QUILLEN'S STRATIFICATION THEOREM

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We fix a finite group G. Our goal is to explain Quillen's result how the maximal ideal spectrum of the group cohomology ring of G over an algebraically closed field of characteristic $p > 0$ can be glued together from group cohomology of its elementary abelian p-subgroups.

Quillen wrote four papers about the structure of the group cohomology ring. The first two [Qui71b] are more general as they treat compact Lie groups and use equivariant cohomology. In [Qui71a], Quillen developed an algebraic approach, but still used equivariant cohomology for some key step. Finally, he provided an algebraic proof for this step in collaboration with Venkov [QV72]. In this short exposition, we follow the algebraic approach. More details can be found in the very readable master's thesis of Amalie Høgenhaven [Høg13].

Quillen's stratification theorem arose as a continuation of establishing the Atiyah-Swan conjecture which states that the Krull dimension of the mod-p cohomology ring of G equals its p-rank. We will encounter the stratification theorem as a crucial input in Henning Krause's forthcoming talk. He will provide an exposition of his work [BIK11] with Dave Benson and Srikanth Iyengar. For further developments motivated by Quillen's work, we refer to Eric Friedlander's discussion [Fri13].

1. Basics of group cohomology

Let R be a commutative ring, G and G' finite groups, $H \subset G$ a subgroup, and M an RG-module.

The group cohomology of G with coefficients in M is the graded R-module given in degree n by $H^n(G, M) = \mathrm{Ext}^n_{RG}(R, M)$. If $M = R$, then $H^*(G, R)$ is a graded commutative ring. The multiplication can be defined via Yoneda splicing if the Ext-groups are defined via extensions or with the help of a diagonal approximation $P_* \to P_* \otimes_R P_*$ if the Ext-groups are defined as the cohomology groups of the cochain complex $\mathrm{Hom}_{RG}(P_*, R)$ for a projective resolution P_* of R over RG.

Group cohomology is functorial. If $\phi\colon G \to G'$ is a group homomorphism, M' an RG'-module and $f\colon M' \to M$ a homomorphism of RG-modules, then we obtain an induced map

$$(\phi, f)^*\colon H^*(G', M') \to H^*(G, M).$$

In particular, the inclusion $H \to G$ induces a natural *restriction map*

$$\mathrm{res}_{G,H}\colon H^*(G, M) \to H^*(H, M).$$

It is induced on cochain level by $\mathrm{Hom}_{RG}(P_*, M) \to \mathrm{Hom}_{RH}(P_*, M)$ using that any projective resolution P_* over RG is also a projective resolution over RH and that RG-module homomorphisms are in particular RH-module homomorphisms.

There is also a natural map in the other direction, called *corestriction* (or transfer)

$$\mathrm{cor}_{H,G}\colon H^*(H, M) \to H^*(G, M).$$

We will only need the fact that $\mathrm{cor}_{H,G} \circ \mathrm{res}_{G,H}$ is multiplication by the index $|G : H|$.

For any $g \in G$, conjugation induces a natural homomorphism

$$g^*\colon H^*(H, M) \to H^*(gHg^{-1}, M)$$

given on cochain level by $\mathrm{Hom}_{RH}(P_n, M) \to \mathrm{Hom}_{R(gHg^{-1})}(P_n, M), \quad f \mapsto (x \mapsto gfg^{-1}(x))$.

If H is normal in G, then we obtain a G/H-action on $H^*(H, M)$ since elements of H act as the identity by construction.

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The Bockstein homomorphism $\beta\colon H^n(G,\mathbb{F}_p) \to H^{n+1}(G,\mathbb{F}_p)$ is the connecting homomorphism in the long exact sequence arising from the short exact sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \to 0.$$

Finally, we recall the group cohomology of an elementary abelian p-group $E = (\mathbb{Z}/p)^n$ of rank n with coefficients in a field k of characteristic p. It is a polynomial algebra for $p = 2$ and a tensor product of a polynomial algebra with an exterior algebra for odd p:

$$H^*(E,k) \cong \begin{cases} k[x_1,\dots,x_n], \text{ with } |x_i| = 1, & p = 2, \\ k[x_1,\dots,x_n] \otimes_k \Lambda(y_1,\dots,y_n), \text{ with } |x_i| = 2, |y_i| = 1, & p \text{ odd.} \end{cases}$$

There is a canonical choice of generators such that $\beta(y_i) = x_i$ for $k = \mathbb{F}_p$ and odd p.

