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Intersection pairings on categories of matrix factorizations

S RLR, $f \in S$ iso. sing.

$R = S/(f)$, M, N f.g. R-mods.

$\mathbb{P} \xrightarrow{\simeq} M$ proj. resln.

$\mathrm{Tor}_i^R(M,N) := H_i(\mathbb{P} \otimes_R N)$

$$0 \leftarrow M \leftarrow \underbrace{F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow F \xleftarrow{A} G \xleftarrow{B} F \leftarrow \cdots}_{\text{even}}$$

$$F_0 \otimes N \leftarrow \cdots \leftarrow F_d \otimes N \leftarrow \overset{\mathbb{P}}{F \otimes N} \leftarrow G \otimes N \leftarrow \cdots$$

$\mathrm{Tor}_{ev}(M,N)$

$\mathrm{Tor}_{odd}(M,N)$

1981 M. Hochster: "θ-pairing / θ-invariant"

$$\theta(M,N) := \ell(\mathrm{Tor}_{ev}(M,N)) - \ell(\mathrm{Tor}_{odd}(M,N)) \in \mathbb{Z}$$

length instead of dim because we don't assume R contains a field

<u>Note</u> over S, $\chi(M',N') = \sum_i (-1)^i \ell(\mathrm{Tor}_i^S(M',N'))$, M', N' f.g. S-mods. $\ell(M' \otimes_S N') < \infty$.

"intersection theory of M', N' over $\operatorname{Spec} S$"

<u>Prop</u> (Hochster) $M = S/I$, $N = S/J$, $S/_{I+J}$ artinian
$f \in I \cap J$

$\Rightarrow \theta_R(M,N) = \chi(M,N)$

<u>Properties of θ:</u>

a) θ defines a pairing $K_0(R) \times K_0(R) \longrightarrow \mathbb{Z}$

b) θ descends to $\dfrac{K_0(R)}{K_0(\mathrm{prj}\,R)} \cong K_0(D_{sg}(R)) \cong K_0(\underline{MF}(f))$

c) θ vanishes on torsion elements (since $\mathbb{Z}$ is t.f.)

<u>Ex</u> For an even diml. ADE sing., $\theta \equiv 0$
since K_0 is torsion in this case.

d) θ kills classes of artinian modules

<u>Pf</u> k = res. field class, M R-module

$0 \leftarrow M \leftarrow \cdots \leftarrow F \xleftarrow{A} G \xleftarrow{B} F \leftarrow \cdots$ free resln. w/ $A, B \in \mathfrak{m}\,\mathrm{Mat}(R)$

$- \otimes_R k$: $\cdots \leftarrow F\otimes k \xleftarrow{0} G \otimes k \xleftarrow{0} F \otimes k \leftarrow \cdots$

So $\theta(k, M) = \mathrm{rk}\,F - \mathrm{rk}\,G = 0$.

For general case of N art., induct on $\ell(N)$.

So θ is defined on $\dfrac{\mathrm{Mod}\,R}{\mathrm{art}\,R} \cong \mathrm{coh}\,U$, $U = \mathrm{Spec}\,R \setminus \{\mathfrak{m}\}$

So "$\theta(M,N) = \theta(M|_U, N|_U)$"

Also θ is zero on divisible elts. of K_0.

Motto θ is extremely degenerate.

Thm (Conj. by Hailong Dao)

$f \in \mathbb{C}\{x_1, \dots, x_{2n+1}\}$ (or $\mathbb{C}[[x_1, \dots, x_{2n+1}]]$)

w/ iso. sing. Then θ vanishes identically.

Pfs ① f homog., M, N graded, no restriction on field — Moore, Piepmeyer, Spiroff, Walker Advances, 2010

② Formal case, deduced from Kapustin-Li — Polishchuk, Vaintrob, see ArXiv.

③ Analytic case ~ Buch. / van Straten

Common feature of these proofs

$\theta : K_0(R) \times K_0(R) \longrightarrow \mathbb{Z}$

factor through cohomology

① $H^{\cdot}_{et}(X)$: étale cohomology on $X = \operatorname{Proj} R \subseteq \mathbb{P}^{2n}$

② $HH_0(MF(f)) \cong S/(\partial_1 f, \dots, \partial_n f)\,[d-1 \bmod 2]$

③ $H^n(L)$, L link of the singularity defined by f

HRR gives all of these; in mixed characteristic no one knows a cohom. theory to use.

