BIREP SUMMER SCHOOL 2023
SUGGESTED TALKS

These are the suggested topics for the talks to be delivered by the participants at the BIREP Summer School on persistence modules. As there are more topics than slots for talks, the final program depends on the choices of the participants. Topics without a speaker are shaded in grey.

The talks with a star ($*$) are optional: the remaining talks without a star are obligatory, since they contain prerequisite material for other talks.

If you have a question or want to suggest a topic that would fit well with the program, send us an email to birep-school@math.uni-bielefeld.de.

Contents

1. Functor Categories

Talk 1 sets up the language of functor categories, to be used in the rest of the summer school. Below we follow Popescu [Pop73, §3.4], but there are other references.

(1) **Functor categories of posets.**
- For a commutative ring R define: *R-linear* categories (resp. functors); and the category $[\mathcal{C}, \mathcal{D}]$ of functors $\mathcal{C} \to \mathcal{D}$ when $\mathcal{C}$ is *small*. Explain why $[\mathcal{C}, \mathcal{D}]$ is R-linear when $\mathcal{D}$ is. Explain how to choose $\mathcal{C}, \mathcal{D}$ so that $[\mathcal{C}, \mathcal{D}] = R-\mathsf{Mod}$.
- Recall the notion of a *preordered set* [BdSS15, §3.1], characterising them in terms of categories which are small and *thin*.
- Write down what objects and morphisms are in the category $[\mathcal{C}, K-\mathsf{Mod}]$ for each of the following the preordered sets $\mathcal{C}$, where K is a field.
  - Any totally ordered set: compare $\mathbb{Z}$ and $\mathbb{R}$.
  - A product of a pair of totally ordered sets.
  - Any finite partially ordered set (consider the Hasse diagram).

(2) **Limits and colimits.**
- Explain how to compute limits and colimits in functor categories of the form $[\mathcal{C}, \mathcal{D}]$, assuming $\mathcal{D}$ has such co/limits (consider *currying*).
- Using the point above explain how to compute certain limits and colimits in $[\mathcal{C}, R-\mathsf{Mod}]$, such as: kernels, cokernels, products and coproducts.
- Recall what it means for a category to be *abelian*, and show that if $\mathcal{C}$ is small and R-linear category then $[\mathcal{C}, R-\mathsf{Mod}]$ is abelian.
- Roughly define *Grothendieck* abelian categories, using $R-\mathsf{Mod}$ as an example. Show that $[\mathcal{C}, \mathcal{D}]$ is Grothendieck if $\mathcal{D}$ is Grothendieck.

(3) **Persistence modules.** Follow (the start of §2 up until §2.1) in [BC20].
- Starting with *persistence modules*, introduce the terminology from [BC20, §2–§2.1]. For each term (e.g. *indecomposable* persistent module M) give a simple non-trivial example (e.g. where $M_x = k$ and $M_y = 0$ for $y \neq x$).
- Combine (1) and (2) to consider limits and colimits of persistence modules.

2. Representations of Finite Quivers and Posets.

Following Schiffler [Sch14], in Talk 2 *quiver representations* are discussed. In this language, or in that of functor categories, one can discuss *poset representations*. Quivers and posets here will be finite: in later talks they can be infinite.

(1) **Representation theory of quivers.**
   - Define the category of *representations* of a finite quiver, and the subcategory of finite-dimensional representations; see [Sch14, §1] and [Sch14, §3.4].
   - Define what it means for a representation of a quiver to be indecomposable, and state the *Krull–Schmidt* theorem for finite-dimensional quiver representations; see for example [Sch14, Theorem 1.2].

(2) **Representations of finite posets.**
   - Define the category of *representation* of a finite partially ordered set $\mathcal{C}$, as studied by Nazarova [Naz81]. Show how it is a subcategory of $[\mathcal{C}, K-\mathsf{Mod}]$.
   - Explain why the category of representations of a poset $\mathcal{C}$ is equivalent to a subcategory of representations of a finite quiver.

(3) **Representations of quivers of type $\mathbb{A}_n$.**
   - Recall what it means for a quiver to be of *Dynkin type* by recalling and drawing the Dynkin graphs. Give examples of quivers of type $\mathbb{A}_n$.
   - Explain how representations of a type-$\mathbb{A}_n$ quiver can be seen as persistence modules for a finite poset. Use an explicit (but small) example of a poset which is not totally ordered.
   - Recall *thin* representations, and the classification of finite-dimensional representations of a type $\mathbb{A}_n$ quiver. Comment on the proof by Ringel [Rin13].

