A GENTLE INTRODUCTION TO FUKAYA CATEGORIES

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1. Introduction

1.1. Symplectic and contact geometry

DEF

A symp. mfd (M, ω) is a smooth mfd with a 2-form $\omega \in \Omega^2(M)$ s.t.

(i) $d\omega = 0$ (closeness)

(ii) ω is non-deg at every pt (i.e. $\iota(\omega_p): T_pM \xrightarrow{\cong} T_p^*M$).

A symplectomorphism $f: (M_1, \omega_1) \to (M_2, \omega_2)$ is a smooth map s.t. $f^*\omega_2 = \omega_1$.

Important features

(a) (ii) $\Rightarrow$ dim M is $\boxed{\text{even}}$

(b) If $\dim(M) = 2$, a symp form is an area form, and a symplectom. is an area preserving map.

(c) If $(T^*\mathcal{U}, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is a coord. chart for T^*M, $\lambda := \sum_{i=1}^{n} \xi_i\, dx_i$ (Liouville 1-form), then $\omega = d\lambda$ is a canonical symp. form.

(d) By (ii), if $H \in \mathcal{C}^\infty(M)$, $\exists!\, X_H \in \mathfrak{X}(M)$ Hamilt. vector field s.t. $i_{X_H}\omega = dH$. Hamiltonian. The Hamiltonian flow φ_t^H is the unique flow along X_H.

(e) A Lagrangian submfd $L \subseteq M$ is a submfd of dim n s.t. $\omega|_L \equiv 0$.

Idea: In $\boxed{\text{odd}}$ dim, its sibling is $\boxed{\text{contact geom.}}$

DEF

(i) A contact str. on M is a smooth field of tangent hyperplanes $H \subseteq TM$ s.t. for any 1-form α, $d\alpha|_H$ is symp.

(ii) The Reeb vector field R of α is given by $\begin{cases} d\alpha(R, -) \equiv 0 \\ \alpha(R) \equiv 1 \end{cases}$.

Rmk: The distinguished submfds in contact geom. are Legendrian submfds (it is everywhere tangent to H).

1.2. Homological Mirror Symmetry

Kontsevich '94: mathematical formulation of mirror symmetry; X and $X^\vee$ are mirror CY-mfds if and only if

$$(*) \quad \underbrace{D^b\mathcal{F}(X)}_{\substack{\text{A-model} \\ \text{(symp. geom)}}} \simeq \underbrace{D^b(\mathrm{Coh}(X^\vee))}_{\substack{\text{B-model} \\ \text{(alg. geom.)}}} \quad \text{(triangulated categories)}$$

Idea: The Fukaya category $\mathcal{F}(X)$ is a global invariant of the symp. mfd, being an A_∞-categ., whose objects are compact Lagrangians and morphisms are given by Floer complexes.

Q (Kontsevich) Can we extend $(*)$ to more gral mfds?

A: Yes, but you have to reformulate it: (matrix factoriz, D_{sg}, LG-models, wrapped Fukaya...)

Lekili – Polishchuk (based on Haiden-Katzarkov-Kontsevich)

$M = \Sigma_{g,n} \equiv$ compact symp. surface with n boundary components $\ast_0$

Burban-Drozd's nodal stacky curve

Obj	name	A-model		B-model
compact Lagrangians	compact Fukaya	$\mathcal{F}(\Sigma_{g,n})$		$\mathrm{Perf}(\mathcal{C})$
		$\downarrow$ full faithful		$\downarrow$ full faithful
compact & non-comp Lagrang.	partially wrapped Fukaya	$\mathcal{W}(\Sigma_{g,n}, \Lambda, \eta) \underset{[\mathrm{HKK}]}{\simeq} D^b(A\text{-mod})$	$\underset{[\text{Burban-Drozd}]}{\simeq}$	$D^b(\Lambda_{\mathcal{C}})$
		localiz. $\downarrow$ [LP]		$\downarrow$
//	wrapped Fukaya	$\mathcal{W}(\Sigma_{g,n})$		$D^b\mathrm{Coh}(\mathcal{C})$

stops, grading

A: graded gentle alg

$\Lambda_{\mathcal{C}}$: Auslander order

Driving idea: $\simeq$ is given by the "endom. alg" of $\mathcal{W}(\Sigma_{g,n}, \Lambda, \eta)$ (which is simpler than the endom. alg of $\mathcal{F}(\Sigma_{g,n})$)

Why this is interesting in repr th? We can use symp. machinery to prove derived equiv of gr. gentle alg. (converse?)

