Derived and Triangulated Categories

Amnon Neeman

Australian National University

amnon.neeman@anu.edu.au

19 April 2023

Overview

Background—inverting morphisms in a category

Let $\mathcal{A}$ be a category, and let S be a class of morphisms in $\mathcal{A}$.
There exists a functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ such that

Background—inverting morphisms in a category

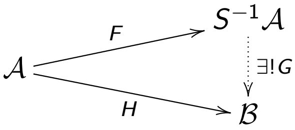
Let $\mathcal{A}$ be a category, and let S be a class of morphisms in $\mathcal{A}$.
There exists a functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ such that

- F takes every morphism in $S \subset \mathcal{A}$ to an isomorphism in $S^{-1}\mathcal{A}$.

Background—inverting morphisms in a category

Let $\mathcal{A}$ be a category, and let S be a class of morphisms in $\mathcal{A}$.
There exists a functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ such that

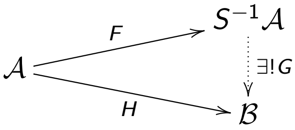
- F takes every morphism in $S \subset \mathcal{A}$ to an isomorphism in $S^{-1}\mathcal{A}$.
- If $H : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor taking every morphism in S to an isomorphism, then there exists a unique functor $G : S^{-1}\mathcal{A} \longrightarrow \mathcal{B}$ rendering commutative the triangle



Background—inverting morphisms in a category

Let $\mathcal{A}$ be a category, and let S be a class of morphisms in $\mathcal{A}$.
There exists a functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ such that

- F takes every morphism in $S \subset \mathcal{A}$ to an isomorphism in $S^{-1}\mathcal{A}$.
- If $H : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor taking every morphism in S to an isomorphism, then there exists a unique functor $G : S^{-1}\mathcal{A} \longrightarrow \mathcal{B}$ rendering commutative the triangle



We call this construction formally inverting the morphisms in S.

Reminder of the construction

As on the previous slide: $\mathcal{A}$ is a category, $S \subset \mathcal{A}$ is a class of morphisms.

Reminder of the construction

As on the previous slide: $\mathcal{A}$ is a category, $S \subset \mathcal{A}$ is a class of morphisms.

Objects of $S^{-1}\mathcal{A}$:

The objects of $S^{-1}\mathcal{A}$ are the same as the objects of $\mathcal{A}$, and on objects the functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ is the identity.

Reminder of the construction

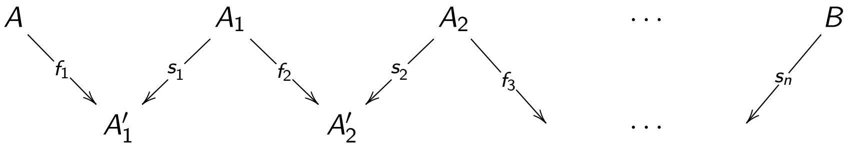
As on the previous slide: $\mathcal{A}$ is a category, $S \subset \mathcal{A}$ is a class of morphisms.

Objects of $S^{-1}\mathcal{A}$:

The objects of $S^{-1}\mathcal{A}$ are the same as the objects of $\mathcal{A}$, and on objects the functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ is the identity.

Morphisms of $S^{-1}\mathcal{A}$:

If A, B are objects of $\mathcal{A}$, then $\mathrm{Hom}_{S^{-1}\mathcal{A}}(A, B)$ is the set of equivalence classes of zigzags



where the s_i belong to S.

Definition of the derived categories $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category. The derived category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ is as follows:

- Objects: cochain complexes of objects in $\mathcal{A}$, that is

$$\cdots \longrightarrow A^{-2} \longrightarrow A^{-1} \longrightarrow A^{0} \longrightarrow A^{1} \longrightarrow A^{2} \longrightarrow \cdots$$

  where the composites $A^i \longrightarrow A^{i+1} \longrightarrow A^{i+2}$ all vanish. The subscript $\mathfrak{C}$ and superscript $\mathfrak{C}'$ stand for conditions.

- Morphisms: cochain maps are examples, that is

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^{0} & \longrightarrow & A^{1} & \longrightarrow & A^{2} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^{0} & \longrightarrow & B^{1} & \longrightarrow & B^{2} & \longrightarrow \cdots
\end{array}$$

  but we formally invert the cohomology isomorphisms.

More generally

If $\mathcal{E}$ is any exact category, we define the categories $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{E})$ analoguously. The objects are still cochain complexes satisfying some conditions.

The issue is with the morphisms—what does it mean for a cochain map to induce an isomorphism in cohomology? Which are the cochain maps we should invert?

More generally

If $\mathcal{E}$ is any exact category, we define the categories $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{E})$ analoguously. The objects are still cochain complexes satisfying some conditions.

The issue is with the morphisms—what does it mean for a cochain map to induce an isomorphism in cohomology? Which are the cochain maps we should invert?

The solution is to invert those maps $f : A^* \longrightarrow B^*$ such that, for every exact functor $F : \mathcal{E} \longrightarrow \mathcal{A}$, with $\mathcal{A}$ abelian, the induced map $F(f) : F(A^*) \longrightarrow F(B^*)$ is an isomorphism in cohomology.

Let R be an associative ring.

