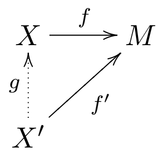
Selected topics in representation theory
– Modules with standard filtration I –
WS 2005/06

Let A be an Artin algebra, and denote the category of finitely generated left A-modules by $\operatorname{mod} A$.

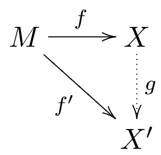
1 Approximations

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} A$, and $M \in \operatorname{mod} A$.

Definition. A *right $\mathcal{X}$-approximation of M* is a map $f : X \to M$ with $X \in \mathcal{X}$ so that for any map $f' : X' \to M$ with $X' \in \mathcal{X}$ there is a map $g : X' \to X$ such that $f' = f \circ g$.



Dually, define a *left $\mathcal{X}$-approximation of M* to be a map $f : M \to X$ with $X \in \mathcal{X}$ so that for any map $f' : M \to X'$ with $X' \in \mathcal{X}$ there is a map $g : X \to X'$ such that $f' = g \circ f$.



A subcategory $\mathcal{X}$ of $\operatorname{mod} A$ is called *functorially finite* if every $M \in \operatorname{mod} A$ has both a right and a left $\mathcal{X}$-approximation.

Notation. Let $\Theta = \{\Theta(1), \dots, \Theta(n)\}$ be a sequence of A-modules with $\operatorname{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ for all $j \geq i$. Denote by $\mathcal{F}(\Theta)$ the full subcategory of $\operatorname{mod} A$ of modules with filtration factors in Θ.

2 Main Theorem

One of the theorems in [2] is the following:

Theorem (Ringel). *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in* $\operatorname{mod} A$*.*

There is also another theorem which assures then the existence of (relative) AR-sequences for a certain full subcategory of $\operatorname{mod} A$ (see [1]):

Theorem (Auslander, Smalø). *A functorially finite subcategory which is closed under extensions and direct summands has relative AR-sequences.*

We denote by $\mathcal{X}(\Theta)$ the full subcategory in $\operatorname{mod} A$ of all modules which are direct summands of modules in $\mathcal{F}(\Theta)$. Since $\mathcal{X}(\Theta)$ is closed under extensions and direct summands and it is also functorially finite in $\operatorname{mod} A$, we obtain immediately:

Corollary. *The category $\mathcal{X}(\Theta)$ has almost split sequences.*

Note that $\mathcal{F}(\Theta)$ is generally *not* closed under direct summands.

Example. *Consider the quiver $Q = \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ}$ and its path algebra kQ.*

Take $\Theta = \{I(2), P(2)\}$. Then $P(1), P(3) \in \mathcal{X}(\Theta)$, but $P(1), P(3) \notin \mathcal{F}(\Theta)$. (Here, $P(i)$ (resp. $I(i)$) denotes the indecomposable projective (resp. injective) kQ-module corresponding to the point i.)

3 Proof of the Theorem

Let $\mathcal{X}$ be an arbitrary full subcategory of $\operatorname{mod} A$, and denote by $\mathcal{Y}$ the full subcategory of $\operatorname{mod} A$ of all modules Y with $\operatorname{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$.

Lemma. *Let $0 \to Y \to X \xrightarrow{f} M \to 0$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ be exact. Then f is a right $\mathcal{X}$-approximation of M.*

Proof. Suppose there is a map $f' : X' \to M$ with $X' \in \mathcal{X}$. Taking the pull back, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f'} & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \underset{f}{\longrightarrow} & M & \longrightarrow & 0
\end{array}$$

The induced exact sequence splits, since $Y \in \mathcal{Y}$ and $X' \in \mathcal{X}$. So there is a map $g : X' \to X$ with $f' = f \circ g$. $\square$

Lemma. *Suppose that $\mathcal{X}$ is closed under extensions and for every $N \in \operatorname{mod} A$ there is an exact sequence $0 \to N \to Y_N \to X_N \to 0$ with $Y_N \in \mathcal{Y}$ and $X_N \in \mathcal{X}$. Then every module $M \in \operatorname{mod} A$ has a right $\mathcal{X}$-approximation.*

Proof. Let $M \in \operatorname{mod} A$.

