INDUCED MODULES: THE MACKEY DECOMPOSITION THEOREM

ROLF FARNSTEINER

Let S be a ring, $R \subseteq S$ be a subring. Given an R-module M, we can form the induced S-module $S \otimes_R M$. In this fashion, we obtain a functor

$$\operatorname{mod} R \longrightarrow \operatorname{mod} S \;\; ; \;\; N \mapsto S \otimes_R N.$$

When the composite of this functor with the restriction functor $\operatorname{mod} S \longrightarrow \operatorname{mod} R \,; M \mapsto M|_R$ can be controlled, one can relate properties of $\operatorname{mod} S$ to those of $\operatorname{mod} R$. By way of illustration, let us consider the following result:

Lemma 1. *Suppose that $R \subseteq S$ are finite-dimensional k-algebras such that*
($*$) *M is a direct summand of $(S \otimes_R M)|_R$ for every finite-dimensional R-module M.*
If S is representation-finite, so is R.

Proof. Let $N_1, \dots, N_r$ be a complete set of representatives for the isoclasses of finite-dimensional indecomposable S-modules. If M is a finite-dimensional indecomposable R-module, then

$$S \otimes_R M \cong n_1 N_1 \oplus \cdots \oplus n_r N_r$$

with $n_i \in \mathbb{N}_0$. In view of ($*$), the Theorem of Krull-Remak-Schmidt implies that M is isomorphic to an indecomposable direct summand of some $N_i|_R$, so that the isoclass of M belongs to the finite set of indecomposable summands of $\bigoplus_{i=1}^{r} N_i|_R$. □

At first sight, condition ($*$) looks awfully contrived and one may wonder about the existence of non-trivial examples. In fact, any representation-infinite algebra R gives rise to a non-example: Since R can be viewed as a subalgebra of some algebra $S = \operatorname{Mat}_n(k)$ of $(n \times n)$-matrices, the resulting extension of algebras cannot satisfy ($*$). The purpose of this lecture is to establish a result for group algebras of finite groups, which greatly refines ($*$).

Let k be a field. In the following, we let $K \subseteq G$ be finite groups with group algebras $kK \subseteq kG$. Given $g \in G$, we let $K^g := \{ghg^{-1} \; ; \; h \in K\}$. If M is a K-module, then M^g denotes the K^g-module with underlying k-space M and action

$$x{\boldsymbol .}m := g^{-1}xg.m \qquad \forall \; x \in K^g, \; m \in M.$$

If M is a G-module, then $M|_K$ denotes the restriction of M to K.

Theorem (Mackey Decomposition Theorem, [3])**.** *Let $H, K \subseteq G$ be subgroups of G, M be a K-module. Then we have an isomorphism*

$$(kG \otimes_{kK} M)|_H \cong \bigoplus_{HgK} kH \otimes_{k(H\cap K^g)} M^g|_{H\cap K^g}$$

of H-modules, where the sum is taken over the double cosets HgK.

Date: November 14, 2007.

Corollary 2. *Suppose that* $\operatorname{char}(k) = p > 0$*. If* kG *has finite representation type, then every Sylow-*p*-subgroup* $P \subseteq G$ *is cyclic.*

Proof. Let $P \subseteq G$ be a Sylow-p-subgroup, M be a kP-module. Setting $H = K = P$ and $g = 1$ in the Theorem, we see that $kP \otimes_{kP} M \cong M$ is a direct summand of the P-module $(kG \otimes_{kP} M)|_P$. In view of Lemma 1, the algebra kP is representation-finite. Since P is a p-group, kP local, whence $\dim_k \operatorname{Rad}(kP)/\operatorname{Rad}(kP)^2 = \dim_k \operatorname{Ext}^1_{kP}(k,k) = 1$. As a result, every element $x \in \operatorname{Rad}(kP) \setminus \operatorname{Rad}(kP)^2$ induces an isomorphism $k[X]/(X^{p^n}) \xrightarrow{\sim} kP$. Since $\operatorname{Rad}(kP) = \sum_{g \in P} k(g-1)$, we conclude that P is cyclic. □

Remarks. (1) The converse of Corollary 2 also holds, see [2].

(2) Property $(*)$ does not require Mackey's Theorem: Let $\mathcal{C} \subseteq G$ be a set of representatives for the left K-cosets $\neq K$. Then $kG \otimes_{kK} M \cong M \oplus (\bigoplus_{g \in \mathcal{C}} g \otimes M)$ is a decomposition of K-modules.