Example

1. $\mathbf{D}(R\text{–Mod})$ has for objects all cochain complexes of left R-modules, no conditions.
2. If R is coherent, $\mathbf{D}(R\text{–mod})$ has for objects all cochain complexes of finitely generated left R-modules.
3. If R is coherent, $\mathbf{D}^b(R\text{–mod})$ has for objects all bounded cochain complexes of finitely generated left R-modules. A complex A^* is bounded if $A^i = 0$ for all but finitely many $i \in \mathbb{Z}$.
4. With R still coherent, $\mathbf{D}^-(R\text{–mod})$ has for objects all bounded above cochain complexes of finitely generated left R-modules. A complex A^* is bounded above if $A^i = 0$ for all $i \gg 0$.
5. With R still coherent, $\mathbf{D}^+(R\text{–mod})$ has for objects all bounded below cochain complexes of finitely generated left R-modules. A complex A^* is bounded below if $A^i = 0$ for all $i \ll 0$.

With R still an associative ring.

Example

1. $\mathbf{D}(R\text{–}\mathrm{Proj})$ has for objects all cochain complexes of projective left R-modules. Note that the category $R\text{–}\mathrm{Proj}$ isn't abelian, it is only an exact category.
2. $\mathbf{D}(R\text{–}\mathrm{proj})$ has for objects all cochain complexes of finitely generated projective left R-modules.
3. $\mathbf{D}^b(R\text{–}\mathrm{proj})$ has for objects all bounded cochain complexes of finitely generated, projective left R-modules.
4. $\mathbf{D}^-(R\text{–}\mathrm{proj})$ has for objects all bounded above cochain complexes of finitely generated, projective left R-modules.

Let X be a scheme.

Example

1. $\mathbf{D_{qc}}(X)$ will be our shorthand for $\mathbf{D_{qc}}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.

Let X be a scheme.

Example

1. $\mathbf{D_{qc}}(X)$ will be our shorthand for $\mathbf{D_{qc}}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
2. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is perfect if it is locally isomorphic to a bounded complex of vector bundles.

Let X be a scheme.

Example

1. $\mathbf{D_{qc}}(X)$ will be our shorthand for $\mathbf{D_{qc}}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
2. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is perfect if it is locally isomorphic to a bounded complex of vector bundles. This means: Let E be an object in $\mathbf{D_{qc}}(X)$. It belongs to the full subcategory $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}(X)$

Let X be a scheme.

Example

1. $\mathbf{D}_{\mathbf{qc}}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc}}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
2. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is perfect if it is locally isomorphic to a bounded complex of vector bundles. This means: Let E be an object in $\mathbf{D}_{\mathbf{qc}}(X)$. It belongs to the full subcategory $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D}_{\mathbf{qc}}(X)$ if X has a cover by open sets U_i such that, for each i, the functor $u_i^* : \mathbf{D}_{\mathbf{qc}}(X) \longrightarrow \mathbf{D}_{\mathbf{qc}}(U_i)$, induced by restriction to U_i, takes E to an object $u_i^*(E)$ isomorphic in $\mathbf{D}_{\mathbf{qc}}(U_i)$ to a bounded complex of vector bundles.

Let X be a scheme.

Example

1. $\mathbf{D_{qc}}(X)$ will be our shorthand for $\mathbf{D_{qc}}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
2. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is perfect if it is locally isomorphic to a bounded complex of vector bundles. This means: Let E be an object in $\mathbf{D_{qc}}(X)$. It belongs to the full subcategory $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}(X)$ if X has a cover by open sets U_i such that, for each i, the functor $u_i^* : \mathbf{D_{qc}}(X) \longrightarrow \mathbf{D_{qc}}(U_i)$, induced by restriction to U_i, takes E to an object $u_i^*(E)$ isomorphic in $\mathbf{D_{qc}}(U_i)$ to a bounded complex of vector bundles.
3. Assume X is noetherian. The objects of $\mathbf{D}^b_{\mathbf{coh}}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Example

Let X be a scheme, and let $Z \subset X$ be a closed subset.

1. $\mathbf{D}_{\mathbf{qc},Z}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc},Z}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent,

Example

Let X be a scheme, and let $Z \subset X$ be a closed subset.

1. $\mathbf{D}_{\mathbf{qc},Z}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc},Z}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to $X - Z$ is acyclic.

Example

Let X be a scheme, and let $Z \subset X$ be a closed subset.

1. $\mathbf{D}_{\mathbf{qc},Z}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc},Z}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to $X - Z$ is acyclic.

2. The objects of $\mathbf{D}_Z^{\mathrm{perf}}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ are the perfect complexes.

Example

Let X be a scheme, and let $Z \subset X$ be a closed subset.

