STABLE REPRESENTATION QUIVERS: THE RIEDTMANN STRUCTURE THEOREM

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Given an artin algebra Λ, it is usually rather difficult to determine the Auslander-Reiten quiver of Λ. The purpose of these lectures is to delineate an approach that has proven to provide valuable information for classes of algebras, such as representation-finite algebras or group algebras of finite groups.

Proceeding in two steps, we begin by investigating abstract representation quivers. Here Riedtmann's Theorem reduces the problem to the consideration of trees and the so-called admissible groups of universal covers. In the concrete situation of stable Auslander-Reiten components, functions that are defined in terms of the underlying modules often allow to pin down the possible trees.

Theorem ([6]). *Let Q be a connected stable representation quiver.*

(1) *There exists a directed tree T_Q and an admissible group $\Pi \subset \mathrm{Aut}(\mathbb{Z}[T_Q])$ such that $Q \cong \mathbb{Z}[T_Q]/\Pi$.*

(2) *The undirected tree $\bar{T}_Q$ of T_Q is determined by Q (up to isomorphism).*

(3) *The group $\Pi \subset \mathrm{Aut}(\mathbb{Z}[T_Q])$ is unique up to conjugation.*

In her paper [6], Riedtmann uses this result to obtain the *tree classes* $\bar{T}_Q$ of the connected components of the stable part of the AR-quiver of a representation-finite algebra Λ, defined over an algebraically closed field. These turn out to be the simply-laced Dynkin diagrams.

In the sequel, we shall only consider quivers without loops and without multiple arrows. Thus, a quiver $Q = (V, A)$ consists of a non-empty set V of vertices and a set $A \subset V \times V$ of arrows. Given $v \in V$, v^+ and v^- are the sets of successors and predecessors of v, respectively.

Definition. Let Q be a quiver. An automorphism $\tau : Q \longrightarrow Q$ is called a *translation* if $v^- = \tau(v)^+$ for every $v \in V$. The pair (Q, τ) is called a *stable representation quiver.*

Remark. The term "stable representation quiver" is a literal translation of Riedtmann's original definition [6]. In his book [2] Benson also uses "translation quiver", while in [1] this notion refers to a quiver affording a "semitranslation".

By definition, homomorphisms of stable representation quivers "commute" with the translations. A stable representation quiver is *connected* if it is not the disjoint union of two stable subquivers. This does not necessarily imply the connectedness of the underlying quiver.

A quiver T that does not contain a subquiver of the form $\bullet \overset{\leftarrow}{\rightarrow} \bullet$ and whose graph $\bar{T}$ is a tree, is called a *directed tree.* To each directed tree (T, A) we associate a stable representation quiver $\mathbb{Z}[T]$: The underlying set of vertices is $V := \mathbb{Z} \times T$. For each arrow $s \to t$ in T we define arrows

$$(n, s) \to (n, t) \text{ and } (n, t) \to (n + 1, s) \qquad \forall\, n \in \mathbb{Z}.$$

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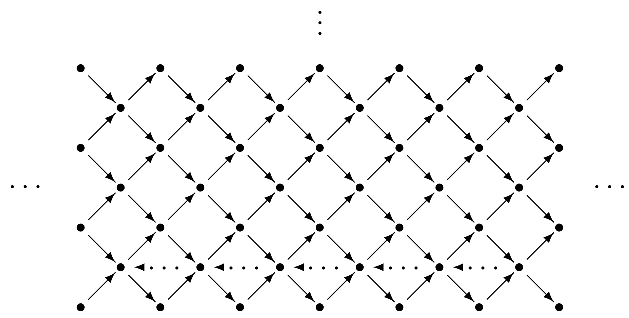
The translation is given by

$$\tau : \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T] \ \ ; \ \ (n,t) \mapsto (n-1,t).$$

Here is the stable representation quiver $\mathbb{Z}[A_\infty]$, where $A_\infty = (\mathbb{N}, \{(n,n+1), n \in \mathbb{N}\})$. The dotted arrows represent the translation.