2. Quillen-Venkov Lemma

The following theorem is the Quillen-Venkov Lemma.

Theorem 2.1. *If $u \in H^*(G,\mathbb{F}_p)$ restricts to $0 \in H^*(E,\mathbb{F}_p)$ for all elementary abelian subgroups E of G, then u is nilpotent.*

It holds more generally over any field k of charcteristic p since $H^*(G,k) \cong H^*(G,\mathbb{F}_p) \otimes_{\mathbb{F}_p} k$. We will use Serre's Theorem.

Theorem 2.2 ([Ser65])**.** *Suppose that G is a finite p-group. If G is not elementary abelian, then there exist cohomology classes $\alpha_1,\dots,\alpha_r \in H^1(G,\mathbb{F}_p) \setminus \{0\}$ such that*

$$\beta(\alpha_1)\dots\beta(\alpha_r) = 0.$$

Example 2.3. The group cohomology of the dihedral group D_8 of order 8 is

$$H^*(D_8,\mathbb{F}_2) \cong \frac{\mathbb{F}_2[x,e,y]}{(xe)}$$

with $|x| = |e| = 1$ and $|y| = 2$. For $p = 2$, the Bockstein of a degree 1 cohomology class is just its square. Thus we can take $\alpha_1 = x$, $\alpha_2 = e$ which multiply to zero even before squaring.

This is no coincidence. Ergün Yalçin proved in [Yal08] that there exist nonzero 1-dimensional cohomology classes with trivial product for any nonabelian 2-group.

In addition to Serre's Theorem we will need the following result whose proof is an application of the Lyndon-Hochschild-Serre spectral sequence. Its statement uses the identification of group cohomology classes of degree one with group homomorphisms.

Lemma 2.4. *Let $v \neq 0$ in $H^1(G,\mathbb{F}_p) \cong Hom(G,\mathbb{F}_p)$ and $G' = \ker v$. If $u \in H^*(G,\mathbb{F}_p)$ restricts to zero on G', then $u^2 \in H^*(G,\mathbb{F}_p) \cdot \beta(v)$.*

We are ready to prove the Quillen-Venkov Lemma.

Proof of Theorem 2.1. By induction on the order of G. Let $u \in H^*(G,\mathbb{F}_p)$ such that $\mathrm{res}_{G,E}(u) = 0$ for all elementary abelian p-subgroups $E \subset G$. By induction hypothesis we assume that $\mathrm{res}_{G,H}(u)$ is nilpotent for all proper subgroups $H \subset G$, and after replacing u by a power, that $\mathrm{res}_{G,H}(u) = 0$ for all such H.

If G is not a p-group, let $H \subset G$ be a p-Sylow subgroup. Then $\mathrm{res}_{G,H}$ is injective since p does not divide the index $|G : H|$, and hence $u = 0$.

If G is a p-group, we may assume that G is not elementary abelian as otherwise $u = 0$ by assumption. Choose $\alpha_1,\dots,\alpha_r$ as in Serre's Theorem 2.2. By Lemma 2.4, the square u^2 is divisible by $\beta(\alpha_i)$ for all $1 \leq i \leq r$. So u^{2r} is divisible by $\beta(\alpha_1)\dots\beta(\alpha_r) = 0$. Hence u is nilpotent. ∎

3. Krull dimension

Let k be a field of characteristic $p > 0$. Instead of working with graded-commutative algebras, Quillen restricts to the commutative part of even-degree classes when p is odd.

Notation 3.1.

$$H(G,k) = \begin{cases} H^*(G,k), & p = 2, \\ \bigoplus_{i \geq 0} H^{2i}(G,k), & p \text{ odd.} \end{cases}$$

Example 3.2. For an elementary abelian p-group E of rank n, we obtain

$$H(E,k) \cong k[x_1, \ldots, x_n], \text{ with } |x_i| = 1,$$

when $p = 2$, and

$$H(E,k) \cong k[x_1, \ldots, x_n] \oplus J, \text{ with } |x_i| = 2,$$

as graded $k[x_1, \ldots, x_n]$-modules, where $J \subset H(E,k)$ is the nilpotent ideal generated by $H^1(E,k) \cdot H^1(E,k)$, when p is odd.

