Analytic ideas applied to triangulated categories, 3

Amnon Neeman

Australian National University

amnon.neeman@anu.edu.au

17 May 2023

Overview

The construction of $\langle G\rangle_\ell^{[m,n]}$, of $\overline{\langle G\rangle}_\ell^{[m,n]}$, of $\langle G\rangle^{[m,n]}$ and of $\overline{\langle G\rangle}^{[m,n]}$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let ℓ, m, n be integers with $\ell > 0$ and with $m \leq n$. In the last talk we went through the construction of four full subcategories of $\mathcal{T}$:

1.

2.

The construction of $\langle G\rangle_\ell^{[m,n]}$, of $\overline{\langle G\rangle}_\ell^{[m,n]}$, of $\langle G\rangle^{[m,n]}$ and of $\overline{\langle G\rangle}^{[m,n]}$

Let $\mathcal{T}$ be a triangulated category. Let $G\in\mathcal{T}$ be an object, and let ℓ, m, n be integers with $\ell>0$ and with $m\leq n$. In the last talk we went through the construction of four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_\ell^{[m,n]}$ and $\overline{\langle G\rangle}_\ell^{[m,n]}$. The construction was by induction on the integer $\ell>0$, starting with $\langle G\rangle_1^{[m,n]}$ and $\overline{\langle G\rangle}_1^{[m,n]}$, which contain all direct summands of (finite) direct sums of shifts of G in the interval $[m,n]$.
2.

The construction of $\langle G\rangle_\ell^{[m,n]}$, of $\overline{\langle G\rangle}_\ell^{[m,n]}$, of $\langle G\rangle^{[m,n]}$ and of $\overline{\langle G\rangle}^{[m,n]}$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let ℓ, m, n be integers with $\ell > 0$ and with $m \leq n$. In the last talk we went through the construction of four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_\ell^{[m,n]}$ and $\overline{\langle G\rangle}_\ell^{[m,n]}$. The construction was by induction on the integer $\ell > 0$, starting with $\langle G\rangle_1^{[m,n]}$ and $\overline{\langle G\rangle}_1^{[m,n]}$, which contain all direct summands of (finite) direct sums of shifts of G in the interval $[m, n]$.
2. $\langle G\rangle^{[m,n]}$ and $\overline{\langle G\rangle}^{[m,n]}$. The shifts allowed were in the interval $[m, n]$, but then one closed with respect to all extensions, (finite) direct sums and direct summands.

Definition (formal definition of (weak) approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if:

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G^\perp$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and with $E \in \overline{\langle G \rangle}^{[-A,A]}$.
- The category $\mathcal{T}$ is declared approximable if, in the triangle $E \longrightarrow F \longrightarrow D$ above, we may assume $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.

The main theorems—sources of examples

1. If $\mathcal{T}$ has a compact generator G such that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.

2. Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then the category $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable.

3. Let X be a quasicompact, separated scheme. Then the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable.

4. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{S} \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

The main theorems—sources of examples

1. If $\mathcal{T}$ has a compact generator G such that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.

2. Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then the category $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable.

3. Let X be a quasicompact, separated scheme. Then the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable.

4. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \rightleftarrows \mathcal{S} \rightleftarrows \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

References for the fact(s) that the nontrivial examples of (weakly) approximable triangulated categories really are examples

Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, https://arxiv.org/abs/1806.05342.

Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^{b}_{\mathbf{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, https://arxiv.org/abs/2202.08861.

We remind the reader what the terms used in the theorems mean.

Some old definitions

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

- G is a classical generator if $\mathcal{S} = \langle G \rangle^{(-\infty,\infty)}$.
-

We remind the reader what the terms used in the theorems mean.

Some old definitions

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

- G is a classical generator if $\mathcal{S} = \langle G \rangle^{(-\infty,\infty)}$.
- G is a strong generator if there exists an integer $\ell > 0$ with $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. The category $\mathcal{S}$ is strongly generated if there exists a strong generator $G \in \mathcal{S}$.

The main theorems

1. Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

2.

The main theorems

1. Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

2. Let X be a finite-dimensional, separated, quasiexcellent noetherian scheme. Then the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated.

Proof of strong generation

The main point is that approximability allows us to easily reduce to Kelly's old theorem. We first remind the reader of Kelly's theorem and its proof.

Theorem (Kelly, 1965)

Suppose R is a ring, and $\mathbf{D}(R)$ its derived category. Let $n \geq 0$ be an integer, and let $F \in \mathbf{D}(R)$ be an object so that the projective dimension of $H^i(F)$ is $\leq n$ for all $i \in \mathbb{Z}$. Then $F \in \overline{\langle R \rangle}_{n+1}^{(-\infty,\infty)}$.

Proof of strong generation

The main point is that approximability allows us to easily reduce to Kelly's old theorem. We first remind the reader of Kelly's theorem and its proof.

Theorem (Kelly, 1965)

Suppose R is a ring, and $\mathbf{D}(R)$ its derived category. Let $n \geq 0$ be an integer, and let $F \in \mathbf{D}(R)$ be an object so that the projective dimension of $H^i(F)$ is $\leq n$ for all $i \in \mathbb{Z}$. Then $F \in \overline{\langle R \rangle}_{n+1}^{(-\infty,\infty)}$.

Before proving the theorem we remind the reader: any morphism $P \longrightarrow H^i(E)$ in $\mathbf{D}(R)$, for any projective R–module P and any $E \in \mathbf{D}(R)$, lifts (uniquely up to homotopy) to a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E^{i-2} & \longrightarrow & E^{i-1} & \longrightarrow & E^{i} & \longrightarrow & E^{i+1} & \longrightarrow & E^{i+2} & \longrightarrow & \cdots
\end{array}$$

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$.

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \longrightarrow H^i(F)$ lifts to a cochain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(F) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \longrightarrow H^i(F)$ lifts to a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(F) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

and when we combine, for every $i \in \mathbb{Z}$, we obtain a cochain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H^{-2}(F) & \xrightarrow{0} & H^{-1}(F) & \xrightarrow{0} & H^0(F) & \xrightarrow{0} & H^1(F) & \xrightarrow{0} & H^2(F) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots
\end{array}$$

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \longrightarrow H^i(F)$ lifts to a cochain map

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(F) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow \cdots
\end{array}$$

and when we combine, for every $i \in \mathbb{Z}$, we obtain a cochain map

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{-2}(F) & \xrightarrow{0} & H^{-1}(F) & \xrightarrow{0} & H^0(F) & \xrightarrow{0} & H^1(F) & \xrightarrow{0} & H^2(F) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow \cdots
\end{array}$$

This is an isomorphism in cohomology, hence an isomorphism in $\mathbf{D}(R)$.

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \longrightarrow H^i(F)$ lifts to a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(F) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

and when we combine, for every $i \in \mathbb{Z}$, we obtain a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{-2}(F) & \xrightarrow{0} & H^{-1}(F) & \xrightarrow{0} & H^0(F) & \xrightarrow{0} & H^1(F) & \xrightarrow{0} & H^2(F) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots
\end{array}$$

This is an isomorphism in cohomology, hence an isomorphism in $\mathbf{D}(R)$. Exhibiting an isomorphism of F with an object in $\overline{\langle R \rangle}_1^{(-\infty,\infty)}$.

Now suppose $n \geq 0$, and we know the result for every ℓ with $0 \leq \ell \leq n$. We wish to show it for $n + 1$.

