Hyperfinite families of modules and (dimensional) expansion

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Outline

Hyperfiniteness and Amenability

Definition

Let k be a field, A be a finite dimensional k-algebra and let $\mathcal{M}$ be a set of A-modules. $\mathcal{M}$ is called **hyperfinite** provided for every $\varepsilon > 0$ there exists a number $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there exists a submodule $P \subseteq M$ such that

$$\dim_k P \geq (1-\varepsilon)\dim_k M, \tag{1}$$

and modules $N_1, N_2, \dots N_t \in \operatorname{mod} A$, with $\dim_k N_i \leq L_\varepsilon$, such that $P \cong \bigoplus_{i=1}^t N_i$.
The k-algebra A is said to be of **amenable representation type** provided the set of all finite dimensional A-modules (or more specific, a set which meets any isomorphism class of finite dimensional A-modules) is hyperfinite.

Motivation

Conjecture (Elek '17)

Let k be a countable algebraically closed field and A be a finite dimensional algebra of infinite representation type over k. Then A is of tame representation type if and only if A is of amenable representation type.

Some (non-)examples

Example (finite representation type)

An algebra A of finite representation type is amenable.

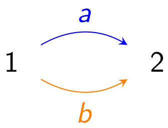
Theorem (Elek '17)

Let k be a countable field. Any string algebra R is of amenable representation type.

Theorem (Elek '17)

The wild Kronecker quiver algebras are not of amenable representation type.

The 2-Kronecker quiver



Let's make this explicit for an easy example.

Example

Let k be any field. Then the path algebra of the 2-Kronecker quiver is of amenable representation type.

Representations of the Kronecker quiver

Question

Given any ε, can we find L_ε such that for all finite dimensional Kronecker-modules M there is a submodule P with $\dim P \geq (1-\varepsilon) \dim M$ which decomposes into summands of dimension bounded by L_ε?

Luckily, there is an easy classification of Kronecker-modules:

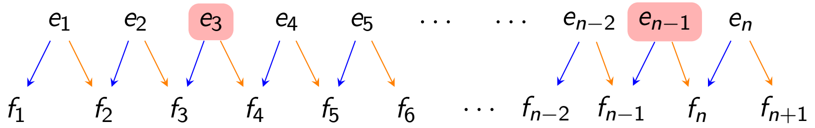
$$P_n\colon k^n \overset{\begin{bmatrix}\mathrm{id}\\0\end{bmatrix}}{\underset{\begin{bmatrix}0\\\mathrm{id}\end{bmatrix}}{\rightrightarrows}} k^{n+1}, \qquad Q_n\colon k^{n+1} \overset{[\mathrm{id}\ 0]}{\underset{[0\ \mathrm{id}]}{\rightrightarrows}} k^n, \qquad R_n(\phi,\psi)\colon k^n \overset{\phi}{\underset{\psi}{\rightrightarrows}} k^n,$$

where $\forall\, n \in \mathbb{N}$ either

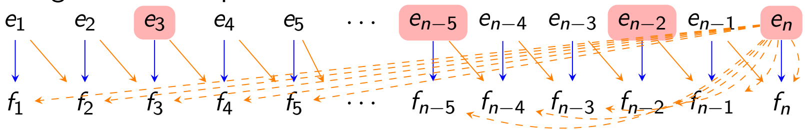
- $\phi = \mathrm{id}$ and ψ is companion matrix of power of monic irreducible over k, or
- $\psi = \mathrm{id}$ and ϕ is given by companion matrix of polynomial λ^m.

Finding a large submodule

for P_n:

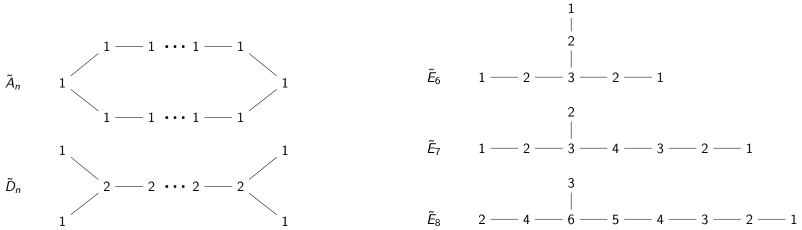


for regular indecomposables



- For the postinjective indecomposables, use the surjective map to the simple injective to find a submodule without postinjective summands.

Tame hereditary path algebras



Main Theorem

Let Q be an acyclic quiver of extended Dynkin type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Let k be any field. Then the path algebra kQ of Q is of amenable representation type.

