Selected topics in representation theory

– Finitistic dimension conjecture: a reduction formula –

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This lecture is based on an article by K. Igusa and G. Todorov on the finitistic global dimension conjecture [1].

Definition 1. Let A be an Artin algebra. The finitistic dimension of A is defined as

$$\operatorname{findim} A := \sup\{\operatorname{pd} M \in \operatorname{mod} A \mid \operatorname{pd} M < \infty\},$$

where $\operatorname{mod} A$ denotes the full subcategory of finitely generated A-modules and $\operatorname{pd} M$ the projective dimension of the module M.

Conjecture 2.

$$\operatorname{findim} A < \infty$$

for all Artin algebras A.

Remark 3. *Known cases where the conjecture is true:*

- for radical cubed zero algebras
- monomial relations' case
- when the category of all modules of finite projective dimension is contravariantly finite in $\operatorname{mod} A$

In this lecture we will prove a theorem which simplifies the calculation of the finitistic dimension in several other cases. This does not concern only those cases mentioned in the end of the article by Igusa and Todorov; their result has also been applied by many other authors.

Lemma 4 (Fitting's Lemma).

- *Let M be a module over a noetherian ring R, $f : M \to M$ an endomorphism of M and X a finitely generated submodule of M. Then there is a smallest $n_{f,X} \in \mathbb{N}_0$ such that $f|_{f^m(X)} : f^m(X) \to f^{m+1}(X)$ is an isomorphism for all $m \geq n_{f,X}$.*
- *If Y is a submodule of X, then $n_{f,Y} \leq n_{f,X}$.*
- *If R is an Artin algebra and M a finitely generated R-module, then there is a decomposition $M = Y \oplus Z$ with $Z = \operatorname{Ker} f^m$ and $Y = \operatorname{Im} f^m$ for all $m \geq n_{f,M}$.*

Let A be an Artin algebra.

Notation 5. We denote by K_0 the free abelian group generated by the isoclasses $[M]$ of finitely generated modules M subject to the following relations:

$$[M] = [X] + [Y], \text{ if } M \cong X \oplus Y$$

and

$$[P] = 0, \text{ if } P \text{ is projective.}$$

Definition 6. Define $L([M]) := [\Omega M]$, where ΩM denotes the first syzygy.

Lemma 7. *The map $L : K_0 \to K_0$ is well-defined.*

Proof. Ω commutes with direct sums and takes projectives to zero. $\square$

Notation 8. Let $M \in \operatorname{mod} A$. By $\langle \operatorname{add} M \rangle$ we denote the finitely generated subgroup of K_0 generated by all indecomposable direct summands of M, and we set $\varphi(M) := n_{L,\langle \operatorname{add} M \rangle}$.

Lemma 9. *Let $M, N \in \operatorname{mod} A$.*

- *If* $\operatorname{pd} M < \infty$, *then* $\varphi(M) = \operatorname{pd} M$.
- *If M is indecomposable and* $\operatorname{pd} M = \infty$, *then* $\varphi(M) = 0$.
- $\varphi(M) \leq \varphi(M \oplus N)$
- $\varphi(M^k) = \varphi(M)$ *for* $k \geq 1$

Proof.

- Let $\operatorname{pd} M = n < \infty$ and $0 \to P_n \xrightarrow{f_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ a minimal projective resolution. $\varphi(M)$ measures exactly, when a minimal projective resolution stops: By definition, $\Omega^{n+k}(M) = \operatorname{Ker} f_{n+k-1} = 0$ for all $k \geq 1$ and $\Omega^n(M) = P_n$ is projective. So the finitely generated additive groups $\langle \operatorname{add} \Omega^{n+k}(M) \rangle \subseteq K_0$ are zero for all $k \geq 0$. But $\Omega^{n-1}(M)$ is neither projective nor zero, so $\langle \operatorname{add} \Omega^{n-1}(M) \rangle \neq 0$.
- Let $\ldots \to P_n \xrightarrow{f_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be a minimal projective resolution of M. If M is indecomposable and $\operatorname{pd} M = \infty$, then $\Omega^k(M) = \operatorname{Ker} f_k \neq 0$ for all $k \geq 0$. By definition of K_0, the additive groups $\langle \operatorname{add} \Omega^k(M) \rangle$, $k \geq 0$, are all isomorphic.
- The third part follows from Lemma 4, part 3, because $\langle \operatorname{add} M \rangle$ is a subgroup of the finitely generated group $\langle \operatorname{add}(M \oplus N) \rangle$.
- The last part follows from: $\langle \operatorname{add} M \rangle = \langle \operatorname{add} M^k \rangle$ for $k \geq 1$.

$\square$

Notation 10. Let $M \in \operatorname{mod} A$. Set

$$\psi(M) := \varphi(M) + \sup\{\operatorname{pd} X \mid \operatorname{pd} X < \infty,\ X \text{ direct summand of } \Omega^{\varphi(M)}(M)\}.$$

Now we can show four basic properties of the function ψ which will allow us to prove the main theorem.

