HOPF MODULES AND INTEGRALS: THE FUNDAMENTAL THEOREM

ROLF FARNSTEINER

In our discussion of Frobenius algebras [2], we mentioned finite dimensional Hopf algebras as an important class of examples. This non-trivial fact is actually a consequence of a theorem by Larson and Sweedler [3], who established a structural result for the so-called *Hopf modules*.

Throughout, H will denote a Hopf algebra, defined over a field k. We let $\Delta : H \longrightarrow H \otimes_k H$, $\eta : H \longrightarrow H$, and $\varepsilon : H \longrightarrow k$ be the comultiplication, the antipode, and the counit of H, respectively.

Hopf modules will be employed to establish the validity of a necessary condition for a finite dimensional H to be a Frobenius algebra. By general theory, the trivial H-module k, defined by ε should occur as a (simple) submodule of H of multiplicity 1. In other words, the space

$$\int_H^\ell := \{x \in H \ ; \ hx = \varepsilon(h)x \ \ \forall \ h \in H\}$$

of *left integrals of* H ought to be one-dimensional. Application to the dual Hopf algebra then yields the existence of a non-degenerate associative bilinear form.

The example of the polynomial ring $k[X]$ in one variable shows that the space of integrals may be trivial for Hopf algebras of infinite dimension. The existence of nonzero integrals for finite dimensional Hopf algebras was first proved by Sweedler [5].

When working with Hopf algebras, it is convenient to use the so-called *Heyneman-Sweedler notation* or *sigma notation*. This takes some getting used to and the advantages may not be obvious at first sight. For an element $h \in H$, we write

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

Iterated coproducts are denoted by

$$[(\mathrm{id}_H \otimes \Delta) \circ \Delta](h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = [(\Delta \otimes \mathrm{id}_H) \circ \Delta(h)](h).$$

Since the antipode η is an anti-homomorphism of coalgebras, we obtain

$$\Delta(\eta(h)) = \sum_{(h)} \eta(h_{(2)}) \otimes \eta(h_{(1)}).$$

We also recall one of the defining properties of η:

$$(*) \qquad \sum_{(h)} \eta(h_{(1)})h_{(2)} = \varepsilon(h)1 = \sum_{(h)} h_{(1)}\eta(h_{(2)}).$$

More details concerning this notation can be found in [4, Chap. I] or [1, Chap. 2, Sec. 1].

Definition. A k-vector space M together with a linear map $\Delta_M : M \longrightarrow H \otimes_k M$ is called a left *H-comodule* if

(1) $(\Delta \otimes \mathrm{id}_M) \circ \Delta_M = (\mathrm{id}_H \otimes \Delta_M) \circ \Delta_M$, and
(2) $(\varepsilon \bar{\otimes} \mathrm{id}_M) \circ \Delta_M = \mathrm{id}_M$.

Date: April 5, 2006.

Given $f \in H^*$, the notation $f \bar{\otimes} \mathrm{id}_M$ refers to the composition of the tensor product $f \otimes \mathrm{id}_M$ with the canonical identification $k \otimes_k M \cong M$. Thus, writing

$$\Delta_M(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)},$$

property (2) reads

$$m = \sum_{(m)} \varepsilon(m_{(0)}) m_{(1)} \quad \forall\, m \in M.$$

Our definition dualizes the notion of an H-module. In fact, any left H-comodule M gives rise to a right H^*-module M by defining

$$m.f := \sum_{(m)} f(m_{(0)}) \otimes m_{(1)} \quad \forall\, f \in H^*, \quad m \in M.$$

We want to consider H-modules which are simultaneously H-comodules and whose structure maps are morphisms in the relevant categories.

Definition. Let H be a Hopf algebra over k. A k-vector space M is called a (left) *Hopf module* if

(1) M is a left H-module, and
(2) M is a left H-comodule, and
(3) $\Delta_M(h.m) = \sum_{(h),(m)} h_{(1)} m_{(0)} \otimes h_{(2)}.m_{(1)} \quad \forall\, h \in H,\ m \in M$.

