Triangle Singularities I

(14:45)

§1 Orlov's theorem

$k = \bar{k}$ alg. closed field

$A = \bigoplus_{i\geq 0} A_i$ positively graded, commutative, noether

- graded isolated singularity;
- graded complete intersection, $k[x_0, \dots, x_n]/(f_1, \dots, f_r)$ reg-seq.

$\Rightarrow$ * $d = n - r$ = K-dim A

* $0 \to A \to E^0 \to \dots \to E^d \to 0$ min. (graded) inj. resolution $E^d = \mathbb{E}(k(a))$

* $a = \sum_{i=1}^{n} \deg(x_i) - \sum_{j=1}^{r} \deg f_j$ Gorenstein parameter

* $D = \mathbb{R}\mathrm{Hom}(-, A) : D^b \mathrm{mod}^{\mathbb{Z}} A \longrightarrow D^b \mathrm{mod}^{\mathbb{Z}} A$ self-duality with $D(k) = k(a)[-d]$

Facts:

a) • $\mathrm{coh}\,\mathbb{X} := \mathrm{mod}^{\mathbb{Z}} A / \mathrm{mod}_0^{\mathbb{Z}} A$ has an interpretation as (graded) coherent sheaves on $\mathrm{Proj}^{\mathbb{Z}} A = \mathbb{X}$ dim $(d-1)$

It is noetherian, Hom-finite & satisfies Serre duality
$$D\,\mathrm{Ext}^{(d-1)-i}(X, Y) = \mathrm{Ext}^i(Y, X(-a))$$

• $\boxed{D^b \mathrm{coh}\,\mathbb{X}} = D^b(\mathrm{mod}^{\mathbb{Z}} A) / D^b(\mathrm{mod}_0^{\mathbb{Z}} A)$ is triangulated with Serre duality (as above).

b) $\boxed{\mathrm{Sing}^{\mathbb{Z}} A} := D^b(\mathrm{mod}^{\mathbb{Z}} A) / D^b(\mathrm{proj}_0^{\mathbb{Z}} A)$ is triangulated Hom-finite. We have seen in previous talks that the singularity category $\mathrm{Sing}^{\mathbb{Z}} A$ appears in many different incarnations, for instance, as $\underline{\mathrm{CM}}^{\mathbb{Z}} A$.

HL2 Situation. $D^b\mathrm{coh}\,\mathbb{X}$ and $\mathrm{Sing}^{\mathbb{Z}} A$ are competing / complementing invariants for an algebraic analysis of singularities.
What is their relationship? This is answered by Orlov's theorem.

Thm (Orlov '09) Assumptions on A as above. Then $D^b\mathrm{coh}\,\mathbb{X}$ and $\mathrm{Sing}^{\mathbb{Z}} A$ are related as follows:

a) $\boxed{a>0}$. Then there is a strong exceptional sequence

$$\mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(a) \text{ in } D^b\mathrm{coh}\,\mathbb{X}, \text{ where}$$

$\mathcal{O}(n)$ is the can. image of $A(n)$ in $\mathrm{coh}\,\mathbb{X}$ such that

$$\mathrm{Sing}^{\mathbb{Z}} A \cong \{\mathcal{O}(1), \ldots, \mathcal{O}(a)\}^{\perp} \subseteq D^b\mathrm{coh}\,\mathbb{X}.$$

b) $\boxed{a=0}$. Then

$$\mathrm{Sing}^{\mathbb{Z}} A \cong D^b\mathrm{coh}\,\mathbb{X}.$$

c) $\boxed{a<0}$. Then

$k(-1), k(-2), \ldots, k(a)$ is an exceptional sequence in $\mathrm{Sing}^{\mathbb{Z}} A$ s.t.

$$D^b\mathrm{coh}\,\mathbb{X} \cong \{k(-1), \ldots, k(a)\}^{\perp} \subseteq \mathrm{Sing}^{\mathbb{Z}} A$$

Sketch of proof. We embed

$$D^b\mathrm{coh}\,\mathbb{X} \text{ and } \mathrm{Sing}^{\mathbb{Z}} X$$

in a larger triang. category as follows.

