Selected topics in representation theory 2
The braid group action on the set of exceptional sequences

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1 The braid group

Definition. Let $n \geq 2$ be a natural number and define

$$B_n := \langle \sigma_i \mid 1 \leq i < n; \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \rangle$$

the group generated by $n-1$ elements, with relations as above.

Remark. The groups B_2 is isomorphic to $\mathbb{Z}$. Morover, there exists a surjective group homomorphism $B_n \longrightarrow S_n$ onto the symmetric group. There is a topological interpretation of this group in terms of equivalence classes of diagrams of braids.

2 The action of the braid group

We consider exceptional sequences of objects in a triangulated category $\mathcal{D}^b$, as introduced in the first lecture. In particular, we assume that all left and right mutations are defined in this category. We denote by $\mathrm{Hom}^\bullet(E,F)$ the finite dimensional vector space $\oplus_l \mathrm{Hom}\,(E, F[l])$. Moreover, we write $\varepsilon = (E_1, \ldots, E_r) \simeq \varepsilon' = (E_1, \ldots, E'_r)$ if $E_i \simeq E'_i$ for all $i = 1, \ldots, r$.

Theorem. *Let $\varepsilon = (E_1, \ldots, E_r)$ be an exceptional sequence. Then*
1) $L_iR_i\varepsilon \simeq R_iL_i\varepsilon \simeq \varepsilon$,
2) $L_iL_{i+1}L_i\varepsilon \simeq L_{i+1}L_iL_{i+1}\varepsilon, \quad R_iR_{i+1}R_i\varepsilon \simeq R_{i+1}R_iR_{i+1}\varepsilon$,
3) $L_iL_j\varepsilon \simeq L_jL_i\varepsilon$, and $R_iR_j\varepsilon \simeq R_jR_i\varepsilon$ for all $|i-j| \geq 2$.

Corollary. *The braid group B_r acts on the set of all exceptional sequences of length r via $\sigma_i\varepsilon := L_i\varepsilon$ and $\sigma_i^{-1}\varepsilon := R_i\varepsilon$.*

Proof. 3) is obvious. We first prove 1). We consider the triangle defining the left mutation of a pair $\varepsilon = (E, F)$, the right mutation of the the pair (L_EF, E), and note that for $L\varepsilon = (L_EF, E)$ we have already shown $\mathrm{Hom}^\bullet(E,F) \simeq \mathrm{Hom}^\bullet(L_EF, E)^*$:

$$\begin{array}{ccccccc} L_EF & \longrightarrow & \mathrm{Hom}^\bullet(E,F) \otimes E & \longrightarrow & F & \longrightarrow & L_EF[1] \\ L_EF & \longrightarrow & \mathrm{Hom}^\bullet(L_EF,E)^* \otimes E & \longrightarrow & R_EL_EF & \longrightarrow & L_EF[1]. \end{array}$$

The two triangles are obviously isomorphic in two positions, consequently we obtain $F \simeq R_EL_EF$. The other isomorphism can be shown analogously.
To show 2) we consider first the mutations of a triple (E, F, G) and obtain

$$L_1L_2L_1(E,F,G) = (L_{L_EF}L_EG, L_EF, E) \text{ and } L_2L_1L_2(E,F,G) = (L_EL_FG, L_EF, E).$$

So we need to show $L_{L_EF}L_EG \simeq L_EL_FG$. For we need the following lemma.

