $\mathrm{Hom}_{\mathcal{T}}(H,M) \to \mathrm{Hom}_{\mathcal{T}}(G,M)$, $f \mapsto fa$

Of great importance later will be the pp-formulas given by divisibility; that is those of the form

$$\begin{tikzcd} & M \\ G \arrow[ur, "x"] \arrow[r, "y"'] & H \arrow[u, "z"'] \end{tikzcd}$$

$$\underset{G\to M}{\varphi(x_G)} = (\exists \underset{H\to M}{y_H} : x_G = y_H \circ \underset{G\to H}{a})$$

<u>Proposition (Garkusha, Prest)</u> Every pp-formula in the canonical language $\mathcal{L}^c_{\mathcal{T}}$ is equivalent to a pp-formula of the form above (divisibity condition).

<u>Definition (Garkusha, Prest)</u> For any object M in $\mathcal{T}$ and any object G in $\mathcal{T}^c$, a pp-definable subgroup of M of sort G is a subgroup of $\mathrm{Hom}_{\mathcal{T}}(G, M)$ of the form

$$Ma = \{ f \circ a \mid f \in \mathrm{Hom}_{\mathcal{T}}(H, M) \}$$

where $a : G \to H$ is some morphism in $\mathcal{T}^c$.

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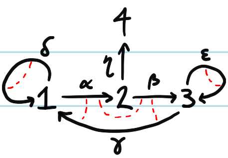
§5: <u>pp-definable subgroups in $K(\Lambda\text{-Proj})$</u>



Back to the gentle algebra $\Lambda = KQ/I$. We want to parallel the story at the start of the talk, which was about Λ-Mod.

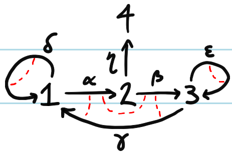
Write $K_{min}(\Lambda\text{-Proj})$ for the full subcategory of $K(\Lambda\text{-Proj})$ consisting of complexes $M^\bullet$ such that $\mathrm{im}(d^n_M) \subseteq \mathrm{rad}(M^{n+1})$ for all $n \in \mathbb{Z}$.

Fact: since Λ is a perfect ring the subcategory $\mathcal{K}_{\min}(\Lambda\text{-Proj})$ is dense inside $\mathcal{K}(\Lambda\text{-Proj})$.

Fix a complex $M^\bullet$ in $\mathcal{K}_{\min}(\Lambda\text{-Proj})$. For all $m \in M = \bigoplus_{n\in\mathbb{Z}} M^n$ we have, for each vertex v in Q,

$$d_M(e_v m) \underset{\text{definition of } \mathcal{K}_{\min}}{\in} e_v \operatorname{rad}(M) \underset{M \text{ is projective}}{=} e_v \operatorname{rad}(\Lambda) M,$$

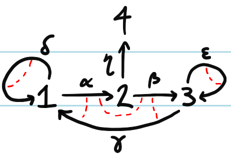
and so



$$d_M(e_v m) \in \begin{cases} \delta M \oplus \gamma M & (v=1) \\ \alpha M & (v=2) \\ \beta M \oplus \varepsilon M & (v=3) \\ \eta M & (v=4) \end{cases}$$

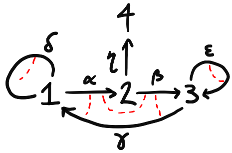
Hence if λ is an arrow in Q with head v there is a K-linear map $d_{\lambda,M} : e_v M \to e_v M$ given by

$$\begin{tikzcd} e_v M \arrow[r, "d_{\lambda,M}"] \arrow[d, "d_M|_{e_v M}"'] \arrow[dr, phantom, "\circlearrowleft"] & e_v M \\ e_v \operatorname{rad}(M) \arrow[r, two heads] & \lambda M \arrow[u, "\subseteq"'] \end{tikzcd}$$



For example, if $d_M(e_1 m) = \delta \ell + \gamma n$ then we have $d_{\delta,M}(m) = \delta\ell$ and $d_{\gamma,M}(m) = \gamma n$.

We can construct homotopy words using the letters of the form d_λ, d_λ^{-1}, p and p^{-1} where p is a non-zero path in Λ and λ is an arrow.