2. (Compact) Fukaya categories

Set up:

$\mathbb{K} \equiv$ field

$(M, \omega) \equiv$ compact symp. mfd with $\begin{cases} \omega = d\lambda & \text{"exact"} \\ 2c_1(M) = 0. \end{cases}$

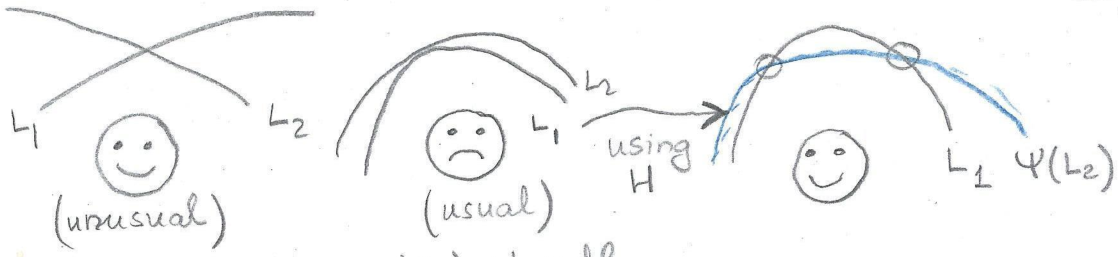
$H \in \mathcal{C}^\infty(M \times [0,1], \mathbb{R})$ (it determines a family of Ham v.f. X_t via $\omega(-, X_t) = dH_t$, where $H_t = H(-,t)$; integrating these vector fields over $t \in [0,1]$ yields the Hamiltonian difeom. Ψ gen. gen. by H)

$J \equiv$ compatible almost-complex str.

Idea: Given L_1, L_2 Lagrangians, Floer cohomology categorifies the classical homological intersection number of L and L' in the sense that

$$\chi(HF^\bullet(L, L')) = (-1)^{\frac{n(n+1)}{2}} [L] \cdot [L']$$

Problem: Lagrangian submfds usually intersect "more" than classical topology suggests



DEF (Compact Fukaya categ.)

$$\mathcal{F}(M) := \begin{cases} \text{Objects: compact, exact } (\lambda|_L = df)\text{, graded Lagrangian submfds} \\ \text{Morphisms: } \mathrm{Hom}_{\mathcal{F}}(L_1, L_2) = \underbrace{CF}_{\text{"Floer complexes"}}(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{K} \cdot p \end{cases}$$

Rmks

(i) Since, by hypothesis, $2c_1(M) = 0$, Floer complexes are $\mathbb{Z}$-graded

(ii) The points $p \in L_1 \cap L_2$ correspond bijectively to the (finite) set of maps $y : [0,1] \to M$ with $y(0) \in L_0$, $L_1 \in y(1)$, and $\frac{dy}{dt} = X(t, y(t))$.

THM (Extremely hard)

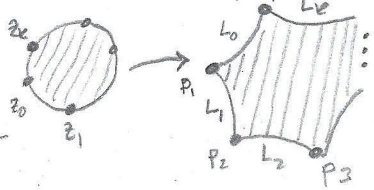
$\mathcal{F}(M)$ is a $\mathbb{K}$-linear pre-triangulated A_∞-category (i.e. given a set of transverse Lagrangians $\{L_i\}_{i=0}^{d}$ there exist higher composition maps

$$\mu^d : CF(L_{d-1}, L_d) \otimes \dots \otimes CF(L_0, L_1) \to CF(L_0, L_d)$$

of degree $2-d$, satisfying:

$$\sum_{i,l} (-1)^{|x_1| + \dots + |x_i| - i} \mu^{d-l+1}(p_d, \dots, p_{i+l+1}, \mu^l(p_{i+l}, \dots, p_{i+1}), p_i, \dots, p_1) = 0).$$

Q: How can we compute μ^d?