Quillen's starting point was the following theorem of Evens-Venkov.

Theorem 3.3 ([Ven59, Eve61])**.** *The group cohomology $H^*(G,k)$ is a finitely generated algebra over k. If M is a finitely generated kG-module, then $H^*(G,M)$ is a finitely generated module over $H^*(G,k)$.*

Since $H^*(H,k) \cong H^*(G, kG \otimes_{kH} k)$ by the Eckmann-Shapiro Lemma, we obtain the following consequence.

Corollary 3.4. *For any subgroup $H \subset G$, the group cohomology $H^*(H,k)$ is a finitely generated module over $H^*(G,k)$ via the restriction map.*

Recall that the Krull dimension of a commutative ring is the longest length l of proper inclusions $p_0 \subset p_1 \subset \ldots \subset p_l$ of prime ideals. In particular the Krull dimension of a polynomial ring over a field is the number of indeterminates. The following result of Quillen establishes a conjecture of Atiyah and Swan.

Theorem 3.5. *The Krull dimension of $H(G,k)$ is the p-rank of G, i.e., the maximal rank of its elementary abelian p-subgroups.*

Proof. The restriction maps $\mathrm{res}_{G,E}$ for the elementary abelian p-subgroups $E \subset G$ induce a ring homomorphism

$$\phi \colon H(G,k) \to \prod_{E \subset G} H(E,k).$$

It factors over its image

$$H(G,k) \longrightarrow \phi(H(G,k)) \longrightarrow \prod_{E \subset G} H(E,k)$$

as a surjection whose kernel is nilpotent by the Quillen-Venkov Lemma, followed by an integral extension since $\prod_{E \subset G} H(E,k)$ is finitely generated as a module over $\phi(H(G,k))$ by Corollary 3.4.

Since nilpotent elements are contained in any prime ideal and integral extensions have the same Krull dimension, we obtain

$$\dim H(G,k) = \dim \phi(H(G,k)) = \dim \prod_E H(E,k) = \max_E \dim H(E,k) = p\text{-rank of } G.$$

□

4. Basics of commutative algebra

Let k be an algebraically closed field. We will work with finitely generated commutative algebras A over k. By Hilbert's Basis Theorem, the ring A is noetherian and we may think of A as a quotient

$$A \cong \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}$$

as in affine algebraic geometry.

The *maximal ideal spectrum* is the set

$$\max(A) = \{\mathfrak{m} \mid \mathfrak{m} \subset A \text{ maximal ideal}\}$$

equipped with the Zariski topology given by the closed sets

$$V(I) = \{\mathfrak{m} \in \max(A) | I \subset \mathfrak{m}\}$$

for the ideals I of A.

Any homomorphism of finitely generated commutative algebras $\phi\colon A \to B$ induces a continuous map

$$\phi^*\colon \max(B) \to \max(A), \quad \mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m}),$$

thus max is a contravariant functor to topological spaces.

Fact 4.1. *Let A, B be finitely generated commutative algebras.*

(1) If $\phi\colon A \to B$ is surjective, then ϕ^* is a closed embedding with image $V(\ker \phi)$.
(2) If $i\colon A \to B$ is an integral extension, then the map $i^*\colon \max(B) \to \max(A)$ is surjective and closed.

We will use the following consequence.

Corollary 4.2. *If $\phi\colon A \to B$ is a homomorphism such that B is integral over $\phi(A)$, then $\phi^*\colon \max(B) \to \max(A)$ is a closed map with image $V(\ker \phi)$.*

5. Quillen stratification

Let k be an algebraically closed field of characteristic $p > 0$. For a finite group G, the group cohomology $H^*(G, k)$ is finitely generated by the Evens-Venkov Theorem. Hence so is its "commutative part" $H(G, k) \subset H^*(G, k)$ which we defined in Notation 3.1. Let

$$V_G = \max(H(G, k))$$

be the maximal ideal spectrum of $H(G, k)$.