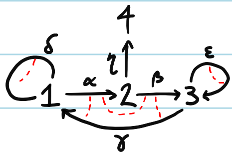


legit because $\beta\alpha = 0$ in Λ

For example, $C = d_\delta^{-1}\,\delta\gamma\,\varepsilon^{-1}\gamma^{-1}\delta^{-1}\alpha^{-1}\,d_\alpha\,\beta^{-1}\,d_\beta\,d_\varepsilon^{-1}\,\varepsilon$ is a homotopy word.

homotopy letters

For any object $M^\bullet$ of $K_{min}(\Lambda\text{-Proj})$ let $M = \bigoplus_{n\in\mathbb{Z}} M^n$.



$$CM = \left\{ m_0 \in e_1 M \;\middle|\; \begin{array}{l} \exists\, m_1 \in e_3 M,\ m_2 \in e_2 M,\ m_3 \in e_3 M,\ m_4 \in e_3 M \text{ such} \\ \text{that } d_{\delta,M}(m_0) = \delta\gamma m_1,\ \alpha\delta\gamma\varepsilon m_1 = d_{\alpha,M}(m_2), \\ \beta m_2 = d_{\beta,M}(m_3), \text{ and } d_{\varepsilon,M}(m_3) = \varepsilon m_4 \end{array} \right\}$$

Here the m_i's fill in the schema

$$m_0 \xrightarrow{d_\delta} \xleftarrow{\delta\gamma} m_1 \xrightarrow{\alpha\delta\gamma\varepsilon} \xleftarrow{d_\alpha} m_2 \xrightarrow{\beta} \xleftarrow{d_\beta} m_3 \xrightarrow{d_\varepsilon} \xleftarrow{\varepsilon} m_4$$

Aim for the remainder today: show that there are compact objects G, H and a morphism $a : G \to H$ in $K_{min}(\Lambda\text{-Proj})$ such that the vectorspace CM is given by the pp-definable subgroup

$$Ma = \{ f \circ a \mid f \in \mathrm{Hom}_{K_{min}(\Lambda\text{-Proj})}(H, M) \}$$

(of M of sort G).

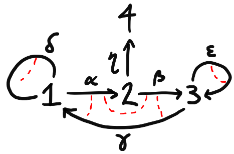
We will use the language of string complexes

The algebras considered here are derived-discrete, but their argument generalises easily.

Lemma (Arnesen, Laking, Pauksztello, Prest '17) Using the Theorem above, due to Jørgensen, the indecomposable compact objects in $K_{min}(\Lambda\text{-Proj})$ are given by (shifts of) string and band complexes in $K^{+,b}_{min}(\Lambda\text{-proj})$

$G^n = 0,\ n \ll 0$

$H^n(G) = 0,\ n \gg 0$



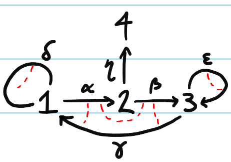
Define the (string) complex $G^\bullet$ in $K^{+,b}_{\min}(\Lambda\text{-proj})$ by

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \Lambda e_1 \arrow[r, "\begin{pmatrix}\gamma\\ \delta\end{pmatrix}"] \arrow[d, equal] & \Lambda e_3 \oplus \Lambda e_1 \arrow[r, "\begin{pmatrix}\beta & 0\\ 0 & \delta\end{pmatrix}"] \arrow[d, equal] & \Lambda e_2 \oplus \Lambda e_1 \arrow[r, "\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}"] \arrow[d, equal] & \Lambda e_1 \oplus \Lambda e_1 \arrow[r, "\begin{pmatrix}\gamma & 0\\ 0 & \delta\end{pmatrix}"] \arrow[d, equal] & \Lambda e_3 \oplus \Lambda e_1 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r, "d_G^0"] & G^1 \arrow[r, "d_G^1"] & G^2 \arrow[r, "d_G^2"] & G^3 \arrow[r, "d_G^3"] & G^4 \arrow[r] & \cdots
\end{tikzcd}$$

Here we abuse notation, writing γ for the Λ-module homomorphism $-\gamma : \Lambda e_1 \to \Lambda e_3$, $z \mapsto z\gamma$.