1. $\mathbf{D}_{\mathbf{qc},Z}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc},Z}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to $X - Z$ is acyclic.
2. The objects of $\mathbf{D}_Z^{\mathrm{perf}}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ are the perfect complexes.
3. Assume X is noetherian. The objects of $\mathbf{D}^b_{\mathbf{coh},Z}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1]: \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

This data must satisfy the following axioms

[TR1]

[TR2]

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$)

We have asserted that the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ is triangulated.
The endofunctor $[1] : \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) \longrightarrow \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$: It takes the cochain complex A^*, i.e.

$$\cdots \longrightarrow A^{-2} \xrightarrow{\partial^{-2}} A^{-1} \xrightarrow{\partial^{-1}} A^0 \xrightarrow{\partial^0} A^1 \xrightarrow{\partial^1} A^2 \longrightarrow \cdots$$

to the cochain complex $\big(A[1]\big)^*$ below:

$$\cdots \longrightarrow A^{-1} \xrightarrow{-\partial^{-1}} A^0 \xrightarrow{-\partial^0} A^1 \xrightarrow{-\partial^1} A^2 \xrightarrow{-\partial^2} A^3 \longrightarrow \cdots$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

If $f^* : A^* \longrightarrow B^*$ is a cochain map

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{-2} & \xrightarrow{\partial_A^{-2}} & A^{-1} & \xrightarrow{\partial_A^{-1}} & A^{0} & \xrightarrow{\partial_A^{0}} & A^{1} & \xrightarrow{\partial_A^{1}} & A^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \\
\cdots & \longrightarrow & B^{-2} & \xrightarrow[\partial_B^{-2}]{} & B^{-1} & \xrightarrow[\partial_B^{-1}]{} & B^{0} & \xrightarrow[\partial_B^{0}]{} & B^{1} & \xrightarrow[\partial_B^{1}]{} & B^{2} & \longrightarrow & \cdots
\end{array}
$$

then $\big(f[1]\big)^*$ is the cochain map

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{-1} & \xrightarrow{-\partial_A^{-1}} & A^{0} & \xrightarrow{-\partial_A^{0}} & A^{1} & \xrightarrow{-\partial_A^{1}} & A^{2} & \xrightarrow{-\partial_A^{2}} & A^{3} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & B^{-1} & \xrightarrow[-\partial_B^{-1}]{} & B^{0} & \xrightarrow[-\partial_B^{0}]{} & B^{1} & \xrightarrow[-\partial_B^{1}]{} & B^{2} & \xrightarrow[-\partial_B^{2}]{} & B^{3} & \longrightarrow & \cdots
\end{array}
$$

For the attentive, careful listeners

Let $\mathcal{A}$ be an abelian category. We let $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ be the category with the same objects as $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, but where the morphisms are the honest cochain maps. And we let S be the class of all morphisms in $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ which induce isomorphisms in cohomology.

By definition $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) = S^{-1}\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$.

For the attentive, careful listeners

Let $\mathcal{A}$ be an abelian category. We let $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ be the category with the same objects as $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, but where the morphisms are the honest cochain maps. And we let S be the class of all morphisms in $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ which induce isomorphisms in cohomology.

By definition $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) = S^{-1}\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$.

$$\begin{array}{ccc}
\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) & \xrightarrow{[1]} & \mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) & & \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})
\end{array}$$

For the attentive, careful listeners

Let $\mathcal{A}$ be an abelian category. We let $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ be the category with the same objects as $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, but where the morphisms are the honest cochain maps. And we let S be the class of all morphisms in $\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ which induce isomorphisms in cohomology.

By definition $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) = S^{-1}\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$.

$$\begin{array}{ccc}
\mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) & \xrightarrow{\ [1]\ } & \mathbf{C}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) & \xrightarrow[\exists ! [1]]{} & \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})
\end{array}$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

The exact triangles: Suppose we are given a commutative diagram in $\mathcal{A}$, where the rows are objects of $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^{0} & \longrightarrow & X^{1} & \longrightarrow & X^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Y^{-2} & \longrightarrow & Y^{-1} & \longrightarrow & Y^{0} & \longrightarrow & Y^{1} & \longrightarrow & Y^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Z^{-2} & \longrightarrow & Z^{-1} & \longrightarrow & Z^{0} & \longrightarrow & Z^{1} & \longrightarrow & Z^{2} & \longrightarrow & \cdots
\end{array}
$$

We may view the above as morphisms $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^*$ in the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$.

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

The exact triangles: Suppose we are given a commutative diagram in $\mathcal{A}$, where the rows are objects of $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^{0} & \longrightarrow & X^{1} & \longrightarrow & X^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Y^{-2} & \longrightarrow & Y^{-1} & \longrightarrow & Y^{0} & \longrightarrow & Y^{1} & \longrightarrow & Y^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Z^{-2} & \longrightarrow & Z^{-1} & \longrightarrow & Z^{0} & \longrightarrow & Z^{1} & \longrightarrow & Z^{2} & \longrightarrow & \cdots
\end{array}$$

We may view the above as morphisms $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^*$ in the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$.

Assume further that, for each $i \in \mathbb{Z}$, the sequence $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i$ is split exact. Choose, for each $i \in \mathbb{Z}$, a splitting $\theta^i : Z^i \longrightarrow Y^i$ of the map $g^i : Y^i \longrightarrow Z^i$.