The stable representation quiver $\mathbb{Z}[T]$ has the following universal property:

Lemma 1. *Let T be a directed tree, Q a stable representation quiver, $\varphi : T \longrightarrow Q$ a morphism of quivers. For each $n_0 \in \mathbb{Z}$ there exists a unique morphism $\hat{\varphi} : \mathbb{Z}[T] \longrightarrow Q$ of stable representation quivers such that*

$$\hat{\varphi}(n_0, t) = \varphi(t) \qquad \forall\ t \in T.$$

Proof. If $\hat{\varphi}$ exists, then we necessarily have

$$\hat{\varphi}(n,t) = \tau^{n_0 - n}(\varphi(t)) \qquad \forall\ (n,t) \in \mathbb{Z}[T].$$

Hence $\hat{\varphi}$ is uniquely determined by φ and n_0. Direct computation shows that the above formula does in fact define a morphism of stable representation quivers. □

Definition. Let $Q = (V, A, \tau)$ be a stable representation quiver. The graph $\bar{Q}$ with underlying set of vertices $V/\langle\tau\rangle$ (the τ-orbits) and edges

$$a \,\text{—}\, b :\Leftrightarrow \ \exists\ x \in a,\ y \in b \text{ with } x \to y \text{ or } y \to x$$

is called the *orbit graph* of Q.

Since τ is a translation, it suffices to require the existence of an arrow $x \to y$.

Let T be a directed tree. Then $\overline{\mathbb{Z}[T]} = \bar{T}$ is the undirected tree associated to T. Thus, if $\mathbb{Z}[T]$ and $\mathbb{Z}[T']$ are isomorphic stable representation quivers, then we have an isomorphism $\bar{T} \cong \bar{T}'$. Showing the converse requires a little more work.

Proposition 2. *Let T and T' be directed trees. If $\bar{T} \cong \bar{T}'$, then $\mathbb{Z}[T] \cong \mathbb{Z}[T']$.*

Proof. Let $\varphi : \bar{T} \longrightarrow \bar{T}'$ be an isomorphism. We consider the set

$$\mathcal{A} := \{(S, \psi) \ ; \ S \subset T \text{ subtree},\ \psi : S \longrightarrow \mathbb{Z}[T'] \text{ morphism with } \mathrm{pr}_2 \circ \psi = \varphi|_S\}$$

and define a partial ordering via

$$(S, \psi) \le (S', \psi') :\Leftrightarrow S \subset S' \text{ and } \psi'|_S = \psi.$$

This set is inductively ordered and Zorn's Lemma provides a maximal element (S_0, ψ_0). Using the fact that T and T' are trees, one shows $S_0 = T$.

We may now apply Lemma 1 to obtain a morphism $\hat{\psi}_0 : \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T']$ with

$$\hat{\psi}_0(0,t) = \psi_0(t) \quad \forall\ t \in T.$$

Since $\mathrm{pr}_2 \circ \psi_0 = \varphi$ there exists a map $\gamma : T \longrightarrow \mathbb{Z}$ with

$$\hat{\psi}_0(m,t) = (m + \gamma(t), \varphi(t)) \quad \forall\ t \in T.$$

Consequently, $\hat{\psi}_0$ is bijective. □

Definition. Let $Q = (V, A, \tau)$ be a stable representation quiver.

(1) A subgroup $\Pi \subset \mathrm{Aut}(Q)$ is called *admissible* if

$$|\Pi.y \cap (\{x\} \cup x^+)| \leq 1 \ \text{ and } \ |\Pi.y \cap (\{x\} \cup x^-)| \leq 1$$

for every $x, y \in V$.

(2) If $\Pi \subset \mathrm{Aut}(Q)$ is admissible, then $Q/\Pi := (V/\Pi, A/\Pi, \bar{\tau})$ with Π acting diagonally on A (and $A/\Pi \hookrightarrow (V/\Pi) \times (V/\Pi)$, by admissibility) and $\bar{\tau}([v]) = [\tau(v)]$ is called the *quotient* of Q by Π.

If T is a directed tree, then any subgroup of $\langle \tau \rangle$ is an admissible group of $\mathbb{Z}[T]$.

The canonical map $\pi : Q \longrightarrow Q/\Pi$ is a morphism of stable representation quivers such that for every $x \in V$, the map

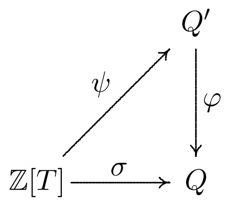
$$(*) \qquad \pi|_{x^+} : x^+ \longrightarrow \pi(x)^+$$

is bijective. Surjective morphisms with this property are referred to as *coverings*. This notion as well as the succeeding results are inspired by topological covering spaces, cf. [5, (I.5)].