HL3

14:59

$$\begin{tikzcd} & D^b(\mathrm{mod}^{\mathbb{Z}} A) & \\ & D^b(\mathrm{mod}^{\geq i} A) \arrow[u, hook] & \\ \mathcal{T}_{\geq i}^{\perp} \arrow[ur] & & {}^{\perp}\mathcal{P}_i \arrow[ul, hook] \end{tikzcd}$$

$$\mathcal{T}_{\geq i} = \mathrm{mod}_0^{\geq i} A$$

$$\mathcal{P}_{\geq i} := \mathrm{proj}^{\geq i} A$$

$$\frac{D^b(\mathrm{mod}^{\geq i} A)}{\mathcal{T}_{\geq i}} = \mathcal{T}_{\geq i}^{\perp} \cong \boxed{D^b \mathrm{coh}\, \mathbb{X}}$$

$${}^{\perp}\mathcal{P}_i = \frac{D^b \mathrm{mod}^{\geq i} A}{\mathcal{P}_{\geq i}} \cong \boxed{\mathrm{Sing}^{\mathbb{Z}} A}$$

Invoking the duality D one proves

$$\boxed{\mathcal{T}_{\geq i}^{\perp} = {}^{\perp}\mathcal{P}_{\geq i+a}}$$

Now
for $a = 0$ $\quad \mathcal{P}_{\geq i+a} = \mathcal{P}_{\geq i}$
for $a > 0$ $\quad \mathcal{P}_{\geq i+a} \subset \mathcal{P}_{\geq i}$
for $a < 0$ $\quad \mathcal{P}_{\geq i+a} \supset \mathcal{P}_{\geq i}$

Passage to the left-perpendicular categories proves the claim. $\square$

§2. Triangle singularities & weighted proj. lines

I restrict to the case of 3 weights.

The polynomial $\boxed{f = x_1^{p_1} + x_2^{p_2} + x_3^{p_3}}$ $\quad p_i \geq 2$
is called a *triangle singularity*; it comes together with the following data

- The *k-algebra* $S = k[x_1, x_2, x_3]/(f)$, naturally graded by the *abelian group*
- $\mathbb{L} = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \mid p_1 \vec{x}_1 = p_2 \vec{x}_2 = p_3 \vec{x}_3 =: \vec{c} \rangle$
  $\vec{c}$ *canonical element*. $\quad (\mathbb{L}, \leq)$ with positive cone $\sum \mathbb{N} \vec{x}_i$.

HL4 | • coh $\mathbb{X}$ ($\mathbb{X} = \mathrm{Proj}^{\mathbb{Z}} S$) category of coherent sheaves, obtained by Serre construction

$\mathrm{coh}\,\mathbb{X} := \mathrm{mod}^{\mathbb{Z}} S / \mathrm{mod}_0^{\mathbb{Z}} S$

* abelian, Hom-finite, noetherian
* Serre duality $D\mathrm{Ext}^1(X,Y) = \mathrm{Hom}(Y, X(\vec{\omega}))$ where $\vec{\omega} := \vec{c} - \sum_{i=1}^{3} \vec{x}_i$ (dualising elt)
* coh $\mathbb{X}$ ($\mathcal{D}^b \mathrm{coh}\,\mathbb{X}$) has almost-split sequences (resp. AR-triangles) with AR-translate = shift ($\vec{\omega}$)
* coh $\mathbb{X}$ hereditary, $\mathrm{Ext}^2 = 0$.
* $(\mathbb{L}, +) \xrightarrow{\sim} (\mathrm{Pic}\,\mathbb{X}, \otimes)$
* $\mathrm{coh}\,\mathbb{X} = \mathrm{vect}\,\mathbb{X} \amalg \mathrm{coh}_0\mathbb{X}$

vect. bundles ⟶ vect $\mathbb{X}$; fin. length ⟶ $\mathrm{coh}_0\mathbb{X}$

Thm (G-L, '87) Sheafification $q: \mathrm{mod}^{\mathbb{L}} S \to \mathrm{coh}\,\mathbb{X}$ induces an equivalence

$\boxed{\sim : \mathrm{CM}^{\mathbb{L}} S \xrightarrow{\sim} \mathrm{vect}\,\mathbb{X}}$

[Inverse: $\Gamma_*(E) = \bigoplus_{\vec{x} \in \mathbb{L}} \mathrm{Hom}(\mathcal{O}(-\vec{x}), E)$]

inducing an eq. $\mathrm{proj}^{\mathbb{L}} S \xrightarrow{\sim} \mathcal{L}$

all line bdles ⟶ $\mathcal{L}$

Cor. Sheafification induces an equivalence

$\underline{\mathrm{CM}}^{\mathbb{L}} S \xrightarrow{\sim} \dfrac{\mathrm{vect}\,\mathbb{X}}{[\mathcal{L}]} =: \underline{\mathrm{vect}}\,\mathbb{X}$

called the stable category of vector bundles,

Consequence: • Obtain further incarnation of $\mathrm{Sing}^{\mathbb{L}} S$ with $\underline{\mathrm{vect}}\,\mathbb{X}$
• particularly suitable for study

of (homotopy) category $\underline{\mathrm{MF}}^{\mathbb{L}}(f)$ of matrix factorizations.