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

Now for each i we have the diagram

$$\begin{array}{ccccc}
Z^i & \xrightarrow{\theta^i} & Y^i & \xrightarrow{g^i} & Z^i \\
\Big\downarrow{\scriptstyle \partial_Z^i} & & \Big\downarrow{\scriptstyle \partial_Y^i} & & \Big\downarrow{\scriptstyle \partial_Z^i} \\
Z^{i+1} & \xrightarrow{\theta^{i+1}} & Y^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1}
\end{array}$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

Now for each i we have the diagram

$$\begin{array}{ccccc} Z^i & \xrightarrow{\theta^i} & Y^i & & \\ \Big\downarrow{\scriptstyle \partial_Z^i} & & \Big\downarrow{\scriptstyle \partial_Y^i} & & \\ Z^{i+1} & \xrightarrow{\theta^{i+1}} & Y^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1} \end{array}$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

Thus the difference $\theta^{i+1}\partial_Z^i - \partial_Y^i\theta^i$ is annihilated by the map $g^{i+1} : Y^{i+1} \longrightarrow Z^{i+1}$, hence must factor uniquely as $Z^i \xrightarrow{h^i} X^{i+1} \xrightarrow{f^{i+1}} Y^{i+1}$. Form the diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle h^{1}} & & \downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots
\end{array}$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle h^{1}} & & \downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & Y^{-1} & \xrightarrow[-\partial_Y^{-1}]{} & Y^{0} & \xrightarrow[-\partial_Y^{0}]{} & Y^{1} & \xrightarrow[-\partial_Y^{1}]{} & Y^{2} & \xrightarrow[-\partial_Y^{2}]{} & Y^{3} & \longrightarrow & \cdots
\end{array}
$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle h^{1}} & & \downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & Y^{-1} & \xrightarrow[-\partial_Y^{-1}]{} & Y^{0} & \xrightarrow[-\partial_Y^{0}]{} & Y^{1} & \xrightarrow[-\partial_Y^{1}]{} & Y^{2} & \xrightarrow[-\partial_Y^{2}]{} & Y^{3} & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccc}
Z^{i} & \xrightarrow{\partial_Z^{i}} & Z^{i+1} \\
{\scriptstyle h^{i}}\downarrow & & \downarrow{\scriptstyle h^{i+1}} \\
X^{i+1} & \xrightarrow[-\partial_X^{i+1}]{} & X^{i+2}
\end{array}$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle h^{-2}} & & \Big\downarrow{\scriptstyle h^{-1}} & & \Big\downarrow{\scriptstyle h^{0}} & & \Big\downarrow{\scriptstyle h^{1}} & & \Big\downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f^{-1}} & & \Big\downarrow{\scriptstyle f^{0}} & & \Big\downarrow{\scriptstyle f^{1}} & & \Big\downarrow{\scriptstyle f^{2}} & & \Big\downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & Y^{-1} & \xrightarrow[-\partial_Y^{-1}]{} & Y^{0} & \xrightarrow[-\partial_Y^{0}]{} & Y^{1} & \xrightarrow[-\partial_Y^{1}]{} & Y^{2} & \xrightarrow[-\partial_Y^{2}]{} & Y^{3} & \longrightarrow & \cdots
\end{array}
$$

$$
\begin{array}{ccc}
Z^{i} & \xrightarrow{\partial_Z^{i}} & Z^{i+1} \\
{\scriptstyle h^{i}}\Big\downarrow & & \Big\downarrow{\scriptstyle h^{i+1}} \\
X^{i+1} & \xrightarrow[-\partial_X^{i+1}]{} & X^{i+2} \\
{\scriptstyle f^{i+1}}\Big\downarrow & & \Big\downarrow{\scriptstyle f^{i+2}} \\
Y^{i+1} & \xrightarrow[-\partial_Y^{i+1}]{} & Y^{i+2}
\end{array}
$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle h^{1}} & & \downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & Y^{-1} & \xrightarrow[-\partial_Y^{-1}]{} & Y^{0} & \xrightarrow[-\partial_Y^{0}]{} & Y^{1} & \xrightarrow[-\partial_Y^{1}]{} & Y^{2} & \xrightarrow[-\partial_Y^{2}]{} & Y^{3} & \longrightarrow & \cdots
\end{array}
$$

$$
\begin{array}{ccc}
Z^{i} & \xrightarrow{\partial_Z^{i}} & Z^{i+1} \\
\downarrow{\scriptstyle h^{i}} & & \downarrow{\scriptstyle h^{i+1}} \\
X^{i+1} & \xrightarrow[-\partial_X^{i+1}]{} & X^{i+2} \\
 & & \downarrow{\scriptstyle f^{i+2}} \\
 & & Y^{i+2}
\end{array}
$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

Thus $h^* : Z^* \longrightarrow X^*[1]$ is a cochain map. We have constructed in the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ a diagram $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{h^*} X^*[1]$. We declare

- The exact triangles in $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ are all the isomorphs, in $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, of diagrams that come from our construction.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

This data must satisfy the following axioms

[TR1] Any isomorph of an exact triangle is an exact triangle. For any object $X \in \mathcal{T}$ the diagram $0 \longrightarrow X \xrightarrow{\mathrm{id}} X \longrightarrow 0$ is an exact triangle.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

This data must satisfy the following axioms

[TR1] Any isomorph of an exact triangle is an exact triangle. For any object $X \in \mathcal{T}$ the diagram $0 \longrightarrow X \xrightarrow{\mathrm{id}} X \longrightarrow 0$ is an exact triangle. Any morphism $f : X \longrightarrow Y$ may be completed to an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a triangulated structure if:

1. It has an invertible additive endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of exact triangles, meaning diagrams in $\mathcal{T}$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

This data must satisfy the following axioms

[TR1] Any isomorph of an exact triangle is an exact triangle. For any object $X \in \mathcal{T}$ the diagram $0 \longrightarrow X \xrightarrow{\mathrm{id}} X \longrightarrow 0$ is an exact triangle. Any morphism $f : X \longrightarrow Y$ may be completed to an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

[TR2] Any rotation of an exact triangle is exact. That is: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an exact triangle if and only if $Y \xrightarrow{-g} Z \xrightarrow{-h} X[1] \xrightarrow{-f[1]} Y[1]$ is.