(4) **Representation-finite quivers and posets.**
   - Define finite quivers which are of *finite representation type*. State [Sch14, Theorem 3.1 (Gabriel's Theorem, Part I)] on the $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ classification.
   - Define posets of *finite representation type*. State a characterisation of such posets by Kleĭner [Kle72, Theorem 1] (see also [Naz81, p. 346, Theorem 2]).

3. Persistence Theory of a Point Cloud

This talk covers the prototypical application of persistent homology to topological data analysis by [ELZ02].

- Define the distance function $f\colon \mathbb{R}^n \to \mathbb{R}$, $u \mapsto d(u, X)$ associated to a finite point cloud $X \subset \mathbb{R}^n$.
- Define sublevel set persistent homology $\mathrm{PH}_1\colon \mathbb{R} \to K-\mathsf{Mod}$ of f in terms of singular homology of sublevel sets as a functor from $\mathbb{R}$ to the category of vector spaces over K; see also [ELZ02, BS14].
- Define the *Voronoi diagram* of a point cloud $X \subset \mathbb{R}^n$ as a closed convex cover $\{\mathrm{Vor}(x, X)\}_{x\in X}$ of $\mathbb{R}^n$; see also [BE17, Section 3.3].
- Define the *Delaunay complex* $\mathrm{Del}(X)$ of a point cloud $X \subset \mathbb{R}^n$ as the nerve of the associated Voronoi diagram $\{\mathrm{Vor}(x, X)\}_{x\in X}$.
- Mention the associated geometric simplicial complex with ambient space $\mathbb{R}^n$ of $\mathrm{Del}(X)$ for a point cloud X in *general spherical position* [BE17, Definition 4.2].
- Define the linear map $\Gamma\colon |\operatorname{Sd}\mathrm{Del}(X)| \to \mathbb{R}^n$ from the barycentric subdivision $|\operatorname{Sd}\mathrm{Del}(X)|$ of $\mathrm{Del}(X)$ to $\mathbb{R}^n$ sending each simplex $\sigma \in \mathrm{Del}(X)$, seen as a vertex of the barycentric subdivision $\operatorname{Sd}\mathrm{Del}(X)$, to the center of the smallest enclosing sphere of $\sigma \subseteq X$; see also [BE17, Section 3.3] and [BKRR22, Section 3].
- Define the *Delaunay–Čech filtration* $\big(\mathrm{Del\check{C}ech}_r(X)\big)_{r\geqslant 0}$ of the Delaunay complex $\mathrm{Del}(X)$ as in [BE17, Section 3.3].
- Show that $\Gamma(|\operatorname{Sd}\mathrm{Del\check{C}ech}_r(X)|) \subseteq f^{-1}([0, r])$ for $r \geqslant 0$.
- Use the nerve theorem [BKRR22, Theorem 3.1] for the closed convex cover $\{\mathrm{Vor}(x, X) \cap f^{-1}([0, r])\}_{x\in X}$ to show that Γ induces a homotopy equivalence $\varphi_r\colon |\mathrm{Del\check{C}ech}_r(X)| \xrightarrow{\sim} f^{-1}([0, r])$, which is natural in $r \geqslant 0$;
- Define the persistent homology $\mathrm{PH}_2\colon \mathbb{R} \to K-\mathsf{Mod}$ of $\big(\mathrm{Del\check{C}ech}_r(X)\big)_{r\geqslant 0}$ in terms of simplicial homology.
- Use the homotopy equivalences φ_r, $r \geqslant 0$ in conjunction with the isomorphism between simplicial and singular homology to provide a natural isomorphism $\mathrm{PH}_2 \cong \mathrm{PH}_1$.
- Explain how PH_2 can be thought of as a representation of a type-$\mathbb{A}_n$ quiver.
- Introduce the *persistence barcode* as the multiset of indecomposable direct summands of PH_2.
- Provide an informal topological interpretation of the persistence barcode of PH_2 as 'features' that *persist* for a range of values $r \geqslant 0$.

4. Decompositions of Persistence Modules

In talk 4 a result of Botnan and Crawley-Boevey [BC20, Theorem 1.1] is discussed.