- if (p_1, p_2, p_3) are pairwise coprime, then $\mathbb{L} = \mathbb{Z}$. In this particular case we deal with a positively $\mathbb{Z}$-graded situation.

$\mathbb{Z} \vec{\omega}$

§3 Influence of the Euler characteristic

The 'shape' of $\mathrm{coh}\,\mathbb{X}$, $\mathcal{D}^b \mathrm{coh}\,\mathbb{X}$ and $\mathrm{vect}\,\mathbb{X}$ is largely influenced by the Euler characteristic

$$\chi_{\mathbb{X}} = 2 - \sum_{i=1}^{3} \left(1 - \frac{1}{p_i}\right) = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 1 .$$

It is a bit tricky to define the Gorenstein parameter properly:

$$a := -\left[(p_1 - 1)\cdot(p_2 - 1)\cdot(p_3 - 1) - (p_1 + p_2 + p_3) + 1\right]$$

Note:
- $a = 0 \iff \chi_{\mathbb{X}} = 0$ (tub. case)
- $a > 0 \iff \chi_{\mathbb{X}} > 0$ (dom. case)
- $a < 0 \iff \chi_{\mathbb{X}} < 0$ (wild case)

Moreover, if $(a \neq 0)$ then

$$|a| = [\mathbb{L} : \mathbb{Z}\vec{\omega}] = \text{number of AR-orbits of line bundles in } \mathrm{vect}\,\mathbb{X}$$

HL6 | **§4. Tilting objects**

Thm. $\mathbb{X} = \mathbb{X}(p_1, p_2, p_3)$. Then the foll. holds.

a) The category $\operatorname{coh}\mathbb{X}$ (resp. $D^b \operatorname{coh}\mathbb{X}$) has a tilting object T consisting of $(p_1+p_2+p_3-1)$ pairwise non-iso $\boxed{\text{line bundles.}}$ $\operatorname{End}(T)$ = can. alg. of type (p_1,p_2,p_3) sense of Ringel.

GL[87]

b) The category $\operatorname{vect}\mathbb{X}$ has a tilting object T' which is a direct sum of $(p_1-1)\cdot(p_2-1)\cdot(p_3-1)$ pairwise non-iso ind. $\boxed{\text{bundles of rank two}}$ (see Kussin's talk for details.)

KLM[10]

Takahashi[10]

c) $\operatorname{rk}_{\mathbb{Z}}(K_0(\operatorname{vect}\mathbb{X})) - \operatorname{rk}(K_0 \operatorname{coh}\mathbb{X})$

$= [(p_1-1)\cdot(p_2-1)\cdot(p_3-1)] - [(p_1+p_2+p_3-1)]$

= Gorenstein parameter of A.

c) is an instance of Orlov's theorem ($\mathbb{L}$-graded version).

§5 Calabi-Yau

Thm The category $\operatorname{vect}\mathbb{X} = \operatorname{HMF}(f)$ is fractional Calabi-Yau of dimension $1 + 2\chi_{\mathbb{X}}$.

Cor. Periodicity of Coxeter transformation ϕ.
If $\bar{p} = \operatorname{lcm}(p_1,p_2,p_3)$ then $\phi^{\bar{p}} = \mathbb{1}$.

§6. Concluding remarks

Remark. Related work by Kajiura–Saito–Takahashi & L–de la Peña on Kleinian singularities.

Those deal with $\boxed{a = \pm 1}$.

15:30

The Orlov effect

Example. $p_1 = 9$, $p_2 = 10$, $p_3 = 11$ gives

- $\mathrm{rk}(K_0(D^b \mathrm{coh}\, \mathbb{X})) = 19$
- $\mathrm{rk}(\underline{\mathrm{vect}}\ \mathbb{X}) = 720$
- $a = -701$

15:35