Definition (definition of triangulated categories—continued)

[TR3+4] Given a commutative diagram, where the rows are exact triangles,

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

Definition (definition of triangulated categories—continued)

[TR3+4] Given a commutative diagram, where the rows are exact triangles,

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}$$

we may complete it to a commutative diagram (also known as a morphism of triangles)

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}$$

Definition (definition of triangulated categories—continued)

[TR3+4] (continued): Moreover: we can do it in such a way that

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -g & 0 \\ v & f' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -h & 0 \\ w & g' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -f[1] & 0 \\ u[1] & h' \end{pmatrix}} Y[1] \oplus X'[1]$$

is an exact triangle.

If $\mathcal{T}$ is triangulated then so is $\mathcal{T}^{\mathrm{op}}$

The endomorphism $[1] : \mathcal{T} \longrightarrow \mathcal{T}$ gets replaced by $[-1] : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$, where $[-1] = [1]^{-1}$.

If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is an exact triangle in $\mathcal{T}$, we declare it to also be an exact triangle in $\mathcal{T}^{\mathrm{op}}$. The point being that the rotation

$$Z[-1] \xrightarrow{-h} X \xrightarrow{-f} Y \xrightarrow{-g} Z$$

has the required form.

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.
- We will lazily abbreviate "exact triangle" to just "triangle".

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.
- We will lazily abbreviate "exact triangle" to just "triangle".
- A full subcategory $\mathcal{S} \subset \mathcal{T}$ is called triangulated if $0 \in \mathcal{S}$, if $\mathcal{S}[1] = \mathcal{S}$, and if, whenever $X, Y \in \mathcal{S}$ and there exists in $\mathcal{T}$ a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$, we must also have $Z \in \mathcal{S}$.

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.
- We will lazily abbreviate "exact triangle" to just "triangle".
- A full subcategory $\mathcal{S} \subset \mathcal{T}$ is called triangulated if $0 \in \mathcal{S}$, if $\mathcal{S}[1] = \mathcal{S}$, and if, whenever $X, Y \in \mathcal{S}$ and there exists in $\mathcal{T}$ a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$, we must also have $Z \in \mathcal{S}$.
- The subcategory $\mathcal{S}$ is thick if it is triangulated, as well as closed in $\mathcal{T}$ under direct summands.

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.
- We will lazily abbreviate "exact triangle" to just "triangle".
- A full subcategory $\mathcal{S} \subset \mathcal{T}$ is called triangulated if $0 \in \mathcal{S}$, if $\mathcal{S}[1] = \mathcal{S}$, and if, whenever $X, Y \in \mathcal{S}$ and there exists in $\mathcal{T}$ a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$, we must also have $Z \in \mathcal{S}$.
- The subcategory $\mathcal{S}$ is thick if it is triangulated, as well as closed in $\mathcal{T}$ under direct summands.
- Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{A}$ be an abelian category. A functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is homological if it takes triangles to long exact sequences.

First Lemmas

Lemma

If $\mathcal{T}$ is a triangulated category, and if $t \in \mathcal{T}$ is an object, then the functor $\mathrm{Hom}(t, -) : \mathcal{T} \longrightarrow Ab$ *is homological.*

First Lemmas

Lemma

If $\mathcal{T}$ is a triangulated category, and if $t \in \mathcal{T}$ is an object, then the functor $\mathrm{Hom}(t, -) : \mathcal{T} \longrightarrow Ab$ is homological.

Proof.

If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is an exact triangle in $\mathcal{T}$, we need to prove that

$$\mathrm{Hom}(t, A) \longrightarrow \mathrm{Hom}(t, B) \longrightarrow \mathrm{Hom}(t, C) \longrightarrow \mathrm{Hom}(t, A[1]) \longrightarrow$$

is a long exact sequence. By [TR2], the axiom saying that any rotation of an exact triangle is an exact triangle, it suffices to prove that

$$\mathrm{Hom}(t, A) \longrightarrow \mathrm{Hom}(t, B) \longrightarrow \mathrm{Hom}(t, C)$$

is exact. □

Proof, continued.

Let f be an element in $\mathrm{Hom}(t, A)$, that is f is a morphism $f : t \longrightarrow A$. Consider the commutative diagram

$$\begin{array}{ccccccc}
t & \xrightarrow{1} & t & \longrightarrow & 0 & \longrightarrow & t[1] \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle uf} & & & & \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

The rows are triangles, and [TR3+4] permits us to extend the commutative diagram to a morphism of triangles

$$\begin{array}{ccccccc}
t & \xrightarrow{1} & t & \longrightarrow & 0 & \longrightarrow & t[1] \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle uf} & & \big\downarrow & & \big\downarrow{\scriptstyle f[1]} \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

The commutativity of the middle square tells us that $vuf = 0$, □

Proof, continued.

which proves the vanishing of the composite

$$\mathrm{Hom}(t,A) \longrightarrow \mathrm{Hom}(t,B) \longrightarrow \mathrm{Hom}(t,C)$$