Sketch of the proof

Recall $T^{\perp} = \{Y \in \text{mod } kQ \colon \operatorname{Hom}(T, Y) = 0 = \operatorname{Ext}^1(T, Y)\}$.

Proposition

Let Q be a quiver of tubular type (p, q, r), where $p > 1$. Let the extended Dynkin quiver of type $(p-1, q, r)$ be amenable. If T is an inhomogeneous simple regular module belonging to a tube of rank p in Γ_{kQ}, then $T^{\perp}$ is hyperfinite.

Q	$\widetilde{A}_{p,q}$	$\widetilde{D}_n$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
(m_i)	(p, q)	$(2, 2, n-2)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$

Kronecker quiver can serve as the base case of an induction-style argument.

Sketch of the proof

- If X is some indecomposable preprojective, pick a tube $\mathbb{T}$ of rank $p \geq 2$ (or maximal rank). Then either $X \in S^{\perp}$ for a regular simple S in $\mathbb{T}$ or we can find Y such that $0 \to Y \to X \to T \to 0$ is exact with $Y \in S^{\perp}$ for regular simples $S, T \in \mathbb{T}$.
- The indecomposable regular modules are either in $S^{\perp}$, via orthogonality of the tubes or have a submodule in $T^{\perp}$ for some regular-simple $T \in \mathbb{T}$.
- For the indecomposable postinjectives, we can do an induction on the defect, showing hyperfiniteness of the families $\mathcal{N}_d :=$ {indecomposable modules of defect $\leq d$}.

Going further

Using similar methods, we can prove the same result for all finite dimensional, tame hereditary algebras.

- Tame concealed works okay.
- There are partial results for tubular canonical algebras: preprojective, postinjective and integral slope modules
- One should try and do it for clannish algebras, as Elek did it for string algebras.

Hyperfiniteness

Definition (Elek)

Collection $\mathcal{G}$ of finite graphs is **hyperfinite** if $\forall \varepsilon > 0\ \exists K_\varepsilon$ finite s.t. $\forall G \in \mathcal{G}\ \exists S \subset E(G)$ s.t. $|S| \leq \varepsilon |V(G)|$ and every connected component of $G \setminus S$ has at most K_ε vertices.

Example

Linear/path graphs are hyperfinite.

Theorem (Lipton-Tarjan '80)

The set of planar graphs with maximal degree at most M is hyperfinite for every $M < \infty$.

Fragmentability

Definition (Edwards-McDiarmid)

Class $\mathcal{G}$ of graphs is fragmentable if $\forall \varepsilon > 0\ \exists n_0, c_\varepsilon \in \mathbb{N}^+$ s.t. $\forall G \in \mathcal{G}$ with $n \geq n_0$ non-isolated vertices $\exists S \subset V(G)$ with $|S| \leq \varepsilon n$ s.t. each connected component of $G \setminus S$ has at most c_ε vertices.

Corollary (Edwards-McDiarmid '94)

The following classes of graphs are fragmentable:

- *trees*
- *graphs of genus at most γ, for any fixed $\gamma \geq 0$*
- *rectangular lattices of dimension at most d, for fixed $d \in \mathbb{Z}$*

Expander Graphs

Definition

$G = (V, E)$ is an ε-**expander** if its Cheeger constant

$$h(G) := \min \left\{ \frac{|\partial A|}{|A|} : A \subseteq V, 0 < |A| \leq \frac{|V|}{2} \right\} \geq \varepsilon$$

for $\varepsilon > 0$, where $\partial(A)$ is the edge boundary of G. A family of (d-regular) graphs $\{G_N\}_{N \in \mathbb{S}}$ of size $|V(G_N)| = N$, $\mathbb{S} \subseteq \mathbb{N}$ infinite is a **family of expander graphs** if $\exists \varepsilon > 0$ s.t. $h(G_N) \geq \varepsilon \; \forall N \in \mathbb{S}$.

Example

The complete graph K_n on $n \geq 2$ vertices is an $\frac{n}{2}$-expander.

Remark

The spectral gap $d - \lambda_2$ (of the spectrum of a d-regular graph's adjacency matrix) yields an estimate on the expansion ratio $h(G)$.