Case 1: **There is an epimorphism $\pi : X \to M$ with $X \in \mathcal{X}$.**

Let $K = \ker \pi$. We get a commutative diagram with exact rows and columns (taking the push out sequences):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & Y_K & \longrightarrow & X_K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & X_K & \longrightarrow & 0 \\
 & & {\scriptstyle \pi}\downarrow & & {\scriptstyle f}\downarrow & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Now, $X, X_K \in \mathcal{X}$ and $\mathcal{X}$ is closed under extensions, so $Z \in \mathcal{X}$. Use the previous lemma for the second row to obtain that f a right $\mathcal{X}$-approximation of M.

Case 2: **There is no epimorphism $X \to M$ with $X \in \mathcal{X}$.**

Consider the submodule $M' \subseteq M$ generated by the images of maps $X' \to M$ with $X' \in \mathcal{X}$. Since M is finitely generated, there exists a finite set of maps $X_i \to M$ with $X_i \in \mathcal{X}$ such that the images generate M'.

Since $\mathcal{X}$ is closed under extensions (and therefore under direct sums), $X = \bigoplus_i X_i \in \mathcal{X}$, and there is an epimorphism $X \to M'$ with $X \in M'$. Now the conditions in *Case 1* are fulfilled for X and M', and we get a right $\mathcal{X}$-approximation for M', say f'. If $i : M' \to M$ denotes the inclusion map, then $i \circ f'$ gives a right $\mathcal{X}$-approximation of M. (Every map $\tilde{X} \to M$ with $\tilde{X} \in \mathcal{X}$ factors via the inclusion i.) □

Let now $\Theta = \{\Theta(1), \dots \Theta(n)\}$ be a sequence of A-modules as above, $\mathcal{X} = \mathcal{F}(\Theta)$, and $\mathcal{Y} = \mathcal{Y}(\Theta) = \{Y \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(X, Y) = 0 \ \forall X \in \mathcal{F}(\Theta)\} = \{Y \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(\Theta(i), Y) = 0 \ \forall i = 1, \dots, n\}$.

Question. How can we assure in our case that we have the exact sequences of the form above, $0 \to N \to Y_N \to X_N \to 0$ with $Y_N \in \mathcal{Y}$ and $X_N \in \mathcal{X}$?

Lemma. *Let $t \in \{1, \dots, n\}$, and $N \in \operatorname{mod} A$ such that $\operatorname{Ext}^1_A(\Theta(j), N) = 0$ for all $j > t$. Then there is an exact sequence $0 \to N \to N' \to Q \to 0$ with $Q = \Theta(t)^{r_N}$ and $\operatorname{Ext}^1_A(\Theta(j), N') = 0$ for all $j \geq t$.*

Proof. Uses universal extensions and a little homological algebra. □

Lemma. *Let $t \in \{1, \dots, n\}$, and $N \in \operatorname{mod} A$ such that $\operatorname{Ext}^1_A(\Theta(j), N) = 0$ for all $j > t$. Then there exists an exact sequence $0 \to N \to Y \to X \to 0$ with $X \in \mathcal{F}(\{\Theta(1), \dots, \Theta(t)\})$ and $Y \in \mathcal{Y}(\Theta)$.*

Note the the special case $t = n$ gives us the “required” sequences, therefore the right $\mathcal{F}(\Theta)$-approximations for any module $M \in \operatorname{mod} A$.

Now the theorem follows if we just take the dual constructions to get the left $\mathcal{F}(\Theta)$-approximations.

References

[1] M. Auslander, S. O. Smalø: *Almost split sequences in subcategories.* J. Algebra **69** (1981), no. 2, 426–454.

[2] C. M. Ringel: *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences.* Math. Z. **208** (1991), no. 2, 209–223.