06/05/11

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Geometric analysis of singularities

$f \in \mathbb{C}\{x_0, \ldots, x_n\}$, $f \in m^2$

$f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ germ of hol. fct with isol. sing at 0

Milnor $B_\varepsilon := \{x \in \mathbb{C}^{n+1} \mid |x| \leq \varepsilon\}$ $\quad \varepsilon > 0$

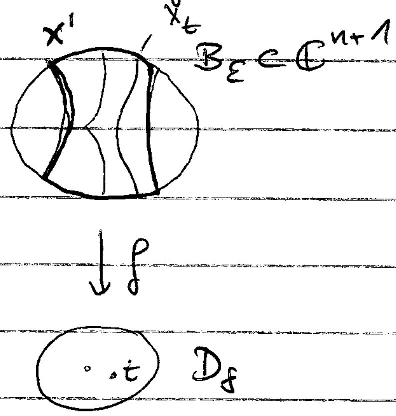
$D_\delta := \{t \in \mathbb{C} \mid |t| < \delta\}$, $D_\delta' = D_\delta \setminus \{0\}$

for $0 < \delta \ll \varepsilon \ll 1$

$f|_{X'} : X' = f^{-1}(D_\delta') \cap B_\varepsilon \longrightarrow D_\delta'$

smooth locally trivial fibration with fibre

$X_t = f^{-1}(t) \cap X'$, $\quad X_t \sim \underbrace{S^n \vee \ldots \vee S^n}_{\mu}$



$H_n(X_t; \mathbb{Z})$ free $\mathbb{Z}$-module of rk μ

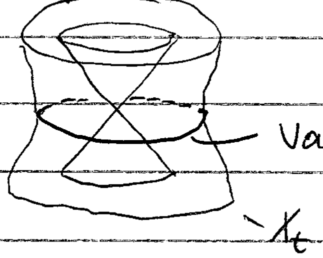
with intersection $\langle \, , \rangle$

n even: symmetric

$h : H_n(X_t, \mathbb{Z}) \longrightarrow H_n(X_t, \mathbb{Z})$ monodromy operator.

Example $f = x_0^2 + \ldots + x_n^2$

$n = 1$:



vanishing cycle $\delta \in H_n(X_t; \mathbb{Z})$

$\mu = 1 \qquad \langle \delta, \delta \rangle = -2$

h Picard-Lefschetz transformation

n even: $h = s_\delta$ reflection

$J_f = \left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \qquad Q_f = \mathcal{O}_{n+1} / J_f$ Milnor algebra

f isol. singularity $\iff \dim_\mathbb{C} Q_f < \infty$

$$Q_f = \mathcal{O}_{n+1}/J_f \longrightarrow \Omega^{n+1}/J_f\Omega^{n+1} = \Omega^{n+1}/df\wedge\Omega^n$$

$$a \longmapsto a\,\underbrace{dx_0\wedge\dots\wedge dx_n}_{\omega_{n+1}}$$

$$df\wedge(x_0\wedge\dots\wedge\hat{x}_i\wedge\dots\wedge dx_n) = \frac{\partial f}{\partial x_i}\omega_{n+1}$$

Brieskorn / Malgrange

$$\Omega^{n+1}/df\wedge\Omega^n \longrightarrow H^n_{dRh}(X_t;\mathbb{C})$$

via Gauß-Manin connection

$\Rightarrow$ ~~dim~~ $\dim_{\mathbb{C}} Q_f = \mu$

Eisenbud-Levine / Klimshiashvili:

$\exists\ b: Q_f\times Q_f \longrightarrow \mathbb{C}$ $\quad b(\varphi,\psi) = \ell(\varphi,\psi)$ bilinear form

ℓ linear form on Q_f which does not vanish Hess(f) (Socle of Q_f)

e.g. $\ell(\varphi) = \lim_{\varepsilon\to 0}\sum_{i=1}^{\tilde{}} \frac{\varphi(p_i)}{\mathrm{Hess}_{p_i}(f)}$

sum over $(\operatorname{grad} f)^{-1}(\varepsilon)$, $\operatorname{grad} f: \mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$

