COMPLEXITY AND KRULL DIMENSION

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Let k be an algebraically closed field of characteristic $p > 0$. In this lecture we want to take a first glance at the geometric approach towards the modular representation theory of finite groups. The results presented here also hold in the wider context of finite group schemes, mainly because the fundamental result, Theorem 1, concerning the cohomology ring also holds in that generality.

In the following, G denotes a finite group with group algebra kG. Let $M \in \operatorname{mod} kG$ be a finite dimensional G-module, $\mathcal{P} := (P_i)_{i \geq 0}$ be a minimal projective resolution of M. Then

$$\operatorname{cx}_G(M) := \min\{s \in \mathbb{N} \cup \{\infty\}\ ;\ \exists\ \lambda > 0 \text{ such that } \dim_k P_n \leq \lambda n^{s-1} \quad \forall\ n \geq 1\}$$

is called the *complexity* of M. This notion, first introduced by Alperin [1] and further developed in [2], has proven to be an effective tool in the modular representation theory of finite groups.

Example. Consider the group algebra

$$k(\mathbb{Z}/(p) \times \mathbb{Z}/(p)) \cong k[X,Y]/(X^p, Y^p) \cong k[X]/(X^p) \otimes_k k[Y]/(Y^p).$$

Let $\mathcal{P} = (P_i)_{i \geq 0}$ be a minimal projective resolution of the trivial $k[X]/(X^p)$-module k, i.e., $P_i = k[X]/(X^p)$ for every $i \geq 0$. Setting $Q_i := \sum_{j=0}^{i} P_j \otimes_k P_{i-j}$, we obtain a minimal projective resolution $\mathcal{Q} := (Q_i)_{i \geq 0}$ of the $k(\mathbb{Z}/(p) \times \mathbb{Z}/(p))$-module $k \otimes_k k \cong k$. Since $\dim_k Q_i = (i+1)p^2$, we have $\operatorname{cx}_{\mathbb{Z}/(p) \times \mathbb{Z}/(p)}(k) = 2$. One can iterate this process to see that $c_{(\mathbb{Z}/(p))^r}(k) = r$.

Let G be a finite group. If M is a G-module, we denote by

$$H^n(G, M) := \operatorname{Ext}^n_G(k, M) \qquad (n \geq 0)$$

the n-th cohomology group of G with coefficients in M. Note that these are just the Hochschild cohomology groups of the augmented algebra (kG, ε).

Given three G-modules X, Y, Z, we recall the *Yoneda product*

$$\operatorname{Ext}^m_G(Y, Z) \times \operatorname{Ext}^n_G(X, Y) \longrightarrow \operatorname{Ext}^{m+n}_G(X, Z).$$

This product endows $\operatorname{Ext}^*_G(X, X) := \bigoplus_{n \geq 0} \operatorname{Ext}^n_G(X, X)$ with the structure of a $\mathbb{Z}$-graded k-algebra. Moreover, the spaces $\operatorname{Ext}^*_G(Y, X)$ and $\operatorname{Ext}^*_G(X, Y)$ are graded left and right $\operatorname{Ext}^*_G(X, X)$-modules, respectively. In particular, $H^*(G, M)$ is a graded right module over the *cohomology ring* $H^*(G, k)$. This ring is known to be *graded commutative*, that is,

$$yx = (-1)^{\deg(x)\deg(y)} xy$$

for any two homogeneous elements $x, y \in H^*(G, k)$. Consequently, the subring

$$H^\bullet(G, k) = \bigoplus_{i \geq 0} H^{2i}(G, k)$$

is a commutative, $\mathbb{Z}$-graded k-algebra (see [4, §6] for details).

The main result to be discussed in this lecture is the following:

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Theorem. [3] *Let M be a G-module. Then there exists an ideal $I_M \subset H^\bullet(G,k)$ such that*

$$\mathrm{cx}_G(M) = \dim H^\bullet(G,k)/I_M.$$

To understand this result, we need to see how the Krull dimension $\dim A$ of a commutative graded ring $A = \bigoplus_{n\geq 0} A_n$ is related to its growth.

Let $(a_i)_{i\geq 0}$ be a sequence of natural numbers. We call

$$\gamma((a_i)_{i\geq 0}) := \min\{s \in \mathbb{N}\cup\{\infty\} \ ; \ \exists\ \lambda > 0 \text{ such that } a_n \leq \lambda n^{s-1} \quad \forall\ n \geq 1\}$$

the *rate of growth* of the sequence $(a_i)_{i\geq 0}$. If $\mathcal{V} := (V_i)_{i\geq 0}$ is a sequence of finite dimensional k-vector spaces, then we write $\gamma(\mathcal{V}) := \gamma((\dim_k V_i)_{i\geq 0})$. Thus, if $\mathcal{P} := (P_i)_{i\geq 0}$ is a minimal projective resolution of a G-module M. Then

$$\mathrm{cx}_G(M) = \gamma(\mathcal{P}).$$

Let

$$A = \bigoplus_{n\geq 0} A_n$$

be a finitely generated, commutative graded k-algebra. We want to find the growth $\gamma(A)$ of the sequence $(A_n)_{n\geq 0}$. By the Noether Normalization Lemma there exists a graded subalgebra

$$R = \bigoplus_{n\geq 0} R_n$$

of A such that

(a) A is a finitely generated R-module, and
(b) $R \cong k[X_1,\dots,X_\ell]$, where $\deg(X_i) = d$ for some $d \geq 1$.