Now suppose $n \geq 0$, and we know the result for every ℓ with $0 \leq \ell \leq n$. We wish to show it for $n+1$. Suppose therefore that we are given an object $F \in \mathbf{D}(R)$ with $H^i(F)$ of projective dimension $\leq n+1$ for every i.

Now suppose $n \geq 0$, and we know the result for every ℓ with $0 \leq \ell \leq n$. We wish to show it for $n+1$. Suppose therefore that we are given an object $F \in \mathbf{D}(R)$ with $H^i(F)$ of projective dimension $\leq n+1$ for every i. Choose for every i a projective module P^i and a surjection $P^i \longrightarrow H^i(F)$, and form the corresponding cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P^i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

Now suppose $n \geq 0$, and we know the result for every ℓ with $0 \leq \ell \leq n$. We wish to show it for $n+1$. Suppose therefore that we are given an object $F \in \mathbf{D}(R)$ with $H^i(F)$ of projective dimension $\leq n+1$ for every i. Choose for every i a projective module P^i and a surjection $P^i \longrightarrow H^i(F)$, and form the corresponding cochain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P^i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

and combine over i to form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{0} & P^{-1} & \xrightarrow{0} & P^0 & \xrightarrow{0} & P^1 & \xrightarrow{0} & P^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots
\end{array}$$

This gives a map $P \longrightarrow F$, which we complete to a triangle $P \longrightarrow F \longrightarrow Q$.

This gives a map $P \longrightarrow F$, which we complete to a triangle $P \longrightarrow F \longrightarrow Q$. By construction, the map $H^i(P) \longrightarrow H^i(F)$ is surjective for every $i \in \mathbb{Z}$, giving short exact sequences

$$0 \longrightarrow H^{i-1}(Q) \longrightarrow H^i(P) \longrightarrow H^i(F) \longrightarrow 0$$

This gives a map $P \longrightarrow F$, which we complete to a triangle $P \longrightarrow F \longrightarrow Q$. By construction, the map $H^i(P) \longrightarrow H^i(F)$ is surjective for every $i \in \mathbb{Z}$, giving short exact sequences

$$0 \longrightarrow H^{i-1}(Q) \longrightarrow H^i(P) \longrightarrow H^i(F) \longrightarrow 0$$

Hence $H^i(Q)$ is of projective dimension $\leq n$.

This gives a map $P \longrightarrow F$, which we complete to a triangle $P \longrightarrow F \longrightarrow Q$. By construction, the map $H^i(P) \longrightarrow H^i(F)$ is surjective for every $i \in \mathbb{Z}$, giving short exact sequences

$$0 \longrightarrow H^{i-1}(Q) \longrightarrow H^i(P) \longrightarrow H^i(F) \longrightarrow 0$$

Hence $H^i(Q)$ is of projective dimension $\leq n$.

Thus $P \in \overline{\langle R\rangle}_1^{(-\infty,\infty)}$ and $Q \in \overline{\langle R\rangle}_{n+1}^{(-\infty,\infty)}$, and the triangle $P \longrightarrow F \longrightarrow Q$ tells us that

$$F \quad \in \quad \overline{\langle R\rangle}_1^{(-\infty,\infty)} * \overline{\langle R\rangle}_{n+1}^{(-\infty,\infty)} \quad \subset \quad \overline{\langle R\rangle}_{n+2}^{(-\infty,\infty)} \quad .$$

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact.

Proof.

□

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact. Then the object $\mathbf{R}u_*\mathcal{O}_U \in \mathbf{D_{qc}}(X)$ belongs to $\overline{\langle G \rangle}_n^{[-n,n]}$ for some integer $n > 0$.

Proof.

□

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact. *Then the object $\mathbf{R}u_*\mathcal{O}_U \in \mathbf{D_{qc}}(X)$ belongs to $\overline{\langle G \rangle}_n^{[-n,n]}$ for some integer $n > 0$.*

Proof.

It is relatively easy to show that there exists an integer $\ell > 0$ with $\mathrm{Hom}\big(\mathbf{R}u_*\mathcal{O}_U \ , \ \mathbf{D_{qc}}(X)^{\leq -\ell}\big) = 0$.

□

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact. *Then the object $\mathbf{R}u_*\mathcal{O}_U \in \mathbf{D_{qc}}(X)$ belongs to $\overline{\langle G\rangle}_n^{[-n,n]}$ for some integer $n > 0$.*

Proof.

It is relatively easy to show that there exists an integer $\ell > 0$ with $\mathrm{Hom}\big(\mathbf{R}u_*\mathcal{O}_U\ ,\ \mathbf{D_{qc}}(X)^{\leq -\ell}\big) = 0$. By the approximability of $\mathbf{D_{qc}}(X)$ we may choose an integer n and a triangle $E \longrightarrow \mathbf{R}u_*\mathcal{O}_U \longrightarrow D$ with $D \in \mathbf{D_{qc}}(X)^{\leq -\ell}$ and $E \in \overline{\langle G\rangle}_n^{[-n,n]}$.

□

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact. Then the object $\mathbf{R}u_*\mathcal{O}_U \in \mathbf{D_{qc}}(X)$ belongs to $\overline{\langle G\rangle}_n^{[-n,n]}$ for some integer $n > 0$.

Proof.

It is relatively easy to show that there exists an integer $\ell > 0$ with $\mathrm{Hom}\big(\mathbf{R}u_*\mathcal{O}_U\ ,\ \mathbf{D_{qc}}(X)^{\leq -\ell}\big) = 0$. By the approximability of $\mathbf{D_{qc}}(X)$ we may choose an integer n and a triangle $E \longrightarrow \mathbf{R}u_*\mathcal{O}_U \longrightarrow D$ with $D \in \mathbf{D_{qc}}(X)^{\leq -\ell}$ and $E \in \overline{\langle G\rangle}_n^{[-n,n]}$.

But the map $\mathbf{R}u_*\mathcal{O}_U \longrightarrow D$ must vanish by the choice of ℓ, making $\mathbf{R}u_*\mathcal{O}_U$ a direct summand of the object $E \in \overline{\langle G\rangle}_n^{[-n,n]}$. □

Sketch of how strong generation follows from the Lemma

Let X be a quasicompact, separated scheme. By hypothesis we may cover X by open subsets $U_i = \mathrm{Spec}(R_i)$ with each R_i of finite global dimension. By the quasicompactness we may choose finitely many U_i which cover.

Sketch of how strong generation follows from the Lemma

Let X be a quasicompact, separated scheme. By hypothesis we may cover X by open subsets $U_i = \mathrm{Spec}(R_i)$ with each R_i of finite global dimension. By the quasicompactness we may choose finitely many U_i which cover. The Lemma tells us that we may choose a compact generator $G \in \mathbf{D_{qc}}(X)$ and an integer n so that

$$\mathbf{R}u_{i*}\mathcal{O}_{U_i} \quad \in \quad \overline{\langle G \rangle}_n^{[-n,n]} \quad \subset \quad \overline{\langle G \rangle}_n^{(-\infty,\infty)}$$

for every i in the finite set.