Proof of the Theorem. We proceed in several steps, beginning with a refinement of the foregoing remark. Given $g \in G$, we put

$$k(HgK) = \sum_{x \in HgK} kx.$$

Let $\{g_1, \dots, g_n\}$ be a complete set of double coset representatives, so that $G = \bigsqcup_{i=1}^n Hg_iK$. We immediately obtain:

(i) $k(HgK)$ *is a* (kH, kK)*-bimodule of* kG *for every* $g \in G$ *and* $kG = \bigoplus_{i=1}^n k(Hg_iK)$*, a direct sum of* (kH, kK)*bimodules.* ◇

(ii) *Let* g *be an element of* G*. Then there is an isomorphism*

$$\varphi_g : kH \otimes_{k(H \cap K^g)} M^g|_{H \cap K^g} \longrightarrow k(HgK) \otimes_{kK} M \quad ; \quad h \otimes m \mapsto hg \otimes m$$

of kH*-modules.*

Direct computation shows the existence of φ_g. Moreover, φ_g is surjective. Let $h_1, \dots, h_\ell$ be a complete set of representatives for the left $H \cap K^g$-cosets of H. Then $\{h_1g, \dots, h_\ell g\}$ is a basis of the right kK-module $k(HgK) \subseteq kG$. Consequently,

$$\dim_k k(HgK) \otimes_{kK} M = \ell \dim_k M = \dim_k kH \otimes_{k(H \cap K^g)} M^g|_{H \cap H^g},$$

implying that φ_g is an isomorphism. ◇

Combining (i) and (ii), we arrive at the following isomorphisms of kH-modules:

$$kG \otimes_{kK} M \cong \bigoplus_{i=1}^n k(Hg_iK) \otimes_{kK} M \cong \bigoplus_{i=1}^n kH \otimes_{k(H \cap K^{g_i})} M^{g_i}|_{H \cap K^{g_i}}.$$

This completes the proof of our theorem. □

Mackey's Theorem is of fundamental importance as it sets the stage for the theory of vertices and sources. Suppose that k is a field of positive characteristic $p > 0$, and let G be a finite group. Let M be a kG-module, $H \subseteq G$ be a subgroup. We say that M is *relatively* H*-projective*, if M is a direct summand of an induced module $kG \otimes_{kH} N$, where N is an H-module. We record the following basic fact:

Lemma 3. *Let* M *be a* G*-module. If* $P \subseteq G$ *is a Sylow-*p*-subgroup, then* M *is relatively* P*-projective.*

Proof. Let $g_1, \ldots, g_n \in G$ be a complete set of representatives for the left P-cosets. Given a P-linear map $\varphi : X \longrightarrow Y$ between two G-modules X and Y, we define

$$\mathrm{Tr}(\varphi) : X \longrightarrow Y \quad ; \quad x \mapsto \sum_{i=1}^{n} g_i \varphi(g_i^{-1} x).$$

Then $\mathrm{Tr}(\varphi)$ does not depend on the choice of $g_1, \ldots, g_n$ and is G-linear (!) with $\mathrm{Tr}(\varphi) = [G\!:\!P]\varphi$ if φ is already G-linear.

We consider the canonical G-linear surjection

$$f : kG \otimes_{kP} M \longrightarrow M \quad ; \quad a \otimes m \mapsto am,$$

which admits a P-linear splitting

$$s : M \longrightarrow kG \otimes_{kH} M \quad ; \quad m \mapsto 1 \otimes m.$$

Since f is G-linear, the identity $f \circ s = \mathrm{id}_M$ implies

$$[G\!:\!P]\,\mathrm{id}_M = \mathrm{Tr}(f \circ s) = f \circ \mathrm{Tr}(s).$$

As p does not divide the index $[G\!:\!P]$, it follows that f is split surjective. □

Definition. Let M be an indecomposable G-module. A subgroup $D \subseteq G$ is called a *vertex* for M if

(a) M is relatively D-projective, and
(b) if $D' \subsetneq D$ is a proper subgroup, then M is not relatively D'-projective.

Definition. Let M be an indecomposable G-module, $D \subseteq G$ be a vertex of M. An indecomposable D-module N is a *source* of M if and only if M is a direct summand of $kG \otimes_{kD} N$.