$\mathcal{M}(p_1, \dots, p_k; q; J) = \Big\{$  $\Big|$ $\underbrace{\bar\partial_J u = 0}_{\text{inhomog. Cauchy-Riemann eq.}}$ + boundary cond's + Energy finiteness $\Big\} \Big/ \mathrm{Aut}(\mathbb{D}) \cong PSL_2(\mathbb{R})$

$$\Rightarrow \mu^k(p_k, \dots, p_1) = \sum_{q \in L_0 \cap L_1} \left(\# \mathcal{M}(p_1, \dots, p_k; q; J)\right) \cdot q$$

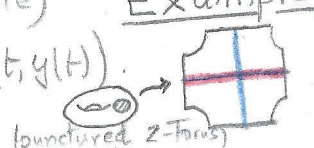
Idea (Seidel): An A_∞-category $\mathcal{A}$ with finitely many objects $X_1, \dots, X_k$ is equivalent to an A_∞-algebra A over the semisimple ring $R = \mathbb{K}e_1 \oplus \dots \oplus \mathbb{K}e_k$, called the endom. algebra

As a graded vector space this algebra is

$$A := \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(X_i, X_j) \text{ w/ the action } e_s \cdot A \cdot e_t = \mathrm{Hom}_{\mathcal{A}}(X_t, X_s)$$

Problem: Even in the simplest case, the endom. algebras (associated to $\mathcal{F}(M)$) are complicated

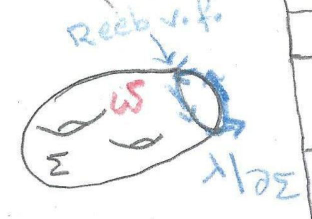
Example (Lekili-Perutz '11, using Python 3.0)



Its endomorphism algebra A is not formal (i.e. not quasi isomorphic to the $\mathbb{K}$-algebra $H^\bullet(A)$).

3. Partially wrapped Fukaya categ. and gentle algebras

$\Sigma := (\Sigma, \omega) \equiv$ connected compact symplectic graded surface with:



→ $\partial\Sigma \neq \emptyset$

→ $\omega = d\lambda$ and $\lambda|_{\partial\Sigma}$ is a contact form

$\Rightarrow$ we have the Reeb vector field on $\partial\Sigma$

→ The orientation of $\partial\Sigma$ is induced from ω.

→ A line field η is a section of the projectivized tangent bundle $\mathbb{P}(T\Sigma)$ $\Rightarrow$ We can define winding numbers.

$\Rightarrow$ We obtain a $\mathbb{Z}$-grading structure on Σ. (in part. immersed curves are $\mathbb{Z}$-grad)

Features of partially wrapped Fukaya categ.

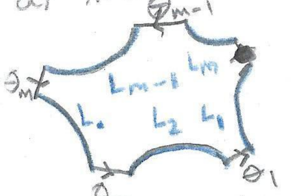
(i) The objects are compact or non-compact exact graded Lagrangians (arcs), with $\partial(L_i) \subseteq \partial\Sigma$;

(ii) $\mathrm{Hom}_{\mathcal{W}}(L_i, L_j)$ is generated by intersection points and Reeb flowlines from ∂L_j to ∂L_i (note the reversal of indices);

(iii) $\Lambda \subseteq \partial\Sigma$ is a finite collection of marked points (stops). The data of Λ enters by disallowing flows that pass through a stop

(iv) (Auroux): a set of pairwise disjoint and non-isotopic Lagrangians $\{L_i\}$ in $\Sigma \setminus \Lambda$ generates the partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda; \eta)$ as a triang. categ. if the complement of the Lagrangians $\Sigma \setminus \{\bigsqcup_i L_i\} = \bigcup_f D_f$ is a union of disks D_f each of which has at most one stop on its boundary.