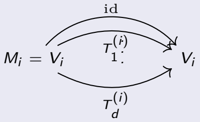
Dimension expanders and non-hyperfinite families

Definition (Barak-Impagliazzo-Shpilka-Wigderson)

k a field, $d \in \mathbb{N}$, $\alpha > 0$, V k-vector space, and $T_1, \dots, T_d$ k-linear endomorphisms of V. The pair $(V, \{T_i\}_{i=1}^d)$ is an α-**dimension expander of degree** d if $\forall W \subset V$ with $\dim W \leq \frac{\dim_k V}{2}$, we have $\dim_k \left(W + \sum_{i=1}^d T_i(W)\right) \geq (1+\alpha) \dim_k W$.

Proposition

k be a field, $d \in \mathbb{N}$ and $\alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^d)\}_{i \in I}$ is a sequence of α-dimension expanders of degree d s.t. $\dim V_i$ is unbounded, then the induced family of $k\Theta(d+1)$-modules



is not hyperfinite.

Constructing an example

Problem (Wigderson '04)

For fixed field k, fixed d, fixed α, find α-dim. expanders of degree d of arbitrarily large dimension.

Solutions

- Lubotzky-Zelmanov '08 for char $k = 0$
- for general k, reduction of Dvir-Shpilka '08/'11 shows that result of Bourgain '09/'13 on "monotone transformations with expansion property" solves it

Corollary

Let k a field, char $k = 0$. *Then the wild Kronecker algebra $K\Theta(3)$ is not of amenable representation type.*

Strictly wild algebras are not amenable

Definition

A f.d. k-algebra. A is **strictly wild** if $\exists$ orthogonal pair (X, Y) of f.d., f.p. modules, s.t. $\mathrm{End}(X)$, $\mathrm{End}(Y)$ are division rings and

$$p = \dim_{\mathrm{End}_A(Y)} \mathrm{Ext}^1_A(X, Y) \cdot \dim_{\mathrm{End}_A(X)} \mathrm{Ext}^1_A(X, Y) \geq 5.$$

Theorem

Let A be a finite dimensional k-algebra. If A is strictly wild, then A is not of amenable representation type.

Sketch of proof

Proposition

k a field, $L|k$ finite. A f.d. L-algebra, B f.d. k-algebras. $\{M_i\}_{i\in I} \subseteq \operatorname{mod} A$ non-hyperfinite family of modules. Let $K_1, K_2 > 0$. If $\forall i \in I$ $\exists$ additive functors $F_i\colon \operatorname{mod} A \to \operatorname{mod} B$, $G_i\colon \operatorname{mod} B \to \operatorname{mod} A$ s.t.

- $G_i F_i(M_i) \cong M_i$ *for all* $i \in I$,
- *all* G_i *are left exact,*
- $K_1 \dim_k F_i(M_i) \leq \dim_L G_i F_i(M_i)$ *for all* $i \in I$,
- $\dim_L G_i(X) \leq K_2 \dim_k X$ *for all* $X \in \operatorname{mod} B$ *and* $i \in I$,

then $\{F_i(M_i)\}_{i\in I}$ *is non-hyperfinite family.*

Lemma

Let A be a finite dimensional k-algebra and $d \geq 3$. If A is strictly wild, then there exists a finite field extension $L|k$ and an A-$L\Theta(d)$-bimodule M s.t. M is of finite L-dimension, projective as a $L\Theta(d)$-module and the functor $F = M \otimes_{L\Theta(d)} -\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ is full and faithful.

Proof of the Theorem.

The functor F above is fully faithful and has a right adjoint G. Dimension estimates work out nicely. Use the Proposition. □

A locally wild example

Theorem

The local wild algebra $A = k\langle x_1, x_2, x_3\rangle / M_2$, where M_2 is the ideal generated by the paths of length two, of dimension four with radical square zero, is not of amenable representation type.

Proof.

The functor $F\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(3)$, with $F(M) = \operatorname{top} M \overset{x_1\cdot -}{\underset{x_3\cdot -}{\xrightarrow{x_2\cdot -}}} \operatorname{rad} M$, is exact and preserves monomorphisms if we ignore simple modules. □

A problem?

Here, we use that A is a radical square zero algebra.
What functor should one use in general?
If the (restricted) functor is not left exact, there is not much hope of preserving being a submodule.