Now let f be an element of the kernel of $\mathrm{Hom}(t,B) \longrightarrow \mathrm{Hom}(t,C)$. That is $f : t \longrightarrow B$ is a morphism such that the composite $t \xrightarrow{f} B \xrightarrow{v} C$ vanishes. Thus we have a commutative diagram

$$\begin{array}{ccccccc} t & \xrightarrow{1} & t & \longrightarrow & 0 & \longrightarrow & t[1] \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1] \end{array}$$

By a rotation of [TR3+4] we may complete to a morphism of triangles □

Proof, continued.

$$\begin{array}{ccccccc}
t & \xrightarrow{1} & t & \longrightarrow & 0 & \longrightarrow & t[1] \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g[1]} \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

and this yields an equality $f = ug$ with $g \in \mathrm{Hom}(t, A)$. That is f is the image of $g \in \mathrm{Hom}(t, A)$ under the map $\mathrm{Hom}(t, A) \longrightarrow \mathrm{Hom}(t, B)$. □

Corollary

Given any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

we have $vu = 0$.

Corollary

Given any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

we have $vu = 0$.

Proof.

The image of $1 \in \mathrm{Hom}(A, A)$ under the exact sequence

$$\mathrm{Hom}(A, A) \xrightarrow{\mathrm{Hom}(A,u)} \mathrm{Hom}(A, B) \xrightarrow{\mathrm{Hom}(A,v)} \mathrm{Hom}(A, C)$$

must vanish. □

In the light of our Lemma, it makes sense to formulate

Definition

Let $\mathcal{T}$ be a triangulated category. A sequence $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is called a weak triangle if, for every object $t \in \mathcal{T}$, the functor $\mathrm{Hom}(t, -)$ takes $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ to a long exact sequence.

In the light of our Lemma, it makes sense to formulate

Definition

Let $\mathcal{T}$ be a triangulated category. A sequence $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is called a weak triangle if, for every object $t \in \mathcal{T}$, the functor $\mathrm{Hom}(t, -)$ takes $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ to a long exact sequence.

Reformulating the first lemma

In terms of the above definition, the first Lemma asserts that every exact triangle is a weak triangle.

Lemma

Let $\mathcal{T}$ be a triangulated category, and let

$$\begin{array}{ccccccc}
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \\
\Big\downarrow f & & \Big\downarrow g & & \Big\downarrow h & & \Big\downarrow f[1] \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

be a commutative diagram where the rows are weak triangles. If f and g are isomorphisms then so is h.

Lemma

Let $\mathcal{T}$ be a triangulated category, and let

$$\begin{array}{ccccccc}
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

be a commutative diagram where the rows are weak triangles. If f and g are isomorphisms then so is h.

Proof.

For any object $t \in \mathcal{T}$, the functor $\mathrm{Hom}(t, -)$ takes the above to a commutative diagram with long exact rows, in which $\mathrm{Hom}(t, f[n])$ and $\mathrm{Hom}(t, g[n])$ are isomorphisms for all $n \in \mathbb{Z}$. The 5-lemma tells us that $\mathrm{Hom}(t, h[n])$ are also isomorphisms for all $n \in \mathbb{Z}$, and by Yoneda's lemma h must be an isomorphism. □

Corollary

Let $\mathcal{T}$ be a triangulated category. If

$$A \xrightarrow{\;u\;} B \xrightarrow{\;v\;} C \xrightarrow{\;w\;} A[1] \;, \qquad\qquad A \xrightarrow{\;u\;} B \xrightarrow{\;v'\;} C' \xrightarrow{\;w'\;} A[1]$$

are exact triangles then they are (non-canonically) isomorphic.

Corollary

Let $\mathcal{T}$ be a triangulated category. If

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]\ , \qquad\qquad A \xrightarrow{u} B \xrightarrow{v'} C' \xrightarrow{w'} A[1]$$

are exact triangles then they are (non-canonically) isomorphic.

Proof.

The commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v'} & C' & \xrightarrow{w'} & A[1] \\
\| & & \| & & & & \\
A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1]
\end{array}$$

has exact triangles for rows, and [TR3+4] permits us to extend to a commutative diagram □

Proof, continued.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v'} & C' & \xrightarrow{w'} & A[1] \\ \| & & \| & & \downarrow h & & \| \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & A[1] \end{array}$$

The identity maps $1 : A \longrightarrow A$ and $1 : B \longrightarrow B$ are isomorphisms, hence so is $h : C' \longrightarrow C$. □

Corollary

Let $\mathcal{T}$ be a triangulated category. If

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]\ , \qquad\qquad A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} A'[1]$$

are exact triangles then so is

$$A \oplus A' \xrightarrow{\ u\oplus u'\ } B \oplus B' \xrightarrow{\ v\oplus v'\ } C \oplus C' \xrightarrow{\ w\oplus w'\ } (A \oplus A')[1]$$

Proof.

By [TR1] we may complete the morphism $u \oplus u'$ to an exact triangle

$$A \oplus A' \xrightarrow{\ u\oplus u'\ } B \oplus B' \xrightarrow{\ \widetilde{v}\ } \widetilde{C} \xrightarrow{\ \widetilde{w}\ } (A \oplus A')[1]$$

□

Proof, continued.

