Nils Carqueville

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Matrix factorisations in topological field & string theory

MFs in physics ($f = W$)

- D-branes and defects in Landau-Ginzburg models

2 dim. field theory

- CFT/LG Corr.

$$S_{LG} = \underbrace{\int_W \left[d^2z\, d^4\theta\, K(\Phi, \bar{\Phi}) + \left(\frac{1}{2} \int d^2z\, d^2\theta\, W(\Phi) \right) \Big|_{\bar{\theta}^+=0} + cc \right]}_{\text{one approach to LG models; we won't use this for this talk}}$$

+ boundary terms

Axiomatic approach

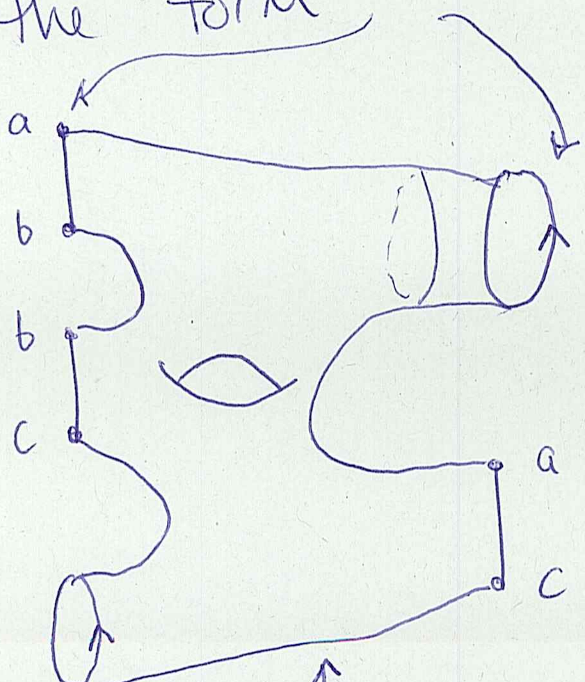
- topological field theory (aka TQFT)
- topological string theory (improvement or enrichment of TFT)
- ~~CFT~~ conformal field theory (LG model not a CFT because of W)

(everything will be 2-dim'l. for us)

A 2-dim'l. open/closed top. field theory is a symm. monoidal functor

$$Z : \mathcal{O}_\Lambda \mathcal{C} \longrightarrow \mathrm{vect}_{\mathbb{Z}/2}(\mathbb{C})$$

w/ Λ a set. Objects in $\mathcal{O}_\Lambda \mathcal{C}$ are of the form



w/ $a, b, c \in \Lambda$

and morphisms are cobordisms between them.

$$\%_c\,TFT = CTFT + OTFT + \text{sewing constraints}$$

$Z(S^1)$ comm. Frobenius algebra

$\{Z(I_{ab})\}_{a,b\in\Lambda}$ Calabi-Yau category

Sewing constraints:

$i_a : Z(S^1) \to Z(I_{aa})$ (boundary - bulk)

$i^a : Z(I_{aa}) \longrightarrow Z(S^1)$ (bulk - boundary)

$\mathrm{Str}(\psi \ldots) \longmapsto \langle \psi_1 \cdot \psi \cdot \psi_2 \rangle = \langle i^a \psi_1, i^b \psi_2 \rangle$ (Cardy cond.)

B-twisted affine LG-Model $W \in \mathbb{C}[x_1, \ldots, x_N]$ iso. sing. :

$\left(\mathrm{Jac}(W), \mathrm{Res} \begin{bmatrix} \phi_1 \phi_2 \, dx_1 \wedge \ldots \wedge dx_N \\ \partial_1 W, \ldots, \partial_n W \end{bmatrix} \right)$

closed state (Vafa)

$(MF(W), \text{KL pairing})$

open state

<u>Thm</u> The above are a TFT.

<u>Pf</u> Hard parts are non-deg. of KL pairing (→ Murfet)

Cardy cond. ⇝ Polishchuk-Vaintrob.

Open top. string theory

"Worldsheet view"

heuristically

$$\langle \dots \rangle_{TST} = \int_{\mathcal{M}} \langle \dots \rangle_{TFT}$$

$\mathcal{M}$ ← moduli space of Riemann surfaces w/ insertions and labels

$$\underline{\text{OTST} \cong \text{CY } A_\infty\text{-category}}$$

An $\underline{A_\infty\text{-algebra}}$ is a graded vector space A w/ a codifferential ∂ of degree 1 on $T_A := \bigoplus_{n \geq 1} A[1]^{\otimes n}$

$$m_n := \pi_{A[1]} \circ \partial \big|_{A[1]^{\otimes n}} : A[1]^{\otimes n} \longrightarrow A[1]$$

(A, m_n) is <u>minimal</u> if $m_1 = 0$ (so no homology)

<u>cyclic</u> wrt $\langle -, - \rangle$ if $\langle a_0, m_n(a_1 \otimes \dots \otimes a_n) \rangle$ is cyclic

<u>Calabi-Yau</u> if it's minimal and cyclic wrt non-deg. pairing.

Let Z be a OTFT and $H = Z(I_{aa}) = \mathbb{C}\{\psi_i\}$

$$\langle \psi_{i_0} \dots \psi_{i_n} \rangle_{OTST} = \langle \psi_{i_0}, m_n(\psi_{i_1} \otimes \dots \otimes \psi_{i_n}) \rangle =: W_{i_0 \dots i_n}$$

for a suitable A_∞-structure (H, m_n).

$\langle , \rangle$ for TFT is correlator

$\langle , \rangle$ for TST is amplitude

$$W_{eff}(u) = \sum_{n \geq 3} \frac{1}{n} W_{i_1 \dots i_n} u_{i_1} \dots u_{i_n}$$

Theorem Any A_∞-algebra is A_∞-quis. to a minimal one.

Pf Constructive!

Fix $D \in MF(W)$, DG-algebra $A = \mathrm{End}_{DG(W)}(D)$

Problem Generically, minimal models on $H^\cdot(A)$ are not acyclic wrt $\langle -,- \rangle^D_{KL}$.

Solution formal non-comm. geom.

[Kontsevich / Soibelman, C. 2009, C/Kay 2011]

Include bulk (or closed) sector

__Theorem__ (Costello) $MF(w)$ (w/ A_∞-structure) together w/ $HH(DG(w))$ as a closed sector gives a $^{o}/_{c}$ TST.

[But at first we're not sure it gives what we want.]

__Thm__ (Dyckerhoff) $HH(DG(w)) \cong Jac(w)$.

But how do we compute amplitudes, i.e. $\langle -,- \rangle_{TST}$?

$$\langle \dots \rangle_{OTST} \rightsquigarrow \langle \dots e^{\sum t_i S\phi_i} \rangle_{OTST}$$

__Theorem__ [Herbst, Lazaroni, Lerche] Bulk-deformed amplitudes are encoded in a __curved__ CY A_∞-category.

[theorem on constructing minl. A_∞-structures, above, doesn't work for curved A_∞-algebras]

__Theorem__ [C/Kay] Bulk-deformed amplitudes can be directly constructed in $MF(w)$.

Pf

$$\begin{tikzcd}
\mathrm{Jac}(w) \arrow[r] & (\text{poly. vector fields}) \arrow[r, "\text{``weak'' def. quant.}"'] & \mathrm{Coderiv}(T_A) \arrow[d, "\text{homl. perturbation lemma}"] \\
 & & \mathrm{Coderiv}(T_H)
\end{tikzcd}$$