Sketch of how strong generation follows from the Lemma

Let X be a quasicompact, separated scheme. By hypothesis we may cover X by open subsets $U_i = \mathrm{Spec}(R_i)$ with each R_i of finite global dimension. By the quasicompactness we may choose finitely many U_i which cover. The Lemma tells us that we may choose a compact generator $G \in \mathbf{D_{qc}}(X)$ and an integer n so that

$$\mathbf{R}u_{i*}\mathcal{O}_{U_i} \quad \in \quad \overline{\langle G \rangle}_n^{[-n,n]} \quad \subset \quad \overline{\langle G \rangle}_n^{(-\infty,\infty)}$$

for every i in the finite set.

Since R_i is of finite global dimension, Kelly's 1965 theorem tells us that we may choose an integer $\ell > 0$ so that $\mathbf{D_{qc}}(U_i) \subset \overline{\langle \mathcal{O}_i \rangle}_\ell^{(-\infty,\infty)}$.

Sketch of how strong generation follows from the Lemma

Let X be a quasicompact, separated scheme. By hypothesis we may cover X by open subsets $U_i = \mathrm{Spec}(R_i)$ with each R_i of finite global dimension. By the quasicompactness we may choose finitely many U_i which cover.
The Lemma tells us that we may choose a compact generator $G \in \mathbf{D_{qc}}(X)$ and an integer n so that

$$\mathbf{R}u_{i*}\mathcal{O}_{U_i} \quad \in \quad \overline{\langle G\rangle}_n^{[-n,n]} \quad \subset \quad \overline{\langle G\rangle}_n^{(-\infty,\infty)}$$

for every i in the finite set.

Since R_i is of finite global dimension, Kelly's 1965 theorem tells us that we may choose an integer $\ell > 0$ so that $\mathbf{D_{qc}}(U_i) \subset \overline{\langle \mathcal{O}_i\rangle}_\ell^{(-\infty,\infty)}$. It follows that

$$\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i) \quad \subset \quad \overline{\langle \mathbf{R}u_{i*}\mathcal{O}_i\rangle}_\ell^{(-\infty,\infty)} \quad \subset \quad \overline{\langle G\rangle}_{\ell n}^{(-\infty,\infty)}$$

Sketch of how strong generation follows from the Lemma—continued

It's an exercise to show that $\mathbf{D_{qc}}(X)$ can be generated from the subcategories $\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i)$ in finitely many steps.

Sketch of how strong generation follows from the Lemma—continued

It's an exercise to show that $\mathbf{D_{qc}}(X)$ can be generated from the subcategories $\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i)$ in finitely many steps. Hence there exists an integer N with $\mathbf{D_{qc}}(X) = \overline{\langle G \rangle}_N^{(-\infty,\infty)}$.

Sketch of how strong generation follows from the Lemma—continued

It's an exercise to show that $\mathbf{D_{qc}}(X)$ can be generated from the subcategories $\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i)$ in finitely many steps. Hence there exists an integer N with $\mathbf{D_{qc}}(X) = \overline{\langle G \rangle}_N^{(-\infty,\infty)}$.

We have proved a statement about $\mathbf{D_{qc}}(X)$, and $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}(X)$ is the subcategory of compact objects.

Sketch of how strong generation follows from the Lemma—continued

It's an exercise to show that $\mathbf{D_{qc}}(X)$ can be generated from the subcategories $\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i)$ in finitely many steps. Hence there exists an integer N with $\mathbf{D_{qc}}(X) = \overline{\langle G\rangle}_N^{(-\infty,\infty)}$.

We have proved a statement about $\mathbf{D_{qc}}(X)$, and $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}(X)$ is the subcategory of compact objects. Standard compactness arguments give that $\mathbf{D}^{\mathrm{perf}}(X) = \langle G\rangle_N^{(-\infty,\infty)}$, which is strong generation.

Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^{b}_{\mathbf{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

Ko Aoki, *Quasiexcellence implies strong generation*, J. Reine Angew. Math. **780** (2021), 133–138.

Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^{b}_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

Next another reminder from Talk 1.

Definition (equivalent *t*–structures)

Let $\mathcal{T}$ be any triangulated category, and let $\big(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0}\big)$ and $\big(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0}\big)$ be two *t*–structures on $\mathcal{T}$. We declare them equivalent if the metrics they induce are equivalent.

Next another reminder from Talk 1.

Definition (equivalent t–structures)

Let $\mathcal{T}$ be any triangulated category, and let $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ be two t–structures on $\mathcal{T}$. We declare them equivalent if the metrics they induce are equivalent.

To spell it out: the two t–structures are equivalent if there exists an integer $A > 0$ with

$$\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}.$$

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ generated by G.

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ generated by G.

More precisely the following formula delivers a t–structure:

$$\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]} , \qquad \mathcal{T}_G^{\geq 0} = \left(\left[\mathcal{T}_G^{\leq 0} \right]^{\perp} \right) [1] .$$

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ generated by G.

More precisely the following formula delivers a t–structure:

$$\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]} , \qquad\qquad \mathcal{T}_G^{\geq 0} = \left(\left[\mathcal{T}_G^{\leq 0}\right]^{\perp} \right)[1] .$$

If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ and $\big(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\big)$ are equivalent.

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ generated by G.

More precisely the following formula delivers a t–structure:

$$\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]}\ , \qquad\qquad \mathcal{T}_G^{\geq 0} = \left(\left[\mathcal{T}_G^{\leq 0}\right]^{\perp} \right)[1]\ .$$

If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ and $\left(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\right)$ are equivalent.

We say that a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ is in the preferred equivalence class if it is equivalent to $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ for some compact generator G, hence for every compact generator.

Structure theorems

Theorem

Let $\mathcal{T}$ be a triangulated category with coproducts. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ such that the hypotheses of weak approximability are satisfied.

Structure theorems

Theorem

Let $\mathcal{T}$ be a triangulated category with coproducts. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ such that the hypotheses of weak approximability are satisfied.

To spell it out:

- $G^{\perp}$ *contains* $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- *For every object* $F \in \mathcal{T}^{\leq 0}$ *there exists a triangle* $E \longrightarrow F \longrightarrow D$, *with* $D \in \mathcal{T}^{\leq -1}$ *and* $E \in \overline{\langle G \rangle}^{[-A,A]}$.

Structure theorems

Theorem

Let $\mathcal{T}$ be a triangulated category with coproducts. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ such that the hypotheses of weak approximability are satisfied.

To spell it out:

- $G^{\perp}$ *contains* $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- *For every object* $F \in \mathcal{T}^{\leq 0}$ *there exists a triangle* $E \longrightarrow F \longrightarrow D$, *with* $D \in \mathcal{T}^{\leq -1}$ *and* $E \in \overline{\langle G \rangle}^{[-A,A]}$.

Then the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class.

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

Then there exists an integer $A > 0$ such that the hypotheses of weak approximability are safisfied.

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

Then there exists an integer $A > 0$ such that the hypotheses of weak approximability are safisfied.

To spell it out:

- *$G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.*
- *For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}^{[-A,A]}$.*

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

Then there exists an integer $A > 0$ such that the hypotheses of weak approximability are safisfied.

To spell it out:

- *$G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.*
- *For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}^{[-A,A]}$.*
- *Assume furthermore that $\mathcal{T}$ is approximable. Then the integer A may be chosen so that the stronger hypotheses of approximability are satisfied.*

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

Then there exists an integer $A > 0$ such that the hypotheses of weak approximability are safisfied.