Lemma. *Let A be an exceptional object. Then L_A is a functor $\mathcal{D}^b \longrightarrow \mathcal{D}^b$.*

Proof. Let $E \longrightarrow F$ be a homomorphism. We can define the objects $L_A E$ and $L_A F$ as the mapping cone of the corresponding canonical map. Then there exists (using one of the axioms of triangulated categories) a map $L_A E \longrightarrow L_A F$. We have to show that this map is unique. For we consider the commutative diagram with triangles in the rows

$$\begin{array}{ccccccc} E[-1] & \longrightarrow & L_A E & \stackrel{can^*}{\longrightarrow} & \mathrm{Hom}\,(A,E)\otimes A & \stackrel{can}{\longrightarrow} & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F[-1] & \longrightarrow & L_A F & \stackrel{can^*}{\longrightarrow} & \mathrm{Hom}\,(A,F)\otimes A & \stackrel{can}{\longrightarrow} & F. \end{array}$$

A homomorphism $E \longrightarrow F$ defines a homomorphism $E[-1] \longrightarrow L_A F$ via composition. We show $\mathrm{Hom}\,(L_A E, L_A F[l]) \simeq \mathrm{Hom}\,(E[-1], L_A F[l])$ for all l, so each homomorphism $E \longrightarrow F[l]$ defines a unique homomorphism $L_A E \longrightarrow L_A F[l]$. For we consider the long exact sequence

$$(A \otimes (A,E), L_A F) \longrightarrow (L_A E, L_A F) \longrightarrow (E[-1], L_A F) \longrightarrow (A[-1] \otimes (A,E), L_A F).$$

Since $(A[l], L_A F) = 0$ for all l we obtain the desired isomorphism $(L_A E, L_A F) \longrightarrow (E[-1], L_A F)$. □

We proceed with the proof of the theorem: we have to show $L_{L_E F} L_E G \simeq L_E L_F G$ for any exceptional sequence (E,F,G). We obtain two triangles, the first one obtained by applying the functor L_E to the triangle defining the left mutation $L_F G$, the second one defines the mutation $L_{L_E F} L_E G$

$$\begin{array}{ccccccc} L_E L_F G & \longrightarrow & \mathrm{Hom}^\bullet(F,G) \otimes L_E F & \longrightarrow & L_E G & \longrightarrow & \\ & & \downarrow & & \downarrow & & \\ L_{L_E F} L_E G & \longrightarrow & \mathrm{Hom}^\bullet(L_E F, L_E G) \otimes L_E F & \longrightarrow & L_E G & \longrightarrow & . \end{array}$$

Since $\mathrm{Hom}^\bullet(F,G) \simeq \mathrm{Hom}^\bullet(L_E F, L_E G)$ both triangles are isomorphic, consequently we have proven the result. □

3 Exceptional Sequences for hereditary algebras

For finite dimensional hereditary algebras we can use some modification for the mutations of exceptional sequences. Let Λ be a hereditary algebra and C be a complex in the derived category $\mathcal{D}^b$ of bounded complexes of finitely generated Λ–modules. If C is indecomposable then C is isomorphic to an indecomposable module M up to some unique shift. If we define a stronger equivalence relation for exceptional sequences: $\varepsilon = (E_1, \ldots, E_r) \sim \varepsilon' = (E'_1, \ldots, E'_r)$ iff $E_i \simeq E'_i[l]$ for some l, then we can represent each exceptional sequence uniquely, up to isomorphism, by a sequence of modules. Since left and right mutations commute with shifts, we obtain an action of the braid group on the set of all exceptional sequences of Λ–modules. So we have already proven the first assertion of the following result.

Theorem. *1) The braid group B_r acts on the set of all exceptional sequences of Λ–modules of length r (via the modified action).*
2) Let n be the number of isomorphism classes of pairwise non-isomorphic simple Λ–modules. Then the braid group B_n acts transitively on the set of all exceptional sequences of Λ–modules of length n.

The proof of the second part is due to Crawley-Boevey ([1]) for basic algebras and due to Ringel ([3]) in general. Instead of mutations the notion of perpendicular categories, developed by Schofield and Geigle, Lenzing ([2]) is used in the proof. The essential step in the proof shows that via a sequence of mutations we can reach the sequence consisting of simple Λ–modules. This sequences is unique, up to reordering. In fact, there is a natural partial order on the set of sequences, so that each sequence of simples is minimal with respect to this order.

References

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