ALGEBRAS OF FINITE GLOBAL DIMENSION: ACYCLIC QUIVERS

ROLF FARNSTEINER

Let Λ be an artinian ring. The category of finitely generated left Λ-modules will be denoted by $\operatorname{mod}\Lambda$. A module M belonging to $\operatorname{mod}\Lambda$ is artinian and noetherian and hence of finite length $\ell(M)$. We let $\mathcal{S}(\Lambda)$ be the set of isoclasses of simple Λ-modules.

By definition, the *Gabriel quiver* Q_Λ has $\mathcal{S}(\Lambda)$ as its set of vertices. There is an arrow $S \to T$ if $\operatorname{Ext}^1_\Lambda(S,T) \neq (0)$. The quiver Q_Λ is referred as *acyclic* if it contains no oriented cycles.

Let $(P_n)_{n\geq 0}$ be a minimal projective resolution of $M \in \operatorname{mod}\Lambda$. Setting $P_{-1} := M$, we write

$$\Omega^n(M) := \ker(P_{n-1} \longrightarrow P_{n-2})$$

for all $n \geq 1$. By general theory, the module $\Omega^n(M) \in \operatorname{mod}\Lambda$ is unique up to isomorphism.

Definition. Let $(0) \neq M \in \operatorname{mod}\Lambda$. Then

$$\operatorname{pd}(M) := \sup\{n \geq 0 \; ; \; \Omega^n(M) \neq (0)\} \in \mathbb{N}_0 \cup \{\infty\}$$

is called the *projective dimension* of M. We put $\operatorname{pd}(0) = 0$.

Note that the modules of projective dimension 0 are the projective modules. Thus, $\operatorname{pd}(M)$ measures the degree of departure from projectivity.

Recall that

$$(*) \qquad \operatorname{Ext}^n_\Lambda(M,S) \cong \operatorname{Hom}_\Lambda(\Omega^n(M),S)$$

for every simple Λ-module S. Consequently, we have

$$\operatorname{pd}(M) = \sup\{n \geq 0 \; ; \; \operatorname{Ext}^n_\Lambda(M,-) \neq (0)\}.$$

Given $M \in \operatorname{mod}\Lambda$ and a simple Λ-module S, we let $[M\!:\!S]$ be the multiplicity of S in a composition series of M. The long exact cohomology sequence now shows that

$$\operatorname{pd}(M) \leq \max\{\operatorname{pd}(S) \; ; \; [M\!:\!S] \neq 0\}.$$

Hence the maximum projective dimension is that of a simple module. This number has turned out to be an important invariant of Λ.

Definition. The number

$$\operatorname{gldim}\Lambda := \max\{\operatorname{pd}(S) \; ; \; S \text{ simple}\} \in \mathbb{N}_0 \cup \{\infty\}$$

is called the *global dimension* of Λ.

Note that Λ is semi-simple if and only if $\operatorname{gldim}\Lambda = 0$. The purpose of this lecture is to prove the following basic result:

Theorem. *If Q_Λ is acyclic, then* $\operatorname{gldim}\Lambda \leq |\mathcal{S}(\Lambda)| - 1$.

Date: October 25, 2007.

Let J denote the Jacobson radical of Λ. If S is a simple Λ-module with projective cover $P(S)$, then $(*)$ specializes to

$$(**) \qquad \mathrm{Ext}^1_\Lambda(S,T) \cong \mathrm{Hom}_\Lambda(JP(S)/J^2P(S),T)$$

for every simple Λ-module T.

Lemma 1. *Let S and T be simple Λ-modules. If $[JP(S)\!:\!T] \neq 0$, then there exists a path in Q_Λ of length ≥ 1 that starts at S and ends at T.*

Proof. Let X be a factor module of $P(S)$ of maximal length, subject to all composition factors of JX being endpoints of paths of lengths ≥ 1 that originate in S. If $P(S)$ is simple, then there is nothing to be shown. Alternatively, formula $(**)$ implies that $\ell(X) \geq 2$. There results a short exact sequence

$$(0) \longrightarrow Y \longrightarrow P(S) \longrightarrow X \longrightarrow (0).$$

Assuming $X \neq P(S)$, we pick a maximal submodule $N \subsetneq Y$ and consider the induced exact sequence

$$(0) \longrightarrow Y/N \longrightarrow P(S)/N \longrightarrow X \longrightarrow (0).$$

Since $P(S)/JP(S) \cong S$ is simple, the middle term is indecomposable, so that the sequence does not split. Writing $T_2 := Y/N$, we thus have $\mathrm{Ext}^1_\Lambda(X,T_2) \neq (0)$ and standard homological algebra provides a composition factor T_1 of X with $\mathrm{Ext}^1_\Lambda(T_1,T_2) \neq (0)$. If $T_1 \cong S$, then there is a path from S to T_2 of length 1. Alternatively, $[JX\!:\!T_1] \neq 0$, so that there is a path from S to T_1, and hence one from S to T_2. Consequently, all composition factors of $J(P(S)/N)$ are endpoints of paths originating in S. Since $\ell(P(S)/N) = \ell(X)+1$, this contradicts the maximality of $\ell(X)$. As a result, $X = P(S)$, as desired. □

Lemma 2. *Let S be a simple Λ-module, $(P_n)_{n\geq 0}$ be a minimal projective resolution of S. If $P(T)$ is a direct summand of P_n, then there exists a path of length $\geq n$ originating in S and terminating in T.*

Proof. We use induction on n, the case $n=0$ being trivial.