To spell it out:

- *$G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.*
- *For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}^{[-A,A]}$.*
- *Assume furthermore that $\mathcal{T}$ is approximable. Then the integer A may be chosen so that the stronger hypotheses of approximability are satisfied. To spell it out: in the triangle $E \longrightarrow F \longrightarrow D$ above, we have $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.*

Theorem

Let $\mathcal{T}$ be a weakly approximable *triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and an integer $A > 0$ such that*

- $G^{\perp}$ *contains* $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- *For every object* $F \in \mathcal{T}^{\leq 0}$ *there exists a triangle* $E \longrightarrow F \longrightarrow D$, *with* $D \in \mathcal{T}^{\leq -1}$ *and* $E \in \overline{\langle G \rangle}^{[-A,A]}$.
-

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and an integer $A > 0$ such that

- $G^{\perp}$ *contains* $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- *For every object* $F \in \mathcal{T}^{\leq 0}$ *there exists a triangle* $E \longrightarrow F \longrightarrow D$, *with* $D \in \mathcal{T}^{\leq -1}$ *and* $E \in \overline{\langle G \rangle}^{[-A,A]}$.
-

Then for any object $F \in \mathcal{T}^{\leq 0}$ *and every integer* $m > 0$, *there exists a triangle* $E_m \longrightarrow F \longrightarrow D_m$ *with* $D_m \in \mathcal{T}^{\leq -m}$ *and with* $E_m \in \overline{\langle G \rangle}^{[-A-m+1,A]}$.

Theorem

Let $\mathcal{T}$ be a approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and an integer $A > 0$ such that

- $G^{\perp}$ *contains* $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- *For every object* $F \in \mathcal{T}^{\leq 0}$ *there exists a triangle* $E \longrightarrow F \longrightarrow D$, *with* $D \in \mathcal{T}^{\leq -1}$ *and* $E \in \overline{\langle G \rangle}^{[-A,A]}$.
- *Suppose the integer A was chosen so that, in the triangle* $E \longrightarrow F \longrightarrow D$ *above, we can guarantee* $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.

Then for any object $F \in \mathcal{T}^{\leq 0}$ *and every integer* $m > 0$, *there exists a triangle* $E_m \longrightarrow F \longrightarrow D_m$ *with* $D_m \in \mathcal{T}^{\leq -m}$ *and with* $E_m \in \overline{\langle G \rangle}_{mA}^{[-A-m+1,A]}$.

‘

Given a t–structure $\big(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0}\big)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}\,, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}\,, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a correponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. These are intrinsic, they're independent of any choice. In the remainder of the slides we only consider the "preferred" $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ in the preferred equivalence class.

Heuristic: the full subcategory $\mathcal{T}_c^-$ should be thought of as the closure of $\mathcal{T}^c$ with respect to the metric—every object of $\mathcal{T}_c^-$ admits arbitrarily good approximations by compacts.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class.

Heuristic: the full subcategory $\mathcal{T}_c^-$ should be thought of as the closure of $\mathcal{T}^c$ with respect to the metric—every object of $\mathcal{T}_c^-$ admits arbitrarily good approximations by compacts.

To spell it out more formally:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } \varepsilon > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and } \mathrm{Length}(\varphi) < \varepsilon \end{array} \right\}$$

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class.

Heuristic: the full subcategory $\mathcal{T}_c^-$ should be thought of as the closure of $\mathcal{T}^c$ with respect to the metric—every object of $\mathcal{T}_c^-$ admits arbitrarily good approximations by compacts.

To spell it out more formally:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } \varepsilon > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and } \mathrm{Length}(\varphi) < \varepsilon \end{array} \right\}$$

We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class.

Heuristic: the full subcategory $\mathcal{T}_c^-$ should be thought of as the closure of $\mathcal{T}^c$ with respect to the metric—every object of $\mathcal{T}_c^-$ admits arbitrarily good approximations by compacts.

To spell it out more formally:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } \varepsilon > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and } \mathrm{Length}(\varphi) < \varepsilon \end{array} \right\}$$

We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

It's obvious that the category $\mathcal{T}_c^-$ is intrinsic. As $\mathcal{T}_c^-$ and $\mathcal{T}^b$ are both intrinsic, so is their intersection $\mathcal{T}_c^b$.

We have defined all this intrinsic structure, assuming only that $\mathcal{T}$ is a triangulated category with coproducts and with a single compact generator. In this generality we know that the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are thick.

We have defined all this intrinsic structure, assuming only that $\mathcal{T}$ is a triangulated category with coproducts and with a single compact generator. In this generality we know that the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are thick.

If we furthermore assume that $\mathcal{T}$ is weakly approximable, then the subcategories $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ are also thick.

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$ and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

-
-

Theorem

Let $\mathcal{T}$ be a weakly approximable *triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$ and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.*

There exists an integer $B > 0$ such that

- *For every object $F \in \left[\mathcal{T}_c^-\right]^{\leq 0}$ and every integer $m > 0$, there exists a triangle $E_m \longrightarrow F \longrightarrow D_m$, with $D_m \in \left[\mathcal{T}_c^-\right]^{\leq -m}$ and $E \in \langle G\rangle^{[-B-m+1,B]}$.*
-

Theorem

Let $\mathcal{T}$ be a weakly approximable triangulated category. Suppose we are given a compact generator $G \in \mathcal{T}$ and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

There exists an integer $B > 0$ such that

- *For every object $F \in \left[\mathcal{T}_c^-\right]^{\leq 0}$ and every integer $m > 0$, there exists a triangle $E_m \longrightarrow F \longrightarrow D_m$, with $D_m \in \left[\mathcal{T}_c^-\right]^{\leq -m}$ and $E \in \langle G\rangle^{[-B-m+1,B]}$.*
- *Suppose our category $\mathcal{T}$ is approximable. Then the integer B above may be chosen so that, in the triangles $E_m \longrightarrow F \longrightarrow D_m$ above, we can guarantee $E_m \in \langle G\rangle^{[-B-m+1,B]}_{mB}$.*

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R)$, with R a coherent ring)

$$\begin{array}{lllllllll} \mathcal{T}^+ & = & \mathbf{D}^+(R), & \mathcal{T}^- & = & \mathbf{D}^-(R), & \mathcal{T}^c & = & \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}^b & = & \mathbf{D}^b(R), & \mathcal{T}^-_c & = & \mathbf{D}^-(R\text{–proj}), & \mathcal{T}^b_c & = & \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a coherent scheme and $Z \subset X$ a closed subset with quasicompact complement)

$$\begin{array}{lllllllll} \mathcal{T}^+ & = & \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- & = & \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c & = & \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b & = & \mathbf{D}^b_{\mathbf{qc},Z}(R), & \mathcal{T}^-_c & = & \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}^b_c & = & \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R)$, with R a coherent ring)

$$\begin{array}{lll} \mathcal{T}^+ = \mathbf{D}^+(R), & \mathcal{T}^- = \mathbf{D}^-(R), & \mathcal{T}^c = \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}^b = \mathbf{D}^b(R), & \mathcal{T}_c^- = \mathbf{D}^-(R\text{–proj}), & \mathcal{T}_c^b = \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a coherent scheme and $Z \subset X$ a closed subset with quasicompact complement)

$$\begin{array}{lll} \mathcal{T}^+ = \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- = \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c = \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b = \mathbf{D}^b_{\mathbf{qc},Z}(R), & \mathcal{T}_c^- = \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}_c^b = \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

The coherence hypothesis isn't essential. If X is quasicompact and quasiseparated, and if $Z \subset X$ is a closed subset with quasicompact complement, the formulas remain true