We record a few basic properties:

- If M is an indecomposable G-module, then any subgroup $D \subseteq G$ of minimal order subject to M being relatively D-projective is a vertex of M. Hence M is a direct summand of some $kG \otimes_{kD} N$, and the Theorem of Krull-Remak-Schmidt provides an indecomposable summand N_0 of N such that M is a direct summand of $kG \otimes_{kD} N_0$. Consequently, vertices and sources exist.
- Let $D \subseteq G$ be a vertex of M, $N \in \mathrm{mod}\, kD$ be a source. Given $g \in G$, we have $M^g \cong M$ and $(kG \otimes_{kD} N)^g \cong kG \otimes_{kD^g} N^g$, so that D^g is also a vertex of M and $N^g \in \mathrm{mod}\, kD^g$ is a source.
- If M is an indecomposable G-module whose vertex is $\{1\}$, then M is a direct summand of kG and hence projective. Thus, vertices measure the degree of departure from projectivity. (Since kG is self-injective, the projective dimension $\mathrm{pd}(M)$ of M is either zero or infinite, so that this notion is useless in our present context.)

For a subgroup $H \subseteq G$, we let $\mathrm{Nor}_G(H) := \{g \in G \ ; \ gHg^{-1} = H\}$ be the normalizer of H in G. Here is a key result from Green's seminal paper [1] on vertices an sources:

Proposition 4. *Let M be an indecomposable G-module, $D \subseteq G$ be a vertex of M.*

(1) *D is a p-group.*

(2) *If $H \subseteq G$ is a subgroup such that M is relatively H-projective, then there exists $g \in G$ such that $D^g \subseteq H$.*

(3) *If $D' \subseteq G$ is a vertex of M, then there exists $g \in G$ with $D' = D^g$.*

(4) *Let N_0 and N_1 be D-modules that are sources of M. Then there exists $g \in \mathrm{Nor}_G(D)$ with $N_1 \cong N_0^g$.*

Proof. (1) Let N be a D-module, which is a source of M. If P is a Sylow-p-subgroup of D, then Lemma 3 implies that N is relatively P-projective. Hence M is a direct summand of $kG \otimes_{kD} N$ and N is a direct summand of $kD \otimes_{kP} N'$. Consequently, M is a direct summand of

$$kG \otimes_{kD} (kD \otimes_{kP} N') \cong kG \otimes_{kP} N'.$$

Since D is a vertex, we obtain $D = P$, so that D is a p-group.

(2) Since M is relatively H-projective, M actually is a direct summand of $kG \otimes_{kH} M|_H$: If $\varphi : kG \otimes_{kH} N \longrightarrow M$ is split surjective, then the map

$$\omega : kG \otimes_{kH} N \longrightarrow kG \otimes_{kH} M \quad ; \quad a \otimes n \mapsto a \otimes \varphi(1 \otimes n)$$

is G-linear and its composite $\psi \circ \omega$ with the canonical map

$$\psi : kG \otimes_{kH} M \longrightarrow M \quad ; \quad a \otimes m \mapsto m$$

equals φ. Hence ψ is also split surjective.

Mackey's Theorem now implies that $M|_D$ is a direct summand of

$$\bigoplus_{DgH} kD \otimes_{k(D\cap H^g)} M^g|_{D\cap H^g}.$$

Since M is also a direct summand of $kG \otimes_{kD} M|_D$, we see that it is a direct summand of

$$\bigoplus_{DgH} kG \otimes_{k(D\cap H^g)} M^g|_{D\cap H^g}.$$

As M is indecomposable, there exists $g \in G$ such that M is a direct summand of $kG \otimes_{k(D\cap H^{g^{-1}})} M^{g^{-1}}|_{D\cap H^{g^{-1}}}$. Since D is a vertex, this implies $D \subseteq H^{g^{-1}}$, whence $D^g \subseteq H$.

(3) This is a direct consequence of (2).

(4) Since M is a direct summand of $kG \otimes_{kD} M|_D$, there exists an indecomposable summand N of $M|_D$ which is a source of M. Then N is an indecomposable summand of

$$(kG \otimes_{kD} N_0)|_D \cong \bigoplus_{DgD} kD \otimes_{k(D\cap D^g)} N_0^g|_{D\cap D^g}.$$

Thus, there is g such that N is a summand of $kD \otimes_{k(D\cap D^g)} N_0^g$. Then M is a summand of $kG \otimes_{k(D\cap D^g)} N_0^g$, so that D being a vertex implies $D = D^g$. Thus, $g \in \mathrm{Nor}_G(D)$ and N is a summand of $kD \otimes_{kD} N_0^g \cong N_0^g$. Consequently, $N \cong N_0^g$, and our assertion follows by applying the same reasoning to N_1. □

References

[1] J. Green, *On the indecomposable representations of a finite group.* Math. Z. **70** (1959), 430–445.
[2] D. Higman, *Indecomposable representations of characteristic p.* Duke J. Math. **21** (1954), 377–381.
[3] G. Mackey, *On induced representations of groups.* Amer. J. Math. **73** (1951), 576–592.