And by [TR3+4] we may complete the commutative diagrams

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\downarrow & & \downarrow & & & & \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & (A \oplus A')[1]
\end{array}$$

and

$$\begin{array}{ccccccc}
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \\
\downarrow & & \downarrow & & & & \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & (A \oplus A')[1]
\end{array}$$

□

Proof, continued.

to commutative diagrams

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & (A \oplus A')[1]
\end{array}$$

and

$$\begin{array}{ccccccc}
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & (A \oplus A')[1]
\end{array}$$

□

Proof, continued.

Combining, we have a commutative diagram

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{v \oplus v'} & C \oplus C' & \xrightarrow{w \oplus w'} & (A \oplus A')[1] \\
\| & & \| & & \downarrow h & & \| \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & (A \oplus A')[1]
\end{array}$$

Since the rows are weak triangles the map h must be an isomorphism. The bottom row is an exact triangle by construction, and [TR1] now tells us that so is the isomorphic top row. □

Corollary

Let $\mathcal{T}$ be a triangulated category. If

$$A \oplus A' \xrightarrow{\quad u \oplus u' \quad} B \oplus B' \xrightarrow{\quad v \oplus v' \quad} C \oplus C' \xrightarrow{\quad w \oplus w' \quad} (A \oplus A')[1]$$

is an exact triangle, then so is the direct summand

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1] \ .$$

Proof.

By [TR1] we may complete the morphism u to an exact triangle

$$A \xrightarrow{\quad u \quad} B \xrightarrow{\quad \widetilde{v} \quad} \widetilde{C} \xrightarrow{\quad \widetilde{w} \quad} A[1]$$

□

Proof, continued.

And by [TR3+4] we may complete the commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;u\;} & B & \xrightarrow{\;\widetilde{v}\;} & \widetilde{C} & \xrightarrow{\;\widetilde{w}\;} & A[1] \\
\downarrow & & \downarrow & & & & \\
A \oplus A' & \xrightarrow{\;u \oplus u'\;} & B \oplus B' & \xrightarrow{\;v \oplus v'\;} & C \oplus C' & \xrightarrow{\;w \oplus w'\;} & (A \oplus A')[1]
\end{array}
$$

to the morphism of triangles

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;u\;} & B & \xrightarrow{\;\widetilde{v}\;} & \widetilde{C} & \xrightarrow{\;\widetilde{w}\;} & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A \oplus A' & \xrightarrow{\;u \oplus u'\;} & B \oplus B' & \xrightarrow{\;v \oplus v'\;} & C \oplus C' & \xrightarrow{\;w \oplus w'\;} & (A \oplus A')[1]
\end{array}
$$

$\square$

Proof, continued.

The commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A \oplus A' & \xrightarrow{u \oplus u'} & B \oplus B' & \xrightarrow{v \oplus v'} & C \oplus C' & \xrightarrow{w \oplus w'} & (A \oplus A')[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1]
\end{array}$$

where the morphism between the second and third row is the projection to a direct summand, composes to give □

Proof, continued.

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & A[1] \\
\| & & \| & & \downarrow h & & \| \\
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1]
\end{array}$$

and as both rows are weak triangles the map h must be an isomorphism. The top row is an exact triangle by construction, and [TR1] now tells us that so is the isomorphic bottom row. □

Theorem (octahedral axiom)

Let $\mathcal{T}$ be a triangulated category. Suppose $A \xrightarrow{f} B \xrightarrow{g} B'$ are two composable morphisms, and choose exact triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad u \quad} & C & \xrightarrow{\quad v \quad} & A[1] \\
A & \xrightarrow{\quad gf \quad} & B' & \xrightarrow{\quad u' \quad} & C' & \xrightarrow{\quad v' \quad} & A[1] \\
B & \xrightarrow{\quad g \quad} & B' & \xrightarrow{\quad g' \quad} & B'' & \xrightarrow{\quad \ell \quad} & B[1]
\end{array}$$

which exist by [TR1].

Then there exist morphisms $h : C \longrightarrow C'$ and $k : C' \longrightarrow B''$ such that

Theorem (octahedral axiom, continued)

the following diagram commutes

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;u\;} & C & \xrightarrow{\;v\;} & A[1] \\
\| & & \downarrow{\scriptstyle g} & & \color{red}{\downarrow{\scriptstyle h}} & & \| \\
A & \xrightarrow{\;gf\;} & B' & \xrightarrow{\;u'\;} & C' & \xrightarrow{\;v'\;} & A[1] \\
 & & \downarrow{\scriptstyle g'} & & \color{red}{\downarrow{\scriptstyle k}} & & \\
 & & B'' & = & B'' & & \\
 & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle u[1]\circ\ell} & & \\
 & & B[1] & \xrightarrow{\;u[1]\;} & C[1] & &
\end{array}
$$

and the third column is an exact triangle.

Proof.