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R)$, with R a coherent ring)

$$\begin{array}{lllllllll} \mathcal{T}^+ & = & \mathbf{D}^+(R), & \mathcal{T}^- & = & \mathbf{D}^-(R), & \mathcal{T}^c & = & \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}^b & = & \mathbf{D}^b(R), & \mathcal{T}^-_c & = & \mathbf{D}^-(R\text{–proj}), & \mathcal{T}^b_c & = & \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a coherent scheme and $Z \subset X$ a closed subset with quasicompact complement)

$$\begin{array}{lllllllll} \mathcal{T}^+ & = & \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- & = & \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c & = & \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b & = & \mathbf{D}^b_{\mathbf{qc},Z}(R), & \mathcal{T}^-_c & = & \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}^b_c & = & \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

The coherence hypothesis isn't essential. If X is quasicompact and quasiseparated, and if $Z \subset X$ is a closed subset with quasicompact complement, the formulas remain true with $\mathbf{D}^b(R\text{–mod})$, $\mathbf{D}^-_{\mathbf{coh},Z}(X)$ and $\mathbf{D}^b_{\mathbf{coh},Z}(X)$ suitably interpreted.

Analogue to keep in mind, for what's coming

Consider the space S of Lebesgue measurable real-valued functions on $\mathbb{R}$. The pairing taking $f, g \in S$ to

$$\langle f, g \rangle = \int fg \, d\mu$$

is a map

$$S \times S \xrightarrow{\langle -,- \rangle} \mathbb{R} \cup \{\infty\}.$$

Analogue to keep in mind, for what's coming

Consider the space S of Lebesgue measurable real-valued functions on $\mathbb{R}$. The pairing taking $f, g \in S$ to

$$\langle f, g\rangle = \int fg \, d\mu$$

is a map

$$S \times S \xrightarrow{\langle -,-\rangle} \mathbb{R} \cup \{\infty\}.$$

If $f \in L^p$ and $g \in L^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, then $\langle f, g\rangle \in \mathbb{R}$ and we deduce two maps

$$L^p \longrightarrow \mathrm{Hom}(L^q, \mathbb{R}), \qquad L^q \longrightarrow \mathrm{Hom}(L^p, \mathbb{R})$$

Analogue to keep in mind, for what's coming

Consider the space S of Lebesgue measurable real-valued functions on $\mathbb{R}$. The pairing taking $f, g \in S$ to

$$\langle f, g\rangle = \int fg \, d\mu$$

is a map

$$S \times S \xrightarrow{\langle -,-\rangle} \mathbb{R} \cup \{\infty\}.$$

If $f \in L^p$ and $g \in L^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, then $\langle f, g\rangle \in \mathbb{R}$

and we deduce two maps, which turn out to be isometries

$$L^p \longrightarrow \mathrm{Hom}(L^q, \mathbb{R}), \qquad L^q \longrightarrow \mathrm{Hom}(L^p, \mathbb{R})$$

Let R be a commutative ring, and assume $\mathcal{T}$ is an R-linear category. The pairing sending $A, B \in \mathcal{T}$ to $\mathrm{Hom}(A, B)$ gives a map

$$\mathcal{T}^{\mathrm{op}} \times \mathcal{T} \longrightarrow R\text{–Mod}$$

and we deduce two ordinary Yoneda maps

$$\mathcal{T} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}^{\mathrm{op}}\ ,\ R\text{–Mod}\Big)$$
$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}\ ,\ R\text{–Mod}\Big)$$

1.

2.

Let R be a commutative ring, and assume $\mathcal{T}$ is an R-linear category. The pairing sending $A, B \in \mathcal{T}$ to $\mathrm{Hom}(A, B)$ gives a map

$$\mathcal{T}^{\mathrm{op}} \times \mathcal{T} \longrightarrow R\text{–}\mathrm{Mod}$$

and we deduce two ordinary Yoneda maps

$$\mathcal{T} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$
$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

If $\mathcal{T}$ is also an approximable triangulated category, we can restrict to obtain restricted Yoneda maps

1. $$\mathcal{T}_c^- \xrightarrow{\ \mathcal{Y}\ } \mathrm{Hom}_R\Big(\big[\mathcal{T}^c\big]^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

2. $$\big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\ \widetilde{\mathcal{Y}}\ } \mathrm{Hom}_R\Big(\mathcal{T}_c^b\ ,\ R\text{–}\mathrm{Mod}\Big)$$

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\, R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\, R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions.

1.

2.

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1.

2.

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2.

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\; R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\; R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\, R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\, R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

More precisely: the assertions about the functors $\mathcal{Y}$ and $\mathcal{Y} \circ i$ are true as stated.

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

More precisely: the assertions about the functors $\mathcal{Y}$ and $\mathcal{Y} \circ i$ are true as stated.

For the assertions about $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$, we need to add the hypothesis that there exists an object $H \in \mathcal{T}_c^b$ and an integer $N > 0$ with $\overline{\langle H \rangle}_N^{(-\infty,\infty)} = \mathcal{T}$.

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\ \ i\ \ } \mathcal{T}_c^- \xrightarrow{\ \ \mathcal{Y}\ \ } \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\ \ \widetilde{\imath}\ \ } \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\ \ \widetilde{\mathcal{Y}}\ \ } \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

A homological functor $H : \mathcal{T}_c^- \longrightarrow R\text{–}\mathrm{Mod}$ is locally finite if, for every object C, the R–module $H^n(C)$ is finite for every $n \in \mathbb{Z}$ and vanishes if $n \gg 0$

Theorem (first general theorem about approximable categories)

Let R be a commutative, noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \xhookrightarrow{\quad i \quad} \mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathrm{Mod}\big)$$

$$\big[\mathcal{T}^c\big]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then the following is almost true:

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

A homological functor $H : \mathcal{T}_c^- \longrightarrow R\text{–}\mathrm{Mod}$ is finite if, for every object C, the R–module $H^n(C)$ is finite for every $n \in \mathbb{Z}$ and vanishes if $n \gg 0$ or $n \ll 0$

What was known before

Theorem

Let R be a commutative, noetherian ring, and let $\mathcal{S}$ be an R-linear triangulated category. Assume

1. *The category $\mathcal{S}$ has a strong generator. This means: there exists an object $G \in \mathcal{S}$ and an integer $N > 0$ with $\langle G \rangle_N^{(-\infty,\infty)} = \mathcal{S}$.*
2. *For any pair of objects $X, Y \in \mathcal{S}$ we have that $\mathrm{Hom}(X, Y)$ is a finite R-module, and $\mathrm{Hom}(X, Y[n])$ vanishes for all but finitely many n.*

Then every finite homological functor $F : \mathcal{S} \longrightarrow R\text{–}\mathrm{mod}$ is representable.

Alexei I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.

What was known before, continued

In the special case where $\mathcal{T} = \mathbf{D_{qc}}(X)$ with X projective over a field k, we had:

Summary

- Bondal and Van den Bergh proved, in the paper cited on the previous slide, that every finite k–linear homological functor on $\left[\mathbf{D}^{\mathrm{perf}}(X)\right]^{\mathrm{op}}$ is of the form $(\mathcal{Y} \circ i)(B) = \mathrm{Hom}(-, B)$ for some $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$.
- Rouquier claims, in the article cited on the previous slide, that every finite k–linear homological functor on $\mathbf{D}^b_{\mathbf{coh}}(X)$ is of the form $(\widetilde{\mathcal{Y}} \circ \widetilde{\imath})(A) = \mathrm{Hom}(A, -)$ for some $A \in \mathbf{D}^{\mathrm{perf}}(X)$.