(1) **Background on ring and module theory.**
  - Recall language from noncommutative ring and module theory: *local* rings, *endomorphism rings*, *indecomposable* modules and the *length* of a module.
  - Characterise local rings in different ways, and state (without proof) *Fitting's lemma*. Note that local rings are *completely primary* as in [Azu48, p. 117].

(2) **Uniqueness and Krull–Remak–Schmidt–Azumaya's theorem.**
  - State (do not prove) Azumaya's generalisation of the Krull–Remak–Schmidt (KRS) theorem, seen in a previous talk; see condition $(*)$ and Theorem 1 in [Azu48, pp. 118–119]. Compare to [Sch14, Theorem 1.2], seen in Talk 2.
  - Explain that the Krull–Remak–Schmidt–Azumaya theorem works for *Grothendieck* categories; see for example [Par70, §4.8, Theorem, p. 193].

(3) **Arbitrary decompositions of persistence modules.**
  - Define *point-wise finite-dimensional* persistence modules. Using lines 8–27 on page 4585 of [BC20], show that the endomorphism ring of a point-wise finite-dimensional persistence module is a local ring.
  - State and complete the proof of Theorem 1.1 of [BC20], found in the remainder of §3 of [BC20], which asserts that any pointwise finite-dimensional persistence module is a direct sum of indecomposable persistence modules.
    - Use Fitting's lemma to prove that endomorphism rings of pointwise finite-dimensional persistence modules are local.
    - Recall *Zorn's lemma* as a set-theoretic foundation and define the set D to which it shall be applied.
    - Complete the proof, time permitted.

5. Representation Theory for Persistence Modules

In talk 5 the representation theory of persistence modules is discussed.

(1) **Interval representations.**
- Introduce the terminology given in §2.1 of [BC20] up to and including what an *interval* is. Define the *constant* representation k_I for a given interval I and explain how it defines a persistence module.
- For intervals I, J describe the morphisms $k_I \to k_J$ of persistence modules. Considering k_I in the category of persistence modules, prove that $\mathrm{End}(k_I) \simeq k$ and hence that k_I is indecomposable [BL18, Proposition 2.2].
- State [BC20, Theorem 1.2]. Define some interval representations for the totally ordered sets $\{1, \ldots, n\}$, $\mathbb{N}$, and $\mathbb{R}$.

(2) **Projectivity and injectivity.**
- Recall what it means for an object in a category $\mathcal{C}$ to be *projective*, and what it means to be *injective*. Consider the case when $\mathcal{C} = R-\mathsf{Mod}$.
- Show *representables* are projective in $[\mathcal{C}, K-\mathsf{Mod}]$. Explain why projectives are representable when $\mathcal{C}$ has *split idempotents*. See [BBOS20, Lemma 3.4].
- Survey results from [HL81] and [Höp83] in order to describe, when $\mathcal{C}$ is a poset, the indecomposable projective and indecomposable injective objects in $[\mathcal{C}, K-\mathsf{Mod}]$. Consider how these indecomposables parameterise injective and projective representations in general.

(3) **Non-totally ordered cases.**
- Draw a picture which summarises the idea of middle-exact persistence modules for a product of two totally ordered sets: see for example [BC20, §5].
- Likewise draw a picture conveying the idea of zig-zag persistence modules for a product of two totally ordered sets: see for example [BC20, §5].
- Mention in passing other situations which have been considered, for example, posets given on a circle $\mathbb{S}^1$; see for example [HR20].

6. Interleaving Distance of Persistence Modules

This talk introduces the interleaving distance of persistence modules by [CdSGO16, BS14] and provides a proof of the *categorical stability theorem* [BS14, Theorem 5.1]. The following notions and results should be covered in this talk:

- Define the notion of a *δ-interleaving*; see [BS14, Definition 3.1].
- Prove the *monotonicity* of δ-interleavings, see [BS14, Lemma 3.4].
- Define the *interleaving distance* of persistence modules; see [BS14, Definition 3.2].
- Show that the interleaving distance is an extended pseudometric on persistence modules; see [BS14, Theorem 3.3].
- Provide a proof of the *categorical stability theorem*; see [BS14, Theorem 5.1].