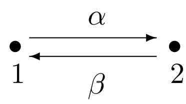
Let $n \geq 1$ and note that P_n is the projective cover of $K_n := \ker(P_{n-1} \longrightarrow P_{n-2}) \subseteq JP_{n-1}$ (Here we set $P_{-1} := S$). Consequently, $P_n/JP_n \cong K_n/JK_n$, so that $P(T)$ being a summand of P_n implies $[JP_{n-1}\!:\!T] \neq 0$. Hence there exists a summand $P(T')$ of P_{n-1} with $[JP(T')\!:\!T] \neq 0$. Lemma 1 provides a path $T' \to T$ of length ≥ 1. By inductive hypothesis, there is a path $S \to T'$ of length $\geq n-1$, and concatenation yields the desired path from S to T. □

Proof of the Theorem. Let S be a simple Λ-module with minimal projective resolution $(P_n)_{n\geq 0}$. Since Q_Λ is acyclic, a path in Q_Λ has length $\leq |\mathcal{S}(\Lambda)|-1 =: n$. By virtue of Lemma 2, we obtain $P_{n+1} = (0)$, whence $\Omega^{n+1}(S) \cong \mathrm{im}(P_{n+1} \longrightarrow P_n) = (0)$. Thus, $\mathrm{pd}(S) \leq n$, so that $\mathrm{gldim}\,\Lambda \leq n$. □

The proof actually shows that the projective dimension $\mathrm{pd}(S)$ of the simple Λ-module S is bounded by the maximum length of all paths originating in S.

The following example shows that algebras of finite global dimension also occur for quivers admitting oriented cycles.

Example. Let k be a field and consider the bound quiver algebra $\Lambda := k[Q]/\langle\beta\alpha\rangle$ with quiver Q given by



We denote the simple modules S_1 and S_2. Then we have $\Omega(S_1) = S_2$ and $\Omega(S_2) = P(S_1)$, so that $\mathrm{pd}(S_2) = 1$ and $\mathrm{pd}(S_1) = 2$, whence $\mathrm{gldim}\,\Lambda = 2$.

Our formula $(**)$ readily yields $Q_\Lambda = Q_{\Lambda/J^2}$. Hence we can hope to get more infomation for algebras satisfying $J^2 = (0)$. We record the following basic observation:

Corollary 3. *Suppose that $J^2 = (0)$. Then the following statements hold:*
(1) *If $\mathrm{gldim}\,\Lambda < \infty$, then Q_Λ has no oriented cycles.*
(2) *If Λ has only one simple module, then Λ is simple.*

Proof. (1) Let S be a simple Λ-module. Since $J^2 = (0)$, the module $\Omega(S) = JP(S) = \bigoplus n_{S'}S'$ is semi-simple and formula $(**)$ implies

$$n_T \operatorname{Hom}_\Lambda(T,T) \cong \operatorname{Hom}_\Lambda(JP(S),T) \cong \operatorname{Ext}^1_\Lambda(S,T).$$

Hence $n_T \neq 0$ whenever there is an arrow $S \to T$, and in that case our Ext-criterion yields

$$\mathrm{pd}(T) \leq \max\{\mathrm{pd}(S') \ ; \ n_{S'} \neq 0\} = \mathrm{pd}(JP(S)) < \mathrm{pd}(S).$$

Consequently, Q_Λ has no oriented cycles.

(2) Part (1) implies that Q_Λ has no arrows. Hence Λ is semi-simple and has only one simple module. By Wedderburn's Theorem, Λ is simple. □

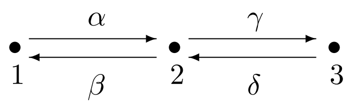
Recall that an arrow starting and terminating at the same vertex is called a *loop*. There are two conjectures relating the structure of the quiver Q_Λ to the various dimensions introduced before.

No loops conjecture. *If $\mathrm{gldim}\,\Lambda < \infty$, then Q_Λ has no loops.*

Strong no loops conjecture. *If S is a simple Λ-module with $\mathrm{pd}(S) < \infty$, then Q_Λ does not possess a loop at the vertex corresponding to S.*

It is of course tempting to verify the first conjecture by comparing the global dimension of Λ with that of a factor algebra $\Lambda/\Lambda e\Lambda$, where e is a suitable idempotent of Λ. The following example illustrates that this approach will in general not be of much avail:

Example. Let Λ be given by the quiver



subject to the relations $\beta\alpha = 0$ and $\alpha\beta = \delta\gamma$. Letting S_i and P_i be the simple and principal indecomposable modules corresponding to the vertex i, we obtain $\Omega^2(S_1) \cong P(S_1)$ and $\Omega^2(S_2) \cong P_2 \cong \Omega(S_3)$, so that $\mathrm{gldim}\,\Lambda = 2$. On the other hand, if e is the idempotent corresponding to the vertex 3, then $\Lambda' := \Lambda/\Lambda e\Lambda$ has quiver the full subquiver with vertices 1 and 2, while the relations are $\alpha\beta = 0 = \beta\alpha$. Being a self-injective algebra which is not semi-simple, Λ' has infinite global dimension.