Application

Let X be a scheme proper over a noetherian ring R. Then $\mathcal{T} = \mathbf{D_{qc}}(X)$ satisfies the hypotheses of the theorem.

Corollary

The functor

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\quad \mathcal{Y} \circ i \quad} \mathrm{Hom}_R\Big(\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–Mod}\Big)$$

gives an equivalence of $\mathbf{D}^b_{\mathbf{coh}}(X)$ *with the category of finite homological functors* $\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} \longrightarrow R\text{–Mod}$.

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Now consider the pairing taking $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to the R–module

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})}\big(\mathcal{L}(A)\ ,\ B\big)$$

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Now consider the pairing taking $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to the R–module

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})}\big(\mathcal{L}(A)\ ,\ B\big)$$

$$\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathrm{Hom}_R\Big(\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–Mod}\Big)$$

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L}:\mathbf{D}^b_{\mathbf{coh}}(X)\longrightarrow\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Now consider the pairing taking $A\in\mathbf{D}^{\mathrm{perf}}(X)$ and $B\in\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to the R–module

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})}\big(\mathcal{L}(A)\ ,\ B\big)$$

The above delivers a map taking $B\in\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to a finite homological functor $\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\longrightarrow R\text{–mod}.$

$$\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})\longrightarrow\mathrm{Hom}_R\Big(\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–Mod}\Big)$$

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Now consider the pairing taking $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to the R–module

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})}\big(\mathcal{L}(A)\ ,\ B\big)$$

The above delivers a map taking $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to a finite homological functor $\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} \longrightarrow R\text{–mod}$.

$$\begin{array}{ccc} \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) & \searrow & \\ & & \mathrm{Hom}_R\Big(\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–Mod}\Big) \\ \mathbf{D}^b_{\mathbf{coh}}(X) & \xrightarrow{\ \mathcal{Y}\circ i\ } \nearrow & \end{array}$$

Why does one care about such representability theorems?

Suppose X is a scheme proper over a noetherian ring R.

Let $\mathcal{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ be the analytification functor.

Now consider the pairing taking $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to the R–module

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})}\big(\mathcal{L}(A)\ ,\ B\big)$$

The above delivers a map taking $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to a finite homological functor $\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} \longrightarrow R\text{–mod}.$

$$\begin{array}{ccc} \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) & \longrightarrow & \\ \Big\downarrow{\scriptstyle \mathcal{R}} & & \mathrm{Hom}_R\Big(\big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–Mod}\Big) \\ \mathbf{D}^b_{\mathbf{coh}}(X) & \xrightarrow{\ \mathcal{Y}\circ i\ } & \end{array}$$

Representablity produced for us a functor $\mathcal{R} : \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X)$.

Representablity produced for us a functor $\mathcal{R} : \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X)$.

The construction gives us, for every pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$, a natural isomorphism

$$\mathrm{Hom}\big(\mathcal{L}(A), B\big) \cong \mathrm{Hom}\big(A, \mathcal{R}(B)\big) \ .$$

Representablity produced for us a functor $\mathcal{R} : \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X)$.

The construction gives us, for every pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$, a natural isomorphism

$$\mathrm{Hom}\big(\mathcal{L}(A), B\big) \cong \mathrm{Hom}\big(A, \mathcal{R}(B)\big) \ .$$

For any pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$, $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$ we deduce a natural map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(A), \mathcal{L}(B)\big) \longrightarrow \mathrm{Hom}\big(A, \mathcal{R}\mathcal{L}(B)\big)$$

Representablity produced for us a functor $\mathcal{R} : \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathbf{D}^b_{\mathbf{coh}}(X)$.

The construction gives us, for every pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$, a natural isomorphism

$$\mathrm{Hom}\big(\mathcal{L}(A), B\big) \cong \mathrm{Hom}\big(A, \mathcal{R}(B)\big) \ .$$

For any pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$, $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$ we deduce a natural map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(A), \mathcal{L}(B)\big) \longrightarrow \mathrm{Hom}\big(A, \mathcal{R}\mathcal{L}(B)\big)$$

which must be induced by a unique morphism $\eta : B \longrightarrow \mathcal{R}\mathcal{L}(B)$.

Representablity produced for us a functor $\mathcal{R} : \mathbf{D}^b_{\mathsf{coh}}(X^{\mathrm{an}}) \longrightarrow \mathbf{D}^b_{\mathsf{coh}}(X)$.

The construction gives us, for every pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathsf{coh}}(X^{\mathrm{an}})$, a natural isomorphism

$$\mathrm{Hom}\big(\mathcal{L}(A), B\big) \cong \mathrm{Hom}\big(A, \mathcal{R}(B)\big) \ .$$

For any pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$, $B \in \mathbf{D}^b_{\mathsf{coh}}(X)$ we deduce a natural map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(A), \mathcal{L}(B)\big) \longrightarrow \mathrm{Hom}\big(A, \mathcal{R}\mathcal{L}(B)\big)$$

which must be induced by a unique morphism $\eta : B \longrightarrow \mathcal{R}\mathcal{L}(B)$.

This allows us to define, for any pair of objects $A \in \mathbf{D}^b_{\mathsf{coh}}(X)$ and $B \in \mathbf{D}^b_{\mathsf{coh}}(X^{\mathrm{an}})$, a natural composite

$$\mathrm{Hom}\big(\mathcal{L}(A), B\big) \longrightarrow \mathrm{Hom}\big(\mathcal{R}\mathcal{L}(A), \mathcal{R}(B)\big) \xrightarrow{\mathrm{Hom}(\eta,-)} \mathrm{Hom}\big(A, \mathcal{R}(B)\big)$$

Now every object $A \in \mathbf{D}^b_{\mathbf{coh}}(X)$ can be approximated, to within arbitrary $\varepsilon > 0$, by objects $A_\varepsilon \in \mathbf{D}^{\mathrm{perf}}(X)$. Recall: this means there exist morphisms $f : A_\varepsilon \longrightarrow A$ with $\mathrm{Length}(f) < \varepsilon$.

Now every object $A \in \mathbf{D}^b_{\mathbf{coh}}(X)$ can be approximated, to within arbitrary $\varepsilon > 0$, by objects $A_\varepsilon \in \mathbf{D}^{\mathrm{perf}}(X)$. Recall: this means there exist morphisms $f : A_\varepsilon \longrightarrow A$ with Length$(f) < \varepsilon$.

For fixed B and ε small enough, the induced vertical maps in the diagram below are isomorphisms

$$\begin{array}{ccc}
\mathrm{Hom}\big(\mathcal{L}(A), B\big) & \longrightarrow & \mathrm{Hom}\big(A, \mathcal{R}(B)\big) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Hom}\big(\mathcal{L}(A_\varepsilon), B\big) & \xrightarrow{\sim} & \mathrm{Hom}\big(A_\varepsilon, \mathcal{R}(B)\big)
\end{array}$$

For a unified proof of the GAGA theorems it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms.

For a unified proof of the GAGA theorems it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms.