Remark. A morphism $\varphi : Q \longrightarrow Q'$ satisfying $(*)$ also induces bijections $\varphi : |_{x^-} : x^- \longrightarrow \varphi(x)^-$. If Q' is connected, then φ is surjective.

The following Lemma, whose proof resembles that of Lemma 1, gives a universal property of $\mathbb{Z}[T]$:

Lemma 3. *Let T be a directed tree, $\sigma : \mathbb{Z}[T] \longrightarrow Q$ a morphism of stable representation quivers, $\varphi : Q' \longrightarrow Q$ a covering. For every vertex (n_0, t_0) of $\mathbb{Z}[T]$ and every vertex q_0' of Q' with $\varphi(q_0') = \sigma(n_0, t_0)$, there exists exactly one morphism $\psi : \mathbb{Z}[T] \longrightarrow Q'$ with $\varphi \circ \psi = \sigma$ and $\psi(n_0, t_0) = q_0'$.* □



Proof of Riedtmann's Theorem. We outline the proof of (1).

(i) Given a stable representation quiver $Q = (V, A, \tau)$, we pick a vertex $v_0 \in V$ and consider the set T_Q of all sectional paths

$$v_0 = x_0 \to x_1 \to \cdots\cdots \to x_n \ \ ; \ \ x_i \neq \tau(x_{i+2}) \qquad 0 \leq i \leq n-2.$$

The arrows are given by

$$(v_0 = x_0 \to x_1 \to \cdots\cdots \to x_n) \to (v_0 = x_0 \to x_1 \to \cdots\cdots \to x_n \to x_{n+1})$$

This makes T_Q a directed tree.

(ii) The map

$$f : T_Q \longrightarrow Q \quad ; \quad f(v_0 \to x_1 \to \cdots\cdots \to x_n) = x_n$$

is a morphism of quivers. By Lemma 1, there is a unique morphism $\hat{f} : \mathbb{Z}[T_Q] \longrightarrow Q$ with $\hat{f}(0,t) = f(t)$ for all $t \in T_Q$. One verifies $(*)$ and obtains that $\hat{f}$ is a covering.

(iii) The "fundamental group" $\Pi := \{g \in \mathrm{Aut}(\mathbb{Z}[T_Q]) \ ; \ \hat{f} \circ g = \hat{f}\}$ is admissible. (The notion derives from algebraic topology, where this group is also referred to as the group of covering transformations. Under suitable hypotheses, this group is isomorphic to a fundamental group, see [5, (I.5.8)].)

(iv) In view of Lemma 3, the group Π acts transitively on the fibres of $\hat{f}$. Hence the fibres of $\hat{f}$ are just the Π-orbits and $\hat{f}$ induces an isomorphism $\bar{f} : \mathbb{Z}[T_Q]/\Pi \xrightarrow{\sim} Q$ of stable representation quivers. □

The "most common" tree class of an Auslander-Reiten component is $T = A_\infty$ (cf. [3, 4, 7]). Since $\mathrm{Aut}(\mathbb{Z}[A_\infty]) = \langle\tau\rangle$, a stable component with tree class A_∞ is isomorphic to $\mathbb{Z}[A_\infty]/\langle\tau^n\rangle$ for some $n \geq 0$.

References

[1] M. Auslander, I. Reiten and S. Smalø. *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics **36**. Cambridge University Press, 1995

[2] D. Benson. *Representations and Cohomology, I.* Cambridge Studies in Advanced Mathematics **30**. Cambridge University Press, 1991

[3] W. Crawley-Boevey. *On tame algebras and bocses.* Proc. London Math. Soc. **56** (1988), 451-483

[4] K. Erdmann. *On Auslander-Reiten components for group algebras.* J. Pure Appl. Algebra **104** (1995), 149-160

[5] M. Greenberg and J. Harper. *Algebraic Topology.* Addison-Wesley, 1981

[6] C. Riedtmann. *Algebren, Darstellungsköcher, Ueberlagerungen und zurück.* Comment. Math. Helvetici **55** (1980), 199-224

[7] C.M. Ringel. *Finite-dimensional hereditary algebras of wild representation type.* Math. Z. **161** (1978), 235-255