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f homog. in $K[x_1, \dots, x_{2m+2}]$, $\deg f = d$

$X = V(f=0) \subseteq \mathbb{P}^{2m+1}$, $Y, Z \subseteq X$ cycles of dim m

<u>Prop</u> (B., v Straten) Set $M = S/I(Y)$, $N = S/I(Z)$, then $\theta(M,N) = \frac{-1}{d} [\![Y]\!] \cdot [\![Z]\!]$, where

$[\![Y]\!] = [Y] - (\deg Y) h^m \in H^{2m}(X)$, h hyp. section, $[Y]$ fund. class of Y

$X \subseteq P$, $H^{\cdot}(\mathbb{P}) \longrightarrow H^{\cdot}(X)$ by restriction

$H^{\cdot}(\mathbb{P}) \cong \dfrac{\mathbb{C}[h]}{(h^{2m+2})}$, $h \in H^2(\mathbb{P})$ hyp. section

$$H^0(\mathbb{P}) \xrightarrow{\cong} H^0(X)$$
$$H^2(\mathbb{P}) \xrightarrow{\cong} H^2(X)$$
$$\vdots$$
$$H^{4m}(\mathbb{P}) \xrightarrow{\cong} H^{4m}(X)$$
$$H^{4m+2}(\mathbb{P}) \longrightarrow 0$$

isom. in every degree except $H^{2m}(\mathbb{P}) \to H^{2m}(X)$

$$0 \to H^{2m}(\mathbb{P}) \to H^{2m}(X) \to P^{2m}H(X) \to 0$$

$P^{2m}H(X)$: primitive cohomology of X

If X is even dimensional, there is no primitive cohomology.
[thus, in accordance w/ the results mentioned above, Hochster's pairing vanishes in that case]

So $[\![Y]\!] \cdot [\![Z]\!]$ is the product in primitive cohomology.

<u>Cor</u> θ on graded modules "is" the intersection pairing on the primitive cohom. of the proj. hypersurface.

Griffiths determined the primitive cohomology and Hodge numbers for any smooth projective hypersurface. Deligne extended this to any smooth complete intersection; see SGA 7.

<u>Thm</u> (P. Griffiths) f and X as above (over $\mathbb{C}$)

$$H^{2m}(X) = \bigoplus_{p+q=2m} H^q(X, \Omega_X^p) \rightsquigarrow \text{Hodge decomp.}$$

$$H^q(X, \Omega_X^p) \cong \Bigg[\overbrace{\frac{k[x_1, \dots, x_{2m+2}]}{(\partial_i f)_{i=1}^{2m+2}}}^{=\mathrm{Jac}(f)} \Bigg]_{d(p+1)-2m-2}$$

In particular, $H^m(X, \Omega_X^m) = [\mathrm{Jac} f]_{(d-2)(m+1)}$

[this is the middle, since $\operatorname{soc} \mathrm{Jac}(f) = \operatorname{hess}(f) = [\mathrm{Jac} f]_{(d-2)(2m+2)}$]

[~~If Hodge conj is true, this is then only part that matters important part for intersection theory~~]

All classes of algebraic cycles of codim. m lie in $H^m(X, \Omega^m)$, and if the Hodge conj. is true they span this v.s. Thus this is the most important piece.

__EY__ f = smooth cubic in $\mathbb{P}^3$, $m=1$, $d=3$

$$\mathrm{Jac}(f) = \frac{k[x_1, \dots, x_4]}{(4 \text{ quadrics: } \partial_i f)}, \quad H = \frac{(1-t^2)^4}{(1-t)^4} = (1+t)^4$$

0	
1	1
2	4
3	$\boxed{6}$
4	4
	1

$\dim \mathbb{P}H(X) = 6$, $\mathbb{P}H(X) \cong H^1(X, \Omega^1_X)$

gives E_6 as lattice w/ its pairing.

Take a line L on X: $f = l_1 q_1 + l_2 q_2$ $\quad$ l_i linear, q_i quadratic

L is $(l_1 = l_2 = 0)$

MF for L is

$\{l_1, q_1\} \otimes \{l_2, q_2\} = M$

For two lines L, L':

1. $L = L'$
2. L meets L' transversely
3. L, L' skew

} gives E_6 lattice

References

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