To prove that the unit is an isomorphism it suffices to find a set of objects $P \in \mathbf{D}^b_{\mathbf{coh}}(X)$, such that $\mathrm{Hom}(P, -)$ takes the unit $\eta : \mathrm{id} \longrightarrow \mathcal{R}\mathcal{L}$ to an isomorphism, and such that $P^\perp = 0$.

For a unified proof of the GAGA theorems it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms.

To prove that the unit is an isomorphism it suffices to find a set of objects $P \in \mathbf{D}^b_{\mathbf{coh}}(X)$, such that $\mathrm{Hom}(P, -)$ takes the unit $\eta : \mathrm{id} \longrightarrow \mathcal{R}\mathcal{L}$ to an isomorphism, and such that $P^\perp = 0$.

Let the counit of adjunction be denoted $e : \mathcal{L}\mathcal{R} \longrightarrow \mathrm{id}$. If we could guarantee that $\mathcal{L}(P)^\perp = 0$, then we'd be done—meaning it would formally follow that $e : \mathcal{L}\mathcal{R} \longrightarrow \mathrm{id}$ is an isomorphism.

For a unified proof of the GAGA theorems it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms.

To prove that the unit is an isomorphism it suffices to find a set of objects $P \in \mathbf{D}^b_{\mathbf{coh}}(X)$, such that $\mathrm{Hom}(P, -)$ takes the unit $\eta : \mathrm{id} \longrightarrow \mathcal{RL}$ to an isomorphism, and such that $P^\perp = 0$.

Let the counit of adjunction be denoted $e : \mathcal{LR} \longrightarrow \mathrm{id}$. If we could guarantee that $\mathcal{L}(P)^\perp = 0$, then we'd be done—meaning it would formally follow that $e : \mathcal{LR} \longrightarrow \mathrm{id}$ is an isomorphism.

The point is that the composite

$$\mathcal{R} \xrightarrow{\eta\mathcal{R}} \mathcal{RLR} \xrightarrow{\mathcal{R}e} \mathcal{R}$$

is the identity, and hence $\mathrm{Hom}(p, -)$ takes it to the identity for all $p \in P$. Now $\mathrm{Hom}(p, \eta\mathcal{R})$ is an isomorphism because η is already known to be an isomorphism,

For a unified proof of the GAGA theorems it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms.

To prove that the unit is an isomorphism it suffices to find a set of objects $P \in \mathbf{D}^b_{\mathbf{coh}}(X)$, such that $\mathrm{Hom}(P, -)$ takes the unit $\eta : \mathrm{id} \longrightarrow \mathcal{RL}$ to an isomorphism, and such that $P^\perp = 0$.

Let the counit of adjunction be denoted $e : \mathcal{LR} \longrightarrow \mathrm{id}$. If we could guarantee that $\mathcal{L}(P)^\perp = 0$, then we'd be done—meaning it would formally follow that $e : \mathcal{LR} \longrightarrow \mathrm{id}$ is an isomorphism.

The point is that the composite

$$\mathcal{R} \xrightarrow{\eta\mathcal{R}} \mathcal{RLR} \xrightarrow{\mathcal{R}e} \mathcal{R}$$

is the identity, and hence $\mathrm{Hom}(p, -)$ takes it to the identity for all $p \in P$. Now $\mathrm{Hom}(p, \eta\mathcal{R})$ is an isomorphism because η is already known to be an isomorphism, forcing $\mathrm{Hom}(p, \mathcal{R}e) = \mathrm{Hom}\big(\mathcal{L}(p), e\big)$ to be an isomorphism.

Summarizing: it suffices to produce a set of objects $P \subset \mathbf{D}^{\mathrm{perf}}(X)$, with $P[1] = P$ and such that

1. $P^\perp = \{0\}$.
2. $\mathcal{L}(P)^\perp = \{0\}$.
3. For every object $p \in P$ and every object $x \in \mathbf{D}^b_{\mathbf{coh}}(X)$, the natural map

$$\mathrm{Hom}(p, x) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(p)\,, \mathcal{L}(x)\big)$$

is an isomorphism.

Summarizing: it suffices to produce a set of objects $P \subset \mathbf{D}^{\mathrm{perf}}(X)$, with $P[1] = P$ and such that

1. $P^{\perp} = \{0\}$.
2. $\mathcal{L}(P)^{\perp} = \{0\}$.
3. For every object $p \in P$ and every object $x \in \mathbf{D}^{b}_{\mathbf{coh}}(X)$, the natural map

$$\mathrm{Hom}(p, x) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(p)\,,\mathcal{L}(x)\big)$$

is an isomorphism.

But this is easy: we let P be the collection of perfect complexes supported at closed points.

Jack Hall, *GAGA theorems*, to appear in Journal de Mathématiques Pures et Appliquées.

Theorem (reminder: theorem of the second talk)

Let $\mathcal{S}$ be a triangulated category with a good metric. In Talk 2 we defined categories

$$\mathfrak{S}(\mathcal{S}) \quad \subset \quad \mathfrak{L}(\mathcal{S})\ .$$

We also defined the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

With this definition of distinguished triangles, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.

Theorem (second general theorem about weakly approximable categories)

Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $\mathcal{T}$ has a preferred equivalence class of t-structures, giving preferred equivalence classes of good metrics on its subcategories $\mathcal{T}^c$ and $\mathcal{T}^b_c$. For the metrics on $\mathcal{T}^c$ we have

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}^b_c.$$

If furthermore $\mathcal{T}$ is coherent, then for the metrics on $\left[\mathcal{T}^b_c\right]^{\mathrm{op}}$ we have

$$\mathfrak{S}\Big(\left[\mathcal{T}^b_c\right]^{\mathrm{op}}\Big) = \left[\mathcal{T}^c\right]^{\mathrm{op}}.$$

Theorem (second general theorem about weakly approximable categories)

Let $\mathcal{T}$ be a weakly approximable triangulated category. Then $\mathcal{T}$ has a preferred equivalence class of t-structures, giving preferred equivalence classes of good metrics on its subcategories $\mathcal{T}^c$ and $\mathcal{T}^b_c$. For the metrics on $\mathcal{T}^c$ we have

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}^b_c.$$

If furthermore $\mathcal{T}$ is coherent, then for the metrics on $\big[\mathcal{T}^b_c\big]^{\mathrm{op}}$ we have

$$\mathfrak{S}\Big(\big[\mathcal{T}^b_c\big]^{\mathrm{op}}\Big) = \big[\mathcal{T}^c\big]^{\mathrm{op}}.$$

Coherent triangulated categories

A weakly approximable triangulated category is coherent if, in the preferred equivalence class, there is a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ such that

$$\big(\mathcal{T}^-_c \cap \mathcal{T}^{\leq 0}\ ,\ \mathcal{T}^-_c \cap \mathcal{T}^{\geq 0}\big)$$

is a t–structure on $\mathcal{T}^-_c$.