7. The Algebraic Stability Theorem

The goal of this talk is to provide a proof of the *Algebraic Stability Theorem* [BL20, BL15] (optionally by using the results covered in Talk 4) thereby also proving the stability of the persistence barcode in conjunction with the results covered in Talk 6. As an example, here is a plan following [BL20]:

- Define the *category of barcodes*; see [BL20, Section 2.3].
- Define and characterize kernels, cokernels, and images of barcodes; see [BL20, Section 2.5].
- Define and characterize δ-trivial kernels and cokernels of barcodes; see [BL20, Definition 1 and Proposition 2].
- Show how a monomorphism of persistence modules induces a matching of persistence barcodes; see [BL20, Proposition 3].
- Explain how duality of finite-dimensional vector spaces and [BL20, Proposition 3] yield a dual result for epimorphisms of pfd persistence modules.
- Provide a characterization of homomorphisms of persistence modules with δ-trivial (co)kernel; see [BL20, Lemma 1].
- Provide a proof of the *Induced Matching Theorem* using the results covered in Talk 4; see [BL20, Section 3.4].
- Provide a proof of the *Algebraic Stability Theorem*; see [BL20, Section 4.2].

8. Persistence Modules for Totally Ordered Sets*

Theorem 1.2 of [BC20] gives a decomposition theorem for persistence modules defined on a totally ordered set, discussed in detail in talk 8.

(1) **Filtered limits.**
- Recall the notions of a *filtered category* and of a *filtered limit*. Explain why filtered colimits of exact sequences (of, say, R-modules) remain exact.
- Recall the notion of a *directed category* and a *directed limit*. Recall (do not prove) a result of Adamek and Rosicky: the first Corollary in [AR94, p. 15].

(2) **Directed ideals.**
- Recall the constant representation modules from talk 5, explain why directed ideals are filtered as categories and state and prove [BC20, Lemma 2.1].
- Define *codirected ideals*, and state and prove [BC20, Lemma 2.2].
- Recall the *hom-dual* of a vector space. Use the notion of the hom-dual, and the points above, to state and prove [BC20, Lemma 2.3].
- State and prove [BC20, Theorem 1.2] (which is in §4).

(3) **Example: the real line.**
- Give an example of an infinite direct sum of interval representations for $\mathbb{R}$ which is pointwise finite-dimensional.
- Using Remark at the end of §2 on page 19 in the book by Oudot [Oud15], comment on how persistence modules relate to modules over path algebras.

9. Levelsets Zigzag Persistent Homology

Ordinary persistent homology, as covered in Talk 3, can be understood as a particular type of invariant of $\mathbb{R}$-indexed filtrations, possibly arising as the sublevel sets of a function. Now provided that we have a function, one might also consider other invariants of functions retaining more information about the function than ordinary persistent homology. In the discretized setting one of such strengthening of persistent homology is *levelsets zigzag persistent homology* by [CdM09]. This talk introduces levelsets zigzag persistent homology and also draws the connection to ordinary persistent homology through *up-down persistent homology* or *extended persistent homology* by [CSEH09]. One key ingredient for making these connections is the *diamond principle*; see [CdM09, CdSKM19, BEMP13]. An overview of different variants of persistent homology applicable to functions and the main aspects relevant to this talk is provided in [BN22].

10. Generalizations of the Interleaving Distance*

The speaker may provide an overview on generalizations of the interleaving distance, or discuss one of [BdSS15, dSMS18, Sco19] in detail, based on personal preference.

As an example, a plan following [BdSS15] is given below.

(1) **Generalisations of persistence modules**
 - Recall categories given by preordered sets from Talk 1. Characterise functors between them, and explain the Thin lemma [BdSS15, Lemma 3.1].
 - Define the preordered monoid of *translations* in the sense of [BdSS15, §3.2], consider an example, and define what it means for there to be an *interleaving* between a pair of persistence modules [BdSS15, Definition 3.4].

(2) **Sublinear projections.**
 - State results concerning *functoriality monotonicity* and the *triangle inequality*; [BdSS15, Propositions 3.9, 3.10 and 3.11]. Prove the triangle inequality.
 - Define *sublinear projections*, and for $\varepsilon \geqslant 0$ define what it means for: a translation to be an ε-*translation*; and a pair of persistence modules to be ε-*interleaved*. Explain these terms with a simple non-trivial example.
 - Recall the notion of an *extended psuedo-metric*, and define the interleaving distance between persistence modules. Prove [BdSS15, Theorem 3.15].

(3) **Superlinear projections.**
 - Introduce the language of *superlinear projections* and an adjusted definition of the interleaving distance [BdSS15, Definition 3.20].
 - State [BdSS15, Theorems 3.21, 3.24] explaining roughly what they mean.