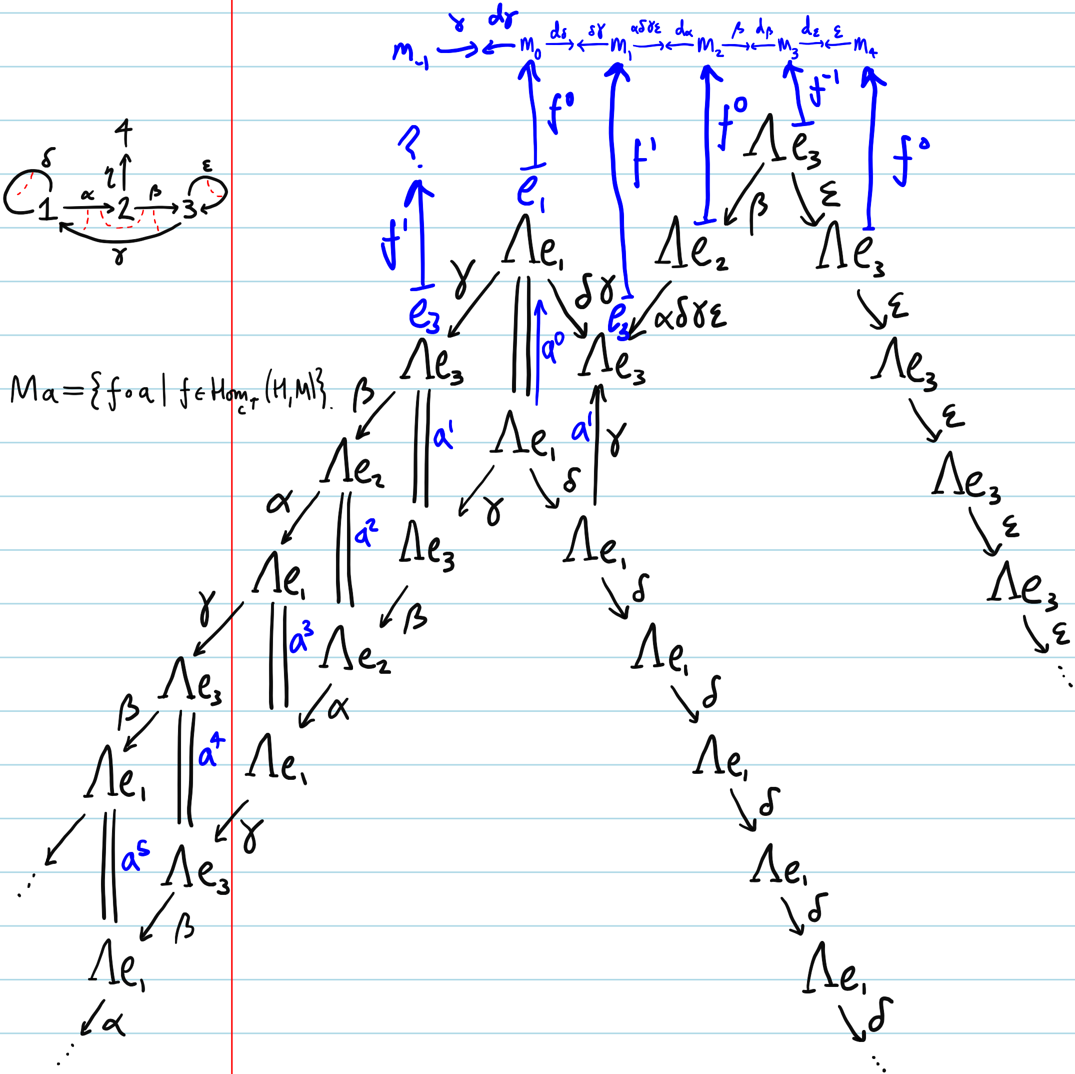
The name string complex is motivated by the redrawing of $G^\bullet$ using the diagram below.

$$\begin{tikzcd}
G^0 \arrow[d, "d_G^0"'] \\
G^1 \arrow[d, "d_G^1"'] \\
G^2 \arrow[d, "d_G^2"'] \\
G^3 \arrow[d, "d_G^3"'] \\
G^4 \arrow[d, "d_G^4"'] \\
G^5 \arrow[d, "d_G^5"'] \\
\vdots
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & & & & & \Lambda e_1 \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & & & & & & \\
 & & & & & \Lambda e_3 \arrow[dl, "\beta"'] & & \Lambda e_1 \arrow[dr, "\delta"] & & & & & \\
 & & & & \Lambda e_2 \arrow[dl, "\alpha"'] & & & & \Lambda e_1 \arrow[dr, "\delta"] & & & & \\
 & & & \Lambda e_1 \arrow[dl, "\gamma"'] & & & & & & \Lambda e_1 \arrow[dr, "\delta"] & & & \\
 & & \Lambda e_3 \arrow[dl, "\beta"'] & & & & & & & & \Lambda e_1 \arrow[dr, "\delta"] & & \\
 & \Lambda e_1 \arrow[dl, "\alpha"'] & & & & & & & & & & \Lambda e_1 \arrow[dr, "\delta"] & \\
  & & & & & & & & & & & & \ddots
\end{tikzcd}$$



So far, $G^\bullet$ seems unrelated to the homotopy word C. The only connection is the 'head' of C is 1, and $G^\bullet$ is the $\mathrm{Hom}(-,\Lambda)$-dual of a min. res. of $\dfrac{e_1\Lambda}{\mathrm{rad}(e_1\Lambda)}$.

Using the depiction of string complexes above, we finally define the morphism $a: G \to H$ in $K^{+,b}_{min}(\Lambda\text{-proj})$ such that CM is given by



$Ma = \{ f \circ a \mid f \in \mathrm{Hom}_{CT}(H, M) \}$.