The case $\mathcal{T} = \mathbf{D}(R)$

Let R be any ring and let $\mathcal{T} = \mathbf{D}(R)$. Then

$$\mathcal{T}^c = \mathbf{D}^b(R\text{–proj}), \qquad \mathcal{T}_c^b = \mathbf{D}^b(R\text{–mod}).$$

The case $\mathcal{T} = \mathbf{D}(R)$

Let R be any ring and let $\mathcal{T} = \mathbf{D}(R)$. Then

$$\mathcal{T}^c = \mathbf{D}^b(R\text{–proj}), \qquad \mathcal{T}_c^b = \mathbf{D}^b(R\text{–mod}).$$

The theorem now gives

$$\mathfrak{S}\big[\mathbf{D}^b(R\text{–proj})\big] = \mathbf{D}^b(R\text{–mod})$$

The case $\mathcal{T} = \mathbf{D}(R)$

Let R be any ring and let $\mathcal{T} = \mathbf{D}(R)$. Then

$$\mathcal{T}^c = \mathbf{D}^b(R\text{–proj}), \qquad \mathcal{T}_c^b = \mathbf{D}^b(R\text{–mod}).$$

The theorem now gives

$$\mathfrak{S}\big[\mathbf{D}^b(R\text{–proj})\big] = \mathbf{D}^b(R\text{–mod})$$

If the ring R is assumed coherent, then one also has

$$\mathfrak{S}\Big(\big[\mathbf{D}^b(R\text{–mod})\big]^{\mathrm{op}}\Big) = \big[\mathbf{D}^b(R\text{–proj})\big]^{\mathrm{op}} .$$

The case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then

$$\mathcal{T}^c = \mathbf{D}_Z^{\mathrm{perf}}(X), \qquad \mathcal{T}_c^b = \mathbf{D}^b_{\mathbf{coh},Z}(X)$$

The case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then

$$\mathcal{T}^c = \mathbf{D}_Z^{\mathrm{perf}}(X), \qquad\qquad \mathcal{T}_c^b = \mathbf{D}^b_{\mathbf{coh},Z}(X)$$

The theorem now gives

$$\mathfrak{S}\left[\mathbf{D}_Z^{\mathrm{perf}}(X)\right] = \mathbf{D}^b_{\mathbf{coh},Z}(X).$$

The case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then

$$\mathcal{T}^c = \mathbf{D}^{\mathrm{perf}}_Z(X), \qquad \mathcal{T}^b_c = \mathbf{D}^b_{\mathbf{coh},Z}(X)$$

The theorem now gives

$$\mathfrak{S}\left[\mathbf{D}^{\mathrm{perf}}_Z(X)\right] = \mathbf{D}^b_{\mathbf{coh},Z}(X).$$

If we add the assumption that the scheme X is coherent, then one also has

$$\mathfrak{S}\left(\left[\mathbf{D}^b_{\mathbf{coh},Z}(X)\right]^{\mathrm{op}}\right) = \left[\mathbf{D}^{\mathrm{perf}}_Z(X)\right]^{\mathrm{op}} .$$

Another approach

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

Another approach

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

The metric we used on $\mathcal{T}^c$ has $B_n = \mathcal{T}^c \cap \mathcal{T}^{\leq -n}$.

Another approach

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

The metric we used on $\mathcal{T}^c$ has $B_n = \mathcal{T}^c \cap \mathcal{T}^{\leq -n}$.

The metric in Krause's paper has $B_n = \mathcal{T}^c \cap \left(\mathcal{T}^{\leq -n} * \mathcal{T}^{\geq n}\right)$

Another approach

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

The metric we used on $\mathcal{T}^c$ has $B_n = \mathcal{T}^c \cap \mathcal{T}^{\leq -n}$.

The metric in Krause's paper has $B_n = \mathcal{T}^c \cap \left(\mathcal{T}^{\leq -n} * \mathcal{T}^{\geq n}\right)$

Where $\mathcal{T}^{\leq -n} * \mathcal{T}^{\geq n}$ is defined by

$$\mathcal{T}^{\leq -n} * \mathcal{T}^{\geq n} = \left\{ Y \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{there exists a triangle } X \longrightarrow Y \longrightarrow Z \\ \text{with } X \in \mathcal{T}^{\leq -n} \text{ and with } Z \in \mathcal{T}^{\geq n} \end{array} \right\}.$$

And now for a totally different example

Example

Let $\mathcal{T}$ be the homotopy category of spectra. Then $\mathcal{T}$ is approximable and coherent.

For the purpose of the formulas that are about to come: $\pi_i(t)$ stands for the ith stable homotopy group of the spectrum t. It can be computed that

1. $$\mathcal{T}^- = \{t \in \mathcal{T} \mid \pi_i(t) = 0 \text{ for } i \ll 0\}$$

2. $$\mathcal{T}^+ = \{t \in \mathcal{T} \mid \pi_i(t) = 0 \text{ for } i \gg 0\}$$

3. $$\mathcal{T}^b = \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \pi_i(t) = 0 \text{ for all but} \\ \text{finitely many } i \in \mathbb{N} \end{array} \right\}$$

4. $\mathcal{T}^c$ is the subcategory of finite spectra.
5. $$\mathcal{T}_c^- \quad = \quad \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \pi_i(t) = 0 \text{ for } i \ll 0, \text{ and} \\ \pi_i(t) \text{ is a finite } \mathbb{Z}\text{–module for all } i \in \mathbb{Z} \end{array} \right\}$$
6. $$\mathcal{T}_c^b \quad = \quad \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \pi_i(t) = 0 \text{ for all but finitely many } i \in \mathbb{Z}, \text{ and} \\ \pi_i(t) \text{ is a finite } \mathbb{Z}\text{–module for all } i \in \mathbb{Z} \end{array} \right\}$$

The general theory applies, telling us (for example)

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}_c^b\,, \qquad\qquad \mathfrak{S}\Big(\big[\mathcal{T}_c^b\big]^{\mathrm{op}}\Big) = \big[\mathcal{T}^c\big]^{\mathrm{op}}\,.$$

It is a theorem of Schwede that the category $\mathcal{T}^c$, that is the homotopy category of finite spectra, has a unique enhancement.

It is a theorem of Schwede that the category $\mathcal{T}^c$, that is the homotopy category of finite spectra, has a unique enhancement.

The reference is:

Stefan Schwede, *The stable homotopy category is rigid*, Ann. of Math. (2) **166** (2007), no. 3, 837–863.

It is a theorem of Schwede that the category $\mathcal{T}^c$, that is the homotopy category of finite spectra, has a unique enhancement.

The reference is:

Stefan Schwede, *The stable homotopy category is rigid*, Ann. of Math. (2) **166** (2007), no. 3, 837–863.

Combining this with the results above

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}_c^b\ , \qquad\qquad \mathfrak{S}\Big(\big[\mathcal{T}_c^b\big]^{\mathrm{op}}\Big) = \big[\mathcal{T}^c\big]^{\mathrm{op}}\ ,$$

It is a theorem of Schwede that the category $\mathcal{T}^c$, that is the homotopy category of finite spectra, has a unique enhancement.

The reference is:

Stefan Schwede, *The stable homotopy category is rigid*, Ann. of Math. (2) **166** (2007), no. 3, 837–863.

Combining this with the results above

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}_c^b\ , \qquad \mathfrak{S}\Big(\big[\mathcal{T}_c^b\big]^{\mathrm{op}}\Big) = \big[\mathcal{T}^c\big]^{\mathrm{op}}\ ,$$

we deduce that the category $\mathcal{T}_c^b$ also has a unique enhancement.

Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^b_{\mathbf{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, https://arxiv.org/abs/1804.02240.

Amnon Neeman, *The category* $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ *as functors on* $\mathcal{T}^b_c$, https://arxiv.org/abs/1806.05777.

Amnon Neeman, *The categories* $\mathcal{T}^c$ *and* $\mathcal{T}^b_c$ *determine each other*, https://arxiv.org/abs/1806.06471.

Thank you!