- What is it?
- Presentations
- stabilization

- $(R, \mathfrak{m}, k)$ local ring, $\mathfrak{m}$ max. ideal
  $k \to R \to R/\mathfrak{m}$
- Assume R is Gorenstein, i.e., $\mathrm{id}_R(R) < \infty$
  e.g.: $R = \mathbb{C}[[\underline{x}]]$, $\mathbb{C}[[\underline{x}]]/w$, $\mathbb{C}[[\underline{x}]]/(\underbrace{f_1, \dots, f_r}_{\text{regular}})$

"perfect cplxes"

$$\mathbb{K}^b(\operatorname{proj} R) \hookrightarrow \mathbb{D}^b(\operatorname{mod} R) \twoheadrightarrow \mathbb{D}^b(\operatorname{mod} R)/\mathbb{K}^b(\operatorname{proj} R) =: \mathbb{D}^b_{sg}(R)$$

equality iff R regular

"stable derived cat."
or: "Δed cat. of singularities"

$\mathbb{D}^b_{sg}(R) =$
- objects: b'ded complexes of f.g. modules
- morphisms: $G \to H$, equiv classes of fractions $G \xleftarrow{\alpha} K \xrightarrow{\beta} H$ s.t. $\mathrm{cone}(\alpha) \in \mathbb{K}^b(\operatorname{proj} R)$

Properties

- $\mathbb{D}^b_{sg}(R) = 0 \iff R$ regular
- $\mathbb{D}^b_{sg}(R)$ is split generated by $\{ R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R$, i.e. $R_{\mathfrak{p}}$ non-regular$\}$
  get every object by using cones, dir. summands
  (Schoutens, Orlov, R. Takahashi)
- Homs in $\mathbb{D}^b_{sg}(R)$ f.g. R-modules ($\iff R$ Gorenstein! (Avramov-Veliche))
- If singularity of R is isolated ($:\iff R_{\mathfrak{p}}$ regular $\forall\ \mathfrak{p} \neq \mathfrak{m}$)
  e.g.: $W \in \mathbb{C}[[x_1, \dots, x_n]]$ has an isolated sing $\Rightarrow R = \mathbb{C}[[\underline{x}]]/W$ has isol. sing as ring.
  (cf. Hom spaces in $\mathbb{D}^b_{sg}(R)$ f. dim $/ k$)
- Isolated singularity: $\mathbb{D}^b_{sg}(R)$ has a Serre functor Σ^{d-1}, $d = \dim R$
  (Auslander) Concretely: $\mathrm{Hom}(G,H)^* \cong \mathrm{Hom}(H, \Sigma^{d-1} G)$ naturally.
- $\mathbb{D}^b_{sg}(R)$ is Δed

All very interesting... But what is the motivation for studying this class of triangulated categories?

MOTIVATION:
- $\mathbb{D}^b_{sg}(R)$ is an invariant (modern perspective, n.c. geom.)
- Topological invariants build ("linearized") by Hom spaces
  $\mathrm{Hom}_{\mathbb{D}^b_{sg}(R)}(G,H) \rightsquigarrow$ stable cohom, Tate cohom.

Presentations:

$\mathbb{K}_{ac}(\mathrm{proj}\, R) :=$ homotopy cat. of acyclic cplxes of f.g. proj R-modules

- Def: $M \in \mathrm{mod}(R)$ is Cohen-Macaulay (CM) if

$$M \cong \mathrm{Ker}(\partial^1 : P^1 \to P^2) \cong \mathrm{Coker}(\partial^{-1} : P^{-1} \to P^0)$$

for some $P \in \mathbb{K}_{ac}(\mathrm{proj}\, R)$.

(Remark: not 2-periodic here, since R not necessarily a hyp. sing.!)

Note: • R is CM.
• If $\mathfrak{p} \in \mathrm{Spec}\, R$, $R_{\mathfrak{p}}$ is regular, $P \in \mathbb{K}_{ac}(\mathrm{proj}\, R)$

$$\cdots \to P^0 \to P^1 \to P^2 \to \cdots \qquad (P^0 \twoheadrightarrow M \hookrightarrow P^1)$$

localiz. is exact $\rightsquigarrow$

$$\cdots \to P^0_{\mathfrak{p}} \to P^1_{\mathfrak{p}} \to P^2_{\mathfrak{p}} \to \cdots \quad \} \text{ contractible} \qquad (P^0_{\mathfrak{p}} \twoheadrightarrow M_{\mathfrak{p}} \hookrightarrow P^1_{\mathfrak{p}})$$

$\rightsquigarrow M_{\mathfrak{p}}$ is R-projective
$\Rightarrow M_{\mathfrak{p}}$ free for $\mathfrak{p}$ regular.

Thus the $\exists$ of a non-free CM module also detects sing.