11. Sheaves and Persistence Theory

The foundations of sheaf-theoretical persistence theory are laid out in [Cur14, KS18]. This talk provides the necessary background from sheaf theory as well as the sheaf-theoretical version of level set persistence.

- Define the notion of a sheaf on a topological space taking values in the category of vector spaces over some field, mention locally constant and skyscraper sheaves as examples, and characterize isomorphisms of sheaves as those homomorphisms inducing isomorphisms on all stalks; see for example [KS90, Section 2.2].
- Define direct images and pullbacks of sheaves; see for example [KS90, Section 2.3].
- Introduce homotopies and homotopy equivalences between complexes of sheaves.
- Introduce the notion of a quasi-isomorphism between complexes of sheaves and the derived category of sheaves.
- Introduce derived levelset persistence of a function $f\colon X \to \mathbb{R}$ as the right-derived pushforward of the sheaf of locally constant functions on X along f.
- Extend this construction to a contravariant functor from $\mathbb{R}$-spaces to derived sheaves on the real numbers using the adjunction between direct image and pullback; see [KS90, Equations (2.7.3) and (2.7.4)].

12. Decompositions of Sheaves*

This talk connects the sheaf-theoretical approach to levelset persistence from Talk 11 and the discrete approach from Talk 9 and also provides a structure theorem for *constructible* sheaves on the reals. As a note of caution, there are two different notions of a constructible sheaf on the real numbers appearing in the outline below.

- Introduce the full subcategory of sheaves on the reals that are *definable* in the sense of [Cur14, Definition 15.3.1] or equivalently *constructible* in the sense of [Gui16, Section 7] and draw the connection to type-$\mathbb{A}_n$ quiver representations.
- Explain how the classification of finite-dimensional type-$\mathbb{A}_n$ quiver representations covered in Talk 2 yields a structure theorem for definable sheaves on the reals; see [Cur14, Section 15.3] or [Gui16, Corollary 7.3].
- Introduce *constructible* derived sheaves on a simplicial complex taking values in the category of vector spaces over some field; see [KS90, Section 8.1].
- Mention the equivalence of constructible derived sheaves and the derived category of constructible sheaves; see [KS90, Theorem 8.1.11].
- As a special case of a constructible sheaf on a simplicial complex introduce the notion of an $\mathbb{R}$-*constructible* sheaf on the reals in the sense of [KS18, Section 1.5].
- Explain how the structure theorem for the first notion of a constructible sheaf on the reals yields a structure theorem for $\mathbb{R}$-*constructible* sheaves on the reals; see [KS18, Theorem 1.17].
- Show that the category of $\mathbb{R}$-constructible sheaves on the reals is hereditary; see [KS18, Corollary 1.18].
- Mention that any object of a derived category of a hereditary category decomposes into its cohomology objects and use this in conjunction with the equivalence between constructible derived sheaves and the derived category of constructible sheaves to conclude with a structure theorem for $\mathbb{R}$-constructible derived sheaves; see also [KS18, Corollary 1.20].

13. Multidimensional Persistence

Provide an introduction to multidimensional persistence theory [CZ07, BL22].

(1) **Multifiltered spaces.**
  - Explain how the relation $\precsim$ on $\mathbb{N}^n$ from [CZ07, §2], restricted to any *multiset* as in [CZ07, §4.1] gives a *quasi-partial order*, which is the same thing as a preordered set in the sense of [BdSS15, §3.1] seen in talks 1 and 10.
  - Hence explain how the corresponding notion of a persistence module [CZ07, Definition 1] is a special case of [BdSS15, §2.2].
  - Give the definition of a *multifiltered space* [CZ07, §3], explain the bifiltered triangle [CZ07, Fig. 2] and sketch details for a different example.

(2) **Graded ring theory and the correspondence theorem.**
  - Introduce, in a minimal way, preliminaries of graded rings, the example $A_n = k[x_1, \ldots, x_n]$ and graded modules from the end of [CZ07, §2]. Give examples and non-examples for $n = 2$.
  - Give the construction of the correspondence α from [CZ07, Definition 2]. For $n = 2$ calculate $\alpha(M)$ where M is given by the bifiltered triangle.
  - State the *Correspondence theorem* [CZ07, Theorem 1] and the *Realization theorem* [CZ07, Theorem 2]. Consider what these say when $n = 1, 2$.