Modify the definition

Definition

k a field, A f.d. k-algebra, $\mathcal{M} \subseteq \operatorname{mod} A$ a family of f.d. A-modules. $\mathcal{M}$ is **weakly hyperfinite** if $\forall \varepsilon > 0 \exists L_\varepsilon > 0$ s.t. $\forall M \in \mathcal{M}\ \exists \theta\colon N \to M$ for some $N \in \operatorname{mod} A$ s.t.

$$\dim_k \ker \theta \leq \varepsilon \dim M, \quad \dim_k \operatorname{coker} \theta \leq \varepsilon \dim M, \tag{2}$$

and $\exists N_1, \ldots, N_t \in \operatorname{mod} A$ with $\dim_k N_i \leq L_\varepsilon$ s.t. $N \cong \bigoplus_{i=1}^t N_i$.
A k-algebra A has **weak amenable representation type** if $\operatorname{mod} A$ itself is a weakly hyperfinite family.

Remark

hyperfinite $\Rightarrow$ weakly hyperfinite

Dimension expanders are good enough

Proposition

k field, $d \in \mathbb{N}$, $\alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^{d})\}_{i \in I}$ is a sequence of α-dimension expanders of degree d s.t. $\dim V_i$ is unbounded, the induced sequence of $k\Theta(d+1)$-modules

$$M_i = V_i \xrightarrow[\;T_1^{(i)},\ \dots,\ T_d^{(i)}\;]{\mathrm{id}} V_i$$

is not weakly hyperfinite.

Finitely controlled wild algebras are not amenable

Let k be alg. closed.

Definition

An algebra A is **(finitely) controlled wild** if for any f.d. algebra B $\exists F : \text{mod } B \to \text{mod } A$ faithful exact and $C \in \text{mod } A$ s.t.

1. $\mathrm{Hom}_A(FM, FN) = F(Hom_B(M, N)) \oplus \mathrm{Hom}_A(FM, FN)_{\text{add } C}$, and
2. $\mathrm{Hom}_A(FM, FN)_{\text{add } C} \subseteq \text{rad}\, \mathrm{End}_A(FM)$.

Theorem

Let A be a finite dimensional k-algebra. If A is finitely controlled wild, then A is not of weakly amenable representation type.

Sketch of proof

Proof.

Use the functor $F\colon \operatorname{mod} k\Theta(d) \to A$ from the definition of controlled wildness. By [GP16, Theorem 4.2], $\exists G\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(d)$ s.t. $(G \circ F)(M) \cong M$ for all $M \in \operatorname{mod} k\Theta(d)$. Indeed, on objects this functor is given by

$$G(X) = {}^{\operatorname{Hom}_A(F(K),X)}\!\big/_{\operatorname{Hom}_A(F(K),X)_{\mathcal{C}}},$$

where $\operatorname{Hom}_A(X,Y)_{\mathcal{C}} = \{A\text{-homs } X \to Y \text{ factoring through } \mathcal{C}\}$. Remains to check estimates on dimensions. □

Bibliography I

.*Tame hereditary path algebras and amenability*. Aug. 2018. arXiv: 1808.02092 [math.RT].

.*(Extended) Kronecker quivers and amenability*. 2020. arXiv: 2011.02040 [math.RT].

Gábor Elek. "The combinatoral cost". In: *Enseign. Math. (2)* 53.3-4 (2007), pp. 225–235. ISSN: 0013-8584; 2309-4672/e.

Gábor Elek. "Infinite dimensional representations of finite dimensional algebras and amenability". In: *Math. Ann.* 369.1 (2017), pp. 397–439. ISSN: 0025-5831. DOI: 10.1007/s00208-017-1552-0.

Keith Edwards and Colin McDiarmid. "New upper bounds on harmonious colorings". In: *J. Graph Theory* 18.3 (1994), pp. 257–267. ISSN: 0364-9024. DOI: 10.1002/jgt.3190180305.

Lorna Gregory and Mike Prest. "Representation embeddings, interpretation functors and controlled wild algebras". In: *J. Lond. Math. Soc. (2)* 94.3 (2016), pp. 747–766. ISSN: 0024-6107. DOI: 10.1112/jlms/jdw055.

Bibliography II

Shlomo Hoory, Nathan Linial and Avi Widgerson. "Expander graphs and their applications". In: *Bull. Am. Math. Soc., New Ser.* 43.4 (2006), pp. 439–561. ISSN: 0273-0979; 1088-9485/e.

Richard J. Lipton and Robert Endre Tarjan. "Applications of a planar separator theorem". In: *SIAM J. Comput.* 9 (1980), pp. 615–627. ISSN: 0097-5397; 1095-7111/e.

Alexander Lubotzky and Efim Zelmanov. "Dimension expanders". In: *J. Algebra* 319.2 (2008), pp. 730–738. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2005.12.033.