The last identity is easier to digest if we recall that the tensor product $M \otimes_k N$ of two H-modules M and N obtains the structure of an H-module via

$$h.(m \otimes n) := \sum_{(h)} h_{(1)}.m \otimes h_{(2)}.n \quad \forall\, h \in H,\ m \in M,\ n \in N.$$

Then (3) simply means that Δ_M is H-linear. Here is an

Example. Let M be a k-vector space, and define on $H \otimes_k M$ the following structure

$$h'.(h \otimes m) := h'h \otimes m \quad ; \quad \Delta_{H \otimes_k M}(h \otimes m) = \Delta(h) \otimes m \quad \forall\, h, h' \in H,\ m \in M.$$

Then M is an H-module and an H-comodule, and condition (3) follows from $\Delta : H \longrightarrow H \otimes_k H$ being a homomorphism of k-algebras. Hopf modules of this type are called *trivial Hopf modules.*

Given a Hopf module M, we denote by

$$M^{\mathrm{co}\, H} := \{m \in M\ ;\ \Delta_M(m) = 1 \otimes m\}$$

the space of *coinvariants* of M.

The following result, customarily referred to as the *Fundamental Theorem of Hopf modules*, states that every Hopf module is trivial.

Theorem ([3], Prop. 1)**.** *Let M be a left Hopf module. Then the restriction*

$$\mu : H \otimes_k M^{\mathrm{co}\, H} \longrightarrow M \quad ; \quad h \otimes m \mapsto h.m$$

of the multiplication is an isomorphism of Hopf modules. In particular, M is a trivial Hopf module and a free H-module.

Proof. Direct computation shows that μ is a homomorphism of Hopf modules (i.e., μ is H-linear and H-colinear). Defining $\varphi(m) := \sum_{(m)} \eta(m_{(0)}).m_{(1)}$ for every $m \in M$, we claim that

$$\Phi : M \longrightarrow H \otimes_k M^{\mathrm{co}\,H} \quad ; \quad m \mapsto \sum_{(m)} m_{(0)} \otimes \varphi(m_{(1)})$$

is the inverse map.

Given $m \in M$, we first show that $\varphi(m)$ is a coinvariant of the comodule M. Observing $(*)$ we have

$$\begin{aligned}\Delta_M(\varphi(m)) &= \sum_{(m)} \Delta_M(\eta(m_{(0)}).m_{(1)}) = \sum_{(m)} \eta(m_{(1)})m_{(2)} \otimes \eta(m_{(0)}).m_{(3)} \\ &= \sum_{(m)} \varepsilon(m_{(1)})1 \otimes \eta(m_{(0)}).m_{(2)} = 1 \otimes \sum_{(m)} \varepsilon(m_{(1)})\eta(m_{(0)}).m_{(2)} \\ &= 1 \otimes \sum_{(m)} \eta(m_{(0)}).m_{(1)} = 1 \otimes \varphi(m).\end{aligned}$$

Let $m \in M$ be an element. Directly from the definitions, we obtain

$$\mu(\Phi(m)) = \sum_{(m)} m_{(0)}.\varphi(m_{(1)}) = \sum_{(m)} m_{(0)}\eta(m_{(1)}).m_{(2)} = \sum_{(m)} \varepsilon(m_{(0)}).m_{(1)} = m.$$

Moreover, if $m \in M^{\mathrm{co}\,H}$ is a coinvariant and $h \in H$, then

$$\Phi(\mu(h \otimes m)) = \Phi(h.m) = \sum_{(h),(m)} h_{(1)}.m_{(0)} \otimes \eta(h_{(2)}m_{(1)})h_{(3)}.m_{(2)}.$$

Since $\Delta_M(m) = 1 \otimes m$, the last expression simplifies to

$$\sum_{(h)} h_{(1)} \otimes \eta(h_{(2)})h_{(3)}.m = \sum_{(h)} h_{(1)} \otimes \varepsilon(h_{(2)}).m = h \otimes m,$$

as desired. □

At first sight, the foregoing result doesn't look very promissing. Its relevance resides in the fact that certain spaces can be identified as Hopf modules. We record the following simple application. Recall that a subsapce $I \subset H$ is a *left coideal* if $\Delta(I) \subset H \otimes_k I$.

Corollary. *Let H be a finite dimensional Hopf algebra. If $I \subset H$ is a left ideal and a left coideal, then $I = H$ or $I = (0)$.*

Proof. Since I is a Hopf submodule of H, the fundamental theorem provides an isomorphism

$$I \cong H \otimes_k I^{\mathrm{co}\,H},$$

leaving only the possibilities $I = (0)$ or $I = H$. □

References

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