Example of D_f:



— $\equiv$ boundary parts on $\partial\Sigma$

— $\equiv$ Lagrangians

(v) The line field η is used to $\mathbb{Z}$-grade the morphism spaces; (A line field on D_f is determined by θ_i given by its winding numbers along the bdry. parts on $\partial\Sigma$). θ_i satisfy $\sum_{i=1}^{m} \theta_i = m-2$ (i.e. η extends to $\mathrm{Int}(D_f)$))

(vi) (Auroux): If each D_f has exactly one stop in its boundary, the assoc. $\mathbb{K}$-algebra

$$A_{L_\bullet} := \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{W}}(L_i, L_j)$$

(whose alg. str. is given by concatenation of flowlines) is formal. Furthermore, the higher products in $A_{L_\bullet}$ vanish (by picking a particularly nice perturbation scheme, he showed that ($\nexists$) holomorphic m-gons for $m \geq 3$).

(vii) $A_{L_\bullet}$ can be described by a graded gentle algebra

(viii) $A_{L_\bullet}$ is always homologically smooth (since so is $\mathcal{W}(\Sigma, \Lambda; \eta)$). It's proper (i.e. f.d) if and only if there is at least one stop on every bdry component

Rmk 1: Lekili-Polishchuk do not impose the condition of f.d. in their def. of gentle algebras. So, they consider locally gentle alg. algebras. They assume that they are homol. smooth.

Rmk 2: Using (see [HKK], p275) the useful observation that $L_1 \xrightarrow{a_1} L_2[|a_1|] \to \cdots \xrightarrow{a_{n-2}} L_{n-1}[|a_1| + \cdots + |a_{n-2}|]$ is isomorphic to $L_n[-|a_n|]$, it can be proved that the (fully) wrapped Fukaya cat. $\mathcal{W}(\Sigma, \eta)$ is the localiz of $\mathcal{W}(\Sigma, \Lambda; \eta)$ given by dividing out by the subcat. gen. by the objects T_1, T_2 supported near the stops. In this way, we overcome the stops and we always can compose flowlines

$T_1 \simeq \{L_0 \xrightarrow{u_1} L_1 \xrightarrow{u_2} L_2\}$

$T_2 \simeq \{L_2 \xrightarrow{v_2} L_1 \xrightarrow{v_1} L_0\}$

EXAMPLE

$A = \Bbbk Q / I$

Graded gentle algebra $\boxed{A}$

$$Q = \begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "d"'] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] & 4 \end{tikzcd} \qquad \boxed{bd = 0}$$

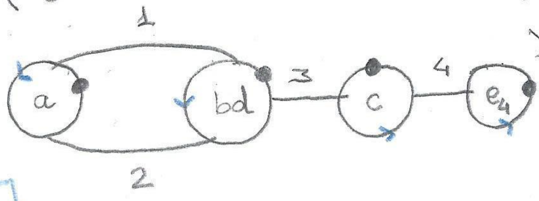
$F = \{a, bd, c, e_4\}$ (forbidden threats)

$T = \{cba, d, e_3, e_4\}$ (permitted threats)

⇝

(Marked) Ribbon graph $\boxed{\mathcal{R}_A}$

(cyclic order at vertices are given by counter clockwise rotation)



vertices: bij w/ forbidden threats

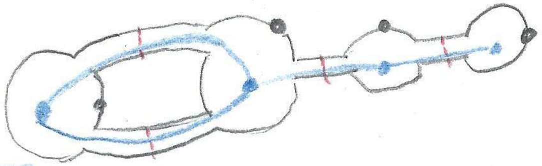
edges: bij w/ vertices of Q

ribbon str: in which these vertices appear in the forbidden threat (recall $(\exists)$ precisely two forbidden threats that pass through each $v \in Q_0$)

⇝

(Marked) thickened surface $\boxed{\Sigma_A}$

(g=0)



