Cohen–Macaulay modules over non–isolated surface singularities

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University of Bielefeld

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See the book of Yoshino and a paper of Kajiura, Saito and Takahashi for explicit lists.

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This description is *rather unexplicit*.

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Here $X \in \mathsf{Mat}_{n\times n}(K)$ and $S, T \in \mathsf{GL}_n(D)$ are such that $S(0) = T(0)$ for $K = \mathsf{k}((t))$ and $D = \mathsf{k}[\![t]\!]$.

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Hence, I is Cohen–Macaulay viewed as A– or R–module.

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  - *Gluing map* $Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_R \widetilde{M}$.

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Remark. This problem of linear algebra is close to classification of representations of $\bullet \longleftarrow \bullet \longrightarrow \bullet$ over the field K.

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It is a result of Buchweitz, Greuel and Schreyer.

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$\big((a_1, b_1), \dots, (a_t, b_t)\big) \in \mathbb{Z}^{2t}$, $I = I_p$ and $J = J_p(\lambda)$.

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Example. In the terms of matrix factorizations

$$J_{2,\lambda} \cong \mathsf{Cok}\left(A^2 \xrightarrow{\begin{pmatrix} x+\mu(\mu+1)z^2 & y+\mu xz \\ y-(\mu+1)xz & -x^2 \end{pmatrix}} A^2 \right).$$

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$$M(\omega, \lambda) := \big\langle J, I(\omega, \lambda) \big\rangle_A^{\vee\vee} \subseteq A.$$

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Moreover, its presentation is

$$A^8 \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & v & u^n & 0 & 0 \\ 0 & v & 0 & 0 & 0 & x & y^p & 0 \\ 0 & 0 & x & 0 & 0 & 0 & u & v^q \\ 0 & 0 & 0 & u & \lambda x^m & 0 & 0 & y \end{pmatrix}} A^4 \longrightarrow M(\omega, \lambda) \longrightarrow 0.$$

Thank you for your attention!