ε reg. value

$\rightsquigarrow Q_f$ Frobenius algebra

Mather-Yau: $f\sim g \iff Q_f\simeq Q_g$

$\Updownarrow$

$\exists\ \varphi: (\mathbb{C}^{n+1},0)\longrightarrow(\mathbb{C}^{n+1},0)$ $\quad f\circ\varphi = g$

Def. Unfolding of f is hol. fct. germ

$F: (\mathbb{C}^{n+1}\times\mathbb{C}^k, 0)\longrightarrow(\mathbb{C},0)$ with $F(x,0) = f(x)$

Def. $\left.\begin{array}{l} F: (\mathbb{C}^{n+1}\times\mathbb{C}^h,0)\to(\mathbb{C},0) \\ G: (\mathbb{C}^{n+1}\times\mathbb{C}^k,0)\longrightarrow(\mathbb{C},0)\end{array}\right\}$ ~~unfolding of f~~

$F \sim G \;:\Leftrightarrow\; \exists$ hol. map germ

$$\psi : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}^{n+1}, 0)$$
$$\psi(x, 0) = x$$
$$\text{s.t.} \quad G(x, u) = F(\psi(x, u), u)$$

<u>Def</u> $F : (\mathbb{C}^{n+1} \times \mathbb{C}^h, 0) \to (\mathbb{C}, 0)$ unfolding of f

$\varphi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ hol. map germ

$G : (\mathbb{C}^{n+1} \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0)$

$$G(x, u) = F(x, \varphi(u))$$

unfolding <u>induced by φ from</u> F

<u>Def</u>. Unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^h, 0) \to (\mathbb{C}, 0)$ <u>versal</u>

$\Leftrightarrow$ every unfolding of f is equivalent to an unfolding induced from F

<u>miniversal</u> $:\Leftrightarrow$ h minimal.

<u>Thm</u> $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ isol. sing.

$g_0 = -1, g_1, \ldots, g_{\mu-1}$ representatives of a basis of Q_f.

Then, $F : (\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$

$$(x, u) \longmapsto f(x) + \sum_{j=0}^{\mu-1} g_j(x) u_j$$

is a miniversal unfolding of f.

"Good" representative

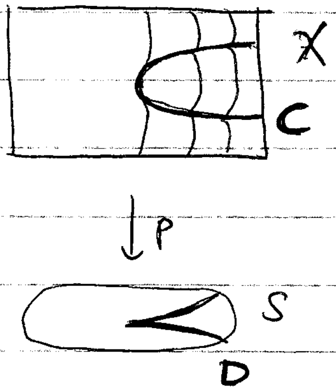
$F : M \times U \to \mathbb{C}, \quad \varepsilon > 0, \; \eta > 0$

$X = \{(x, u) \in M \times U \mid F(x, u) = 0, \; |x| \leq \varepsilon, \; |u| < \eta\}$

$\downarrow p$

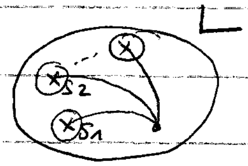
$S = \{u \in U \mid |u| < \eta\}$

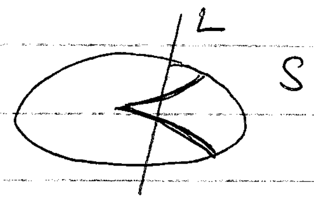
$C = \{(x,u) \in X \mid x$ critical point of $F(-,\mu)\}$



$D = p(C)$ discriminant

(i) $p : X \longrightarrow S$ proper
(ii) C non-singular analytic subset of X
(iii) $p|_C : C \longrightarrow D$ finite
(iv) D irreducible hypersurface in S
(v) $p|_{X'} : X' = X \setminus C \longrightarrow S \setminus D$ smooth locally trivial fibration.





<u>Example</u> Simple $f(x,y,z)$
n even: Vanishing cycles $\longleftrightarrow$ roots of root lattice $A_\mu, D_\mu, E_6, E_7, E_8$
reflections generate Weyl group W

$$(S, D) \xrightarrow{\sim} (\mathbb{C}^M/W, \mathcal{H}/W)$$