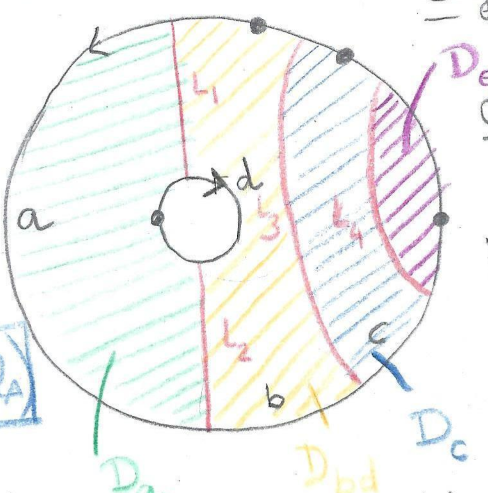
$\mathcal{R}_A$ is embedded as a def retract of Σ_A

vertices of $\mathcal{R}_A \rightsquigarrow$ 2-disk $\mathbb{D}$

edges of $\mathcal{R}_A \longrightarrow$ strip (i.e. a thin oriented rectangle $[-\varepsilon, \varepsilon] \times [0,1]$).

⇝

Wrapped Fukaya category $\boxed{\mathcal{W}(\Sigma_A, \Lambda_A, \eta_A)}$



Objects: dual to the edges of R_A we obtain a disjoint collection of noncompact arcs L_v, $v \in Q_0$

Morphisms: arrows of Q

$\# \Lambda_A = \# F$ ⚠ Keep track of the gradings

Rmk

As $g = 0$, the line field is determined by the winding numbers along the boundary components (of type I). The combinatorial boundary comp. are given by $\{p_3 f_3 p_2 f_2 p_1 f_1, p_4 f_4\}$ con $\begin{cases} p_4 = d \\ f_4 = a \end{cases} \Rightarrow \begin{array}{l} |p_4 f_4| = |a| - |d| \\ |p_3 f_3 p_2 f_2 p_1 f_1| = -|p_4 f_4| = |d| - |a|. \end{array}$

Lessons from the example:

(i) We have

$\Sigma_A \setminus \{\bigsqcup_v L_v\} = \bigcup_{f \text{ forbidden}} D_f =$ 

$\overset{\text{Auroux}}{\Longrightarrow} \{L_v\}$ gives a generating set of $\mathcal{W}(\Sigma_A, \Lambda_A, \eta_A)$

(ii) By construction, $(\exists)$ a bijection b/w $\alpha \in Q_1$ and the generators of $A_L := \bigoplus_{v,w} \mathrm{Hom}(L_v, L_w)$ (since each edge $\alpha \in Q_1$ is in exactly one forbidden thread f, and the corresponding D_f has a flow associated to α).

(iii) Two flows $\alpha_1 : L_{v_2} \to L_{v_1}$ and $\alpha_2 : L_{v_3} \to L_{v_2}$ can be composed in $A_L \iff \forall f_i \in F$, $i = 1,2$ s.t. the disks D_{f_1} and D_{f_2} are glued along the edge corresp. to v_2.

$\Longrightarrow$ the corresp. elements of A satisfy $\alpha_2 \alpha_1 \notin I$ (otherwise A will ~~not~~ be a gentle algebra)

Ex: $\boxed{b}d \in I$

Now, $a : L_1 \to L_2$ and $b : L_2 \to L_3$ are composable. $\boxed{\text{If}}$ we declare $\boxed{b}a \in I$,

$\Rightarrow$ A is not a gentle algebra (recall that for each arrow α, there is at most one arrow β s.t. $\alpha\beta \in I$).

$\Longrightarrow A \simeq A_L^{op}$ as ~~un~~graded algebras.

Since $\{L_v\}$ generates $\mathcal{W}(\Sigma_A, \Lambda_A; \eta_A)$,

bounded derived categ. of perfect (left) dg-modules

THM (Lekili–Polishchuk)

$\mathcal{D}(A) \simeq \mathcal{D}(A_L^{op}) \simeq \mathcal{W}(\Sigma, \Lambda; \eta)$