We are given the commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;u\;} & C & \xrightarrow{\;v\;} & A[1] \\
\| & & \downarrow{\scriptstyle g} & & & & \\
A & \xrightarrow{\;gf\;} & B' & \xrightarrow{\;u'\;} & C' & \xrightarrow{\;v'\;} & A[1]
\end{array}$$

where the rows are exact triangles. [TR3+4] permits us to extend to a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;u\;} & C & \xrightarrow{\;v\;} & A[1] \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \| \\
A & \xrightarrow{\;gf\;} & B' & \xrightarrow{\;u'\;} & C' & \xrightarrow{\;v'\;} & A[1]
\end{array}$$

and do it in such a way that □

Proof, continued.

$$
\begin{array}{ccccc}
B \oplus A & \xrightarrow{\begin{pmatrix} -u & 0 \\ g & gf \end{pmatrix}} & C \oplus B' & \xrightarrow{\begin{pmatrix} -v & 0 \\ h & u' \end{pmatrix}} & A[1] \oplus C' \\
 & & & & \Big\downarrow {\scriptstyle \begin{pmatrix} -f[1] & 0 \\ 1 & v' \end{pmatrix}} \\
 & & & & B[1] \oplus A[1]
\end{array}
$$

is an exact triangle. □

Proof, continued.

This triangle is isomorphic to the direct sum of

$$B \xrightarrow{\begin{pmatrix} -u \\ g \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} h & u' \end{pmatrix}} C' \xrightarrow{fv'} B[1]$$

and

$$A \longrightarrow 0 \longrightarrow A[1] = A[1]$$

and both must be exact triangles. □

Proof, continued.

And now the commutative diagram

$$
\begin{array}{ccccccc}
B & \xrightarrow{\begin{pmatrix} -u \\ g \end{pmatrix}} & C \oplus B' & \xrightarrow{\begin{pmatrix} h & u' \end{pmatrix}} & C' & \xrightarrow{fv'} & B[1] \\
\| & & \big\downarrow{\scriptstyle (0 \quad 1)} & & & & \\
B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' & \xrightarrow{\ell} & B[1]
\end{array}
$$

has exact triangles for rows, and [TR3+4] permits us to extend to a commutative diagram

$$
\begin{array}{ccccccc}
B & \xrightarrow{\begin{pmatrix} -u \\ g \end{pmatrix}} & C \oplus B' & \xrightarrow{\begin{pmatrix} h & u' \end{pmatrix}} & C' & \xrightarrow{fv'} & B[1] \\
\| & & \big\downarrow{\scriptstyle (0 \quad 1)} & & \big\downarrow{\scriptstyle k} & & \| \\
B & \xrightarrow{g} & B' & \xrightarrow{g'} & B'' & \xrightarrow{\ell} & B[1]
\end{array}
$$

Proof, continued.

and do it in such a way that

$$
\begin{array}{c}
C \oplus B' \oplus B \xrightarrow{\begin{pmatrix} -h & -u' & 0 \\ 0 & 1 & g \end{pmatrix}} C' \oplus B' \xrightarrow{\begin{pmatrix} -fv' & 0 \\ k & g' \end{pmatrix}} B[1] \oplus B'' \\
\Big\downarrow {\scriptstyle \begin{pmatrix} u[1] & 0 \\ -g[1] & 0 \\ 1 & \ell \end{pmatrix}} \\
(C \oplus B' \oplus B)[1]
\end{array}
$$

is an exact triangle. And this exact triangle is isomorphic to the direct sum of

□

Proof, continued.

$$
\begin{array}{ccccccc}
C & \xrightarrow{h} & C' & \xrightarrow{k} & B'' & \xrightarrow{u[1]\circ\ell} & C[1] \\
\\
B' & = & B' & \longrightarrow & 0 & \longrightarrow & B'[1] \\
\\
B & \longrightarrow & 0 & \longrightarrow & B[1] & = & B[1]
\end{array}
$$

which must all be exact triangles.

□

Flaws of triangulated categories

Lemma

If $\mathcal{T}$ is a triangulated category and $g : B \longrightarrow C$ is an epimorphism, then $B \cong A \oplus C$ and g is the split surjection $A \oplus C \longrightarrow C$.

Proof.

Complete g to an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$. The composite $hg : B \longrightarrow A[1]$ vanishes, and as g is an epimorphism we deduce that $h = 0$. □

Proof, continued.

But now consider the commutative diagram where the rows are triangles

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{0} & A[1] \\
 & & & & \| & & \| \\
A & \longrightarrow & A \oplus C & \longrightarrow & C & \xrightarrow{0} & A[1]
\end{array}
$$

By [TR3+4] we may complete to a commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{0} & A[1] \\
\| & & \downarrow{\scriptstyle \rho} & & \| & & \| \\
A & \longrightarrow & A \oplus C & \longrightarrow & C & \xrightarrow{0} & A[1]
\end{array}
$$

and ρ must be an isomorphism. □

No cokernels

Suppose $f : X \longrightarrow B$ is a morphism in a triangulated category $\mathcal{T}$, and $g : B \longrightarrow C$ is its cokernel. Then g is an epimorphism, and the above lemma says it must be isomorphic to the projection $A \oplus C \longrightarrow C$.

The fact that $f : X \longrightarrow A \oplus C$ has cokernel $A \oplus C \longrightarrow C$ means that map f must factor as $X \xrightarrow{g} A \xrightarrow{i} A \oplus C$, and the map $X \longrightarrow A$ must be an epimorphism. Hence the map $g : X \longrightarrow A$ is isomorphic to the projection $\pi : Y \oplus A \longrightarrow A$.

Thus the morphism $f : X \longrightarrow B$ is isomorphic to the composite $Y \oplus A \xrightarrow{\pi} A \xrightarrow{i} A \oplus C$, where π is the projection and i is the inclusion.

Summarizing: morphisms in triangulated categories rarely have cokernels.

Thank you!