(3) **Representation theory of the grid poset.**
  - Explain roughly the meaning of *tame representation type* by considering the Jordan-block classification of matrices up to simultaneous similarity, and then consideringer representations of the *Kronecker quiver*.
  - Loosely explain what *wild* representation type is: [BL22, Definition 8.7(iii)]
  - Explain [BL22, Theorem 8.9] which characterises grid posets in terms of their *representation type*. Draw three different grids displaying finite, tame and wild representation types.
  - Explain Figure 8.1 and the diagram (8.2) from [BL22].

14. Signed Barcodes for Multidimensional Persistence*

This talk introduces the signed barcode for multi-parameter persistence modules [MP22, BOO22] and discusses one or more aspects such as their rank decompositions [BOO22], effective computation [CGR$^+$22], relative homological algebra [BOO22, BBH21, CGR$^+$22], or one of the approaches to obtain algebraic stability for signed barcodes: [MP22, OS21, BOOS22].

15. Saecular Persistence*

Most decomposition results in persistence theory assume field coefficients. Saecular persistence generalizes some of these results to persistence modules taking values in categories other than the category of vector spaces over some field; see [GH21].

16. Computation of Persistent Homology*

There are several methods to compute persistent homology [OPT+17, BDM13]. The speaker may provide an overview or discuss one of these in detail based on personal preference.

17. Application: Clustering with *ToMATo* or *Persistable**

Persistent homology can help to prove clustering algorithms. In [COSG11] a method is discussed how persistent homology in dimension 0 can be used to improve a classical mode-seeking clustering approach and how persistence diagrams then help with finding an adequate clustering among many possible. Persitable, the theory of which is developed in [RS20], is a more recent approach to clustering related to ToMATo.

18. Clustering and Tilting Theory*

Ordinary persistent homology as covered in Talk 3 is not robust with respect to outliers. One approach to overcome this limitation is to filter by distance as well as some form of density, effectively obtaining a filtration in two parameters. However, two-parameter persistence modules are far more complex than one-parameter persistence modules. In degree 0, the internal maps in the direction of increasing distance and constant density are surjective. Thus, one may hope that such two-parameter persistence modules are easier to classify. However, the article by [BBOS20] shows that such persistence modules are of similar complexity.

In talk 18 a connection between clustering, tilting theory and persistent modules is described, following [BBOS20].

(1) **Subcategories of grid representations.**
  - Recall the poset defined by the product of two finite totally ordered sets. Define the subcategory of representations of this poset with additional conditions on the horizontal or vertical morphisms; see [BBOS20, p. 7].
  - Use [BBOS20, Construction 1.4] to explain how to pass between the subcategories discussed above and representations of grids of smaller size.
  - Recall [BBOS20, Theorem 1.3], already seen in talk 13. Combine this description with [BBOS20, Theorem 1.5] to state and prove [BBOS20, Corollary 1.6].

(2) **Torsion theory.**
  - Recall what it means for a category to have *enough projective objects* or *enough injective objects*.
  - Define *torsion pairs* for an abelian category [BBOS20, Definition 2.3], and discuss [BBOS20, Example 2.5], which considers torsion pairs for the abelian category of persistence modules (see Talk 1(2)).
  - Define *cotorsion pair* for an abelian category [BBOS20, Definition 2.6], and discuss [BBOS20, Example 2.8], which considers cotorsion pairs for the abelian category of persistence modules.
  - State [BBOS20, Theorem 3.12, Corollaries 3.14 and 3.15].
  - Explain the results discussed so far for [BBOS20, Examples 3.19 and 3.20].

19. Application: Complex Networks and Dynamic State Detection*

In [MMK19] a new method is presented to study graph representations of time series of dynamical systems. In particular it is shown how persistent homology allows to distinguish between periodic and chaotic behaviour.

20. Middle Exactness and Block Decomposition*

Talk 9 covers level set zigzag persistent homology, which is an invariant retaining more information than sublevel set persistent homology, both on the level of objects as well as on the level of homomorphisms. Another invariant retaining yet more information on the level homomorphisms (but not on the level of objects) is *interlevel set persistent homology*; see [CdM09, BEMP13, BL18, CdSKM19, BGO19, CO20, BC20, BL22, Section 10.2]. First explain how interlevel set persistence yields *middle-exact* representations, see for example [CO20, Section 9.3]. Optionally, also explain how interlevel set persistent homology retains more information on the level of homomorphisms. Then discuss the block decomposition of interlevel set persistent homology as in [BC20, Section 5].

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