If $H \subset G$ is a subgroup, then the restriction $\mathrm{res}_{G,H}\colon H(G, k) \to H(H, k)$ induces a map

$$\mathrm{res}^*_{G,H}\colon V_H \to V_G.$$

Theorem 5.1. *The topological space V_G is the union*

$$V_G = \bigcup_{E \subset G} \mathrm{res}^*_{G,E}(V_E)$$

over all elementary abelian p-subgroups $E \subset G$.

Proof. As in the proof of Theorem 3.5, let $\phi\colon H(G, k) \to \prod_{E \subset G} H(E, k)$ be the map induced by the restrictions $\mathrm{res}_{G,E}$. It suffices to show that

$$\coprod_{E \subset G} V_E \cong \max(\prod_{E \subset G} H(E, k)) \xrightarrow{\phi^*} \max(H(G, k)) \cong V_G$$

is surjective. This follows from Corollary 4.2 since ϕ factors over its image $\phi(H(G, k))$ as a surjection with nilpotent kernel followed by an integral extension as explained in the proof of Theorem 3.5. □

Remark 5.2. The subspaces $\mathrm{res}^*_{G,E}(V_E)$ in the stratification are closed and are identical for conjugate elementary abelian p-subgroups. Indeed, Corollary 4.2 applied to $\mathrm{res}_{G,E}$ yields

$$\mathrm{res}^*_{G,E}(V_E) = V(\ker(\mathrm{res}_{G,E})) \subset V_G.$$

Moreover, since the conjugation action by an element $g \in G$ induces a commutative diagram

$$\begin{array}{ccc} H(E, k) & \xrightarrow[\cong]{g^*} & H(gEg^{-1}, k) \\ \uparrow{\scriptstyle \mathrm{res}_{G,E}} & & \uparrow{\scriptstyle \mathrm{res}_{G,gEg^{-1}}} \\ H(G, k) & \xrightarrow{g^* = \mathrm{id}} & H(G, k), \end{array}$$

it follows that $\mathrm{res}^*_{G,E}(V_E) = \mathrm{res}^*_{G,gEg^{-1}}(V_{gEg^{-1}})$.

With a more detailed analysis of the pieces, Quillen established the following refined stratification theorem.

Theorem 5.3. *The restriction maps* $\mathrm{res}_{G,E}$ *induce a homeomorphism*

$$\operatorname*{colim}_{E} V_E \cong V_G,$$

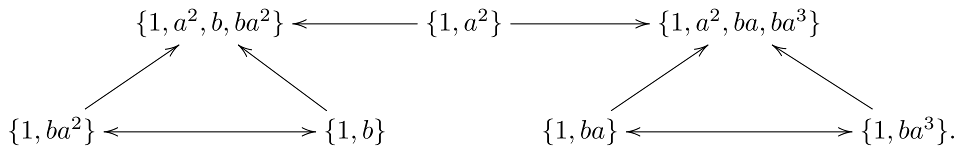
where the colimit is taken over the category with objects the elementary abelian p-subgroups of G and morphisms $E \to E'$ the group homomorphisms of the form $x \mapsto gxg^{-1}$ for some $g \in G$.

Instead of providing a proof, we illustrate it in an example.

Example 5.4. Let $p = 2$ and thus k of characteristic 2, and G the dihedral group of order 8

$$D_8 = \langle a, b \mid a^4 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

The elementary abelian 2-subgroups of D_8 together with inclusions and conjugations are



The subgroups $E_1 = \{1, a^2, b, ba^2\}$ and $E_2 = \{1, a^2, ba, ba^3\}$ are normal in D_8. Their intersection $Z = \{1, a^2\}$ is the center of D_8. For any elementary abelian p-group E and subgroup E', the restriction homomorphism $\mathrm{res}_{E,E'}\colon H^*(E,k) \to H^*(E',k)$ is surjective. Hence $\mathrm{res}^*_{E,E'}\colon V_{E'} \to V_E$ is a closed embedding. It follows that the colimit V_G simplifies to a pushout of two planes glued together along a line

$$V_G \cong \operatorname{colim}_E V_E \cong V_{E_1} \coprod_{V_Z} V_{E_2}.$$

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