Lemma 11. *Let $M, N, Y, Z \in \operatorname{mod} A$.*

- *If* $\operatorname{pd} M < \infty$, *then* $\varphi(M) = \psi(M) = \operatorname{pd} M$.
- $\psi(M^k) = \psi(M)$ *for* $k \geq 1$
- $\psi(M) \leq \psi(M \oplus N)$

- *If* $Z \mid \Omega^m(Y)$ *with* $m \leq \varphi(Y)$ *and* $\operatorname{pd} Z < \infty$*, then* $\operatorname{pd} Z + m \leq \psi(Y)$*.*

Proof.

- If $\operatorname{pd} M = n < \infty$, then $\Omega^{\varphi(M)}(M) = \Omega^n(M)$ is projective. So the last summand is zero.
- The second part follows from part 4 of Lemma 9 and the additivity of Ω^k for all $k \geq 0$.
- Let us first prove the fourth part: By the additivity of Ω^k for $k \geq 0$ and the assumption $Z \mid \Omega^m(Y)$, we have that $\Omega^{\varphi(Y)-m}(Z) \mid \Omega^{\varphi(Y)}(Y)$. By definition of $\psi(Y)$, we get immediately that $\varphi(Y) + \operatorname{pd} \Omega^{\varphi(Y)-m}(Z) \leq \psi(Y)$. It follows that $\operatorname{pd} Z + m = \operatorname{pd} \Omega^{\varphi(Y)-m}(Z) + \varphi(Y) - m + m \leq \psi(Y)$.
- Let us now prove the third part. Use the fourth part of this Lemma, and set $Z := M$, $Y := M \oplus N$, and $m := \varphi(M)$. By definition, the left hand side is $\psi(M)$, and so $\psi(M) \leq \psi(M \oplus N)$.

□

Now we are able to prove the Main Theorem.

Theorem 12. *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence in* $\operatorname{mod} A$ *such that* $\operatorname{pd} C < \infty$*. Then* $\operatorname{pd} C \leq \psi(A \oplus B) + 1$*.*

Proof. Since $\operatorname{pd} C < \infty$, we have that $\Omega^n A \cong \Omega^n B$ for some $n \geq 0$. Take n minimal with $[\Omega^n A] = L^n[A] = L^n[B] = [\Omega^n B]$. Clearly, $n \leq \operatorname{pd} C$. Also, $[A], [B] \in \langle add(A \oplus B) \rangle$, so $n \leq \varphi(A \oplus B)$. The the n-th syzygies provide a short exact sequence

$$\varepsilon : 0 \to X \oplus P \xrightarrow{f} X \oplus Q \to M \to 0$$

with P, Q projective and $M = \Omega^n C$. Let us denote the $X - X$-component of the map f by $\tilde{f}$. Then $\tilde{f}$ is an endomorphism of X. Due to Lemma 4, part 3, $X = Y \oplus Z$ where $Z = \operatorname{Ker}^m \tilde{f}$ and $Y = \operatorname{Im}^m \tilde{f}$ for $m \gg 0$. So ε induces a short exact sequence

$$\eta : 0 \to Z \oplus P \to Z \oplus Q \to M \to 0,$$

where the $Z - Z$-component $\hat{f}$ of the first map in η is nilpotent.

Claim: $\operatorname{pd} Z < \infty$.

Suppose $\operatorname{Ext}^k_A(Z, S) \neq 0$. Apply $\operatorname{Hom}^k_A(-, S)$ to η.

We get the long exact homology sequence, part of which is

$$\ldots \to \operatorname{Ext}^k_A(Z \oplus Q, S) \to \operatorname{Ext}^k_A(Z \oplus P, S) \to \operatorname{Ext}^{k+1}_A(M, S) \to \ldots$$

Since $\hat{f}$ is nilpotent, the endomorphism of $\operatorname{Ext}^k_A(Z, S)$ induced by $\hat{f}$ cannot be surjective. So, $\operatorname{Ext}^{k+1}_A(M, S) \neq 0$.

$Z \mid \Omega^n(A)$, so we can apply Lemma 11 and get:

$$\operatorname{pd} Z + n \leq \psi(A) \leq \psi(A \oplus B).$$

Also, $\operatorname{pd} M \leq \operatorname{pd} Z + 1$, since η is exact.

So we get
$$\operatorname{pd} C = \operatorname{pd} M + n \leq \operatorname{pd} Z + n + 1 \leq \psi(A \oplus B) + 1.$$

$\square$

References

[1] K. Igusa and G. Todorov, *On the finitistic global dimension conjecture for artin algebras*, Representations of algebras and related topics, Fields Inst. Commun. **45** (2005), 201–204.