Owing to (a) we have $\gamma(R) = \gamma(A)$. Condition (b) implies $\gamma(R) = \ell$. On the other hand, the number ℓ is the Krull dimension $\dim R$ of R, which, by Cohen-Seidenberg theory (cf. [6, §9]), coincides with $\dim A$. We therefore obtain

$$\gamma(A) = \dim A.$$

The following fundamental result provides the finitely generated commutative k-algebra we want to work with:

Theorem 1 ([7, 5])**.** *Let G be a finite group, M a finite dimensional G-module. Then the following statements hold:*

(1) *$H^\bullet(G,k)$ is a finitely generated k-algebra.*
(2) *$H^*(G,M)$ is a finitely generated $H^\bullet(G,k)$-module.* □

Let M be a finite dimensional G-module, $(P_i,\partial_i)_{i\geq 0}$ be a projective resolution of the trivial module k. Since $(P_i \otimes_k M, \partial_i \otimes \mathrm{id}_M)_{i\geq 0}$ is a projective resolution of M, we obtain a homomorphism

$$\Phi_M : H^\bullet(G,k) \longrightarrow \mathrm{Ext}^*_G(M,M) \ \ ; \ \ [f] \mapsto [f \hat{\otimes} \mathrm{id}_M]$$

of graded k-algebras. The natural equivalence

$$\mathrm{Hom}_G(M,N) \cong \mathrm{Hom}_G(k, \mathrm{Hom}_k(M,N))$$

gives rise to $\mathrm{Ext}^*_G(M,M) \cong H^*(G, \mathrm{Hom}_k(M,N))$. Thus, Theorem 1 says that the map Φ_M endows the Yoneda algebra with the structure of a finitely generated $H^\bullet(G,k)$-module. Moreover, our map Φ_M is induced by the canonical homomorphism $k \longrightarrow \mathrm{Hom}_k(M,M)$ sending α to $\alpha\,\mathrm{id}_M$.

The following result, due to Alperin-Evens [2], relates the complexity of a module to the growth of certain Ext-groups. We let $\mathcal{S}$ be a complete set of representatives of the simple G-modules, and denote the projective cover of $S \in \mathcal{S}$ by $P(S)$.

Proposition 2. *Let M be a finite dimensional G-module. Then*
$$\mathrm{cx}_G(M) = \max_{S\in\mathcal{S}} \gamma((\mathrm{Ext}^n_G(M,S))_{n\geq 0}).$$

Proof. Given a minimal projective resolution $(P_n)_{n\geq 0}$ of M, we decompose each P_n into its indecomposable constituents and write $P_n = \bigoplus_{T\in\mathcal{S}} \ell_{n,T} P(T)$. Basic properties of Ext and Schur's Lemma yield
$$\dim_k \mathrm{Ext}^n_G(M,S) = \sum_{T\in\mathcal{S}} \ell_{n,T} \dim_k \mathrm{Hom}_G(P(T),S) = \ell_{n,S} \dim_k \mathrm{Hom}_G(S,S) = \ell_{n,S}$$
for every $S \in \mathcal{S}$. Consequently,
$$\mathrm{cx}_G(M) = \max_{S\in\mathcal{S}} \gamma((\ell_{n,S})_{n\geq 0}) = \max_{S\in\mathcal{S}} \gamma((\mathrm{Ext}^n_G(M,S))_{n\geq 0}),$$
as desired. □

We now turn to the proof of our Theorem:

Proof. Thanks to Theorem 1, the Yoneda algebra $\mathrm{Ext}^*_G(M,M)$ is a finitely generated $H^\bullet(G,k)$-module. Consequently, we have
$$\dim H^\bullet(G,k)/\ker\Phi_M = \gamma(H^\bullet(G,k)/\ker\Phi_M) = \gamma(\mathrm{Ext}^*_G(M,M)) \leq \max_{S\in\mathcal{S}} \gamma(\mathrm{Ext}^*_G(M,S)).$$
In view of Proposition 2, the latter number coincides with $\mathrm{cx}_G(M)$.

To verify the reverse inequality, we let S be a simple G-module. Owing to Theorem 1, the space $\mathrm{Ext}^*_G(M,S) \cong H^*(G,\mathrm{Hom}_k(M,S))$ is a finitely generated $(H^\bullet(G,k)$-module. Since this action is induced by the scalar multiplication of k on $\mathrm{Hom}_k(M,S)$, the identity
$$\alpha f = \alpha f \circ \mathrm{id}_M = f \circ (\alpha\, \mathrm{id}_M) \quad \forall\ f \in \mathrm{Hom}_k(M,S),\ \alpha \in k$$
implies that it factors through the right action of $\mathrm{Ext}^*_G(M,M)$ on $\mathrm{Ext}^*_G(M,S)$. As a result, $\mathrm{Ext}^*_G(M,S)$ is a finitely generated $(H^\bullet(G,k)/\ker\Phi_M)$-module, whence
$$\gamma(\mathrm{Ext}^*_G(M,S)) \leq \gamma(H^\bullet(G,k)/\ker\Phi_M).$$
Another application of Proposition 2 now yields $\mathrm{cx}_G(M) \leq \dim H^\bullet(G,k)/\ker\Phi_M$, as desired. □

Example. Suppose that $p \geq 3$. The Künneth formula furnishes an isomorphism $H^*((\mathbb{Z}/(p))^n,k) \cong k[X_1,\dots,X_n]\otimes_k \Lambda(Y_1,\dots,Y_n)$, where the generators X_i and Y_i have degrees 2 and 1, respectively (cf. [4, (7.6)]). Consequently, $k[X_1,\dots,X_n]$ is a Noether normalization of $H^\bullet((\mathbb{Z}/(p))^n,k)$, and $\mathrm{cx}_{(\mathbb{Z}/(p))^n}(k) = n$. Note that this agrees with our earlier observations.

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