Example: $R = \mathbb{C}[x]/x^3$

$$P : \cdots \to R \xrightarrow{x} R \xrightarrow{x^2 = \partial^0} R \xrightarrow{x = \partial^1} R \xrightarrow{x^2} R \to \cdots$$

$$\rightsquigarrow \left.\begin{aligned} \mathrm{Ker}(\partial^0) &= \mathbb{C}x \oplus \mathbb{C}x^2 \cong_R \mathbb{C}[x]/x^2 \\ \mathrm{Ker}(\partial^1) &= \mathbb{C}x^2 \cong_R \mathbb{C} \end{aligned}\right\} \text{CM modules over } R.$$

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$$\begin{array}{ccc} \mathcal{C}_{ac}(\mathrm{proj}\, R) & \xrightarrow{\mathrm{Ker}(\partial^1)} & \mathrm{CM}(R) \subset \mathrm{mod}\, R \\ \downarrow & & \downarrow \\ \mathbb{K}_{ac}(\mathrm{proj}\, R) & \xrightarrow{\simeq} & \underline{\mathrm{CM}}(R) = \mathrm{CM}(R) / \left\{ \begin{array}{l} \text{ideal of maps factoring through} \\ \text{a free}: M \to R^{\oplus n} \to N \end{array} \right\} \end{array}$$

"stable cat. of CM-modules"

Theorem (Buchweitz, Orlov) The composite

$$\mathrm{CM}(R) \hookrightarrow \mathbb{D}^b(\mathrm{mod}\, R) \twoheadrightarrow \mathbb{D}^b_{sg}(R)$$

induces an equivalence

$$\underline{\mathrm{CM}}(R) \xrightarrow{\simeq} \mathbb{D}^b_{sg}(R) \quad \text{"generated by CM"}$$

R Gorenstein: $\mathbb{K}_{ac}(\mathrm{proj}\, R) \simeq \underline{\mathrm{CM}}(R) \simeq \mathbb{D}^b_{sg}(R)$ ← each has its own advantages

• For simplicity, Hypersurfaces : $S := \mathbb{C}[\![\underline{x}]\!]$, $W \in S$

$R := \mathbb{C}[\![\underline{x}]\!]/W$

$\mathrm{hmf}(S,W) :=$ homotopy cat. of matrix factorizations of W over S

($\equiv$ $\mathbb{Z}_2$-graded free $\mathbb{C}[\![\underline{x}]\!]$-modules $X \underset{}{\overset{d}{\rightleftarrows}} X$, $\deg d = 1$, $d^2 = W \cdot 1_X$)

$(X, d_X) \in \mathrm{hmf}(S,W)$: $X^1 \xrightarrow{d_X^1} X^0 \longrightarrow \mathrm{Coker}(d_X^1) =: M \longleftarrow$ CM R-module

$W \cdot \bar{m} = \overline{W \cdot m} = \overline{d^1 d^0 . m} = 0$

⇝ get functors:

$$\begin{array}{ccc} \mathrm{mf}(S,W) & \xrightarrow{\mathrm{Coker}} & \mathrm{CM}(R) \\ \downarrow & & \downarrow \\ \mathrm{hmf}(S,W) & \xrightarrow[\text{(Eisenbud)}]{\simeq} & \underline{\mathrm{CM}}(R) \end{array}$$

Yet another equivalence! →

• We have e.g. $k \in D^b_{sg}(R)$. What is the corr. object in the 3 other categories ?!?

Not obvious... but $\exists$ standard technique:

⊙ Stabilization Let $M \in \mathrm{mod}\, R$.

⇝ $\exists$ a CM-approximation (Auslander-Buchweitz)

$$0 \to K \to M^{CM} \xrightarrow{\pi} M \to 0$$

K: f. proj. dim. M^{CM}: is CM

if map this to the stable category, it becomes a Δ and $K \cong 0$ ⇒ π is iso in $D^b_{sg}(R)$.

⇒ M^{CM} is going to be our CM-module mapping to M.

Def: $\{a,b\}$ is the factorization of ab:

$$\Big(\underset{X^0}{S} \xrightarrow{a} \underset{X^1}{S} \xrightarrow{b} S \Big) \in \mathrm{hmf}(ab), \text{ the "Koszul fact."}$$

Idea: combine these to construct more complicated m.f.'s.

Def: Given $\underline{a} = (a_1, \dots, a_n)$ $\underline{b} = (b_1, \dots, b_n)$ in S

⇝ $F := S\theta_1 \oplus \dots \oplus S\theta_n$, declare $|\theta_i| := -1$

differentials on ΛF: $\delta_+ = (\sum_i b_i \theta_i^*) \lrcorner -$

$\delta_- = (\sum_i a_i \theta_i) \wedge -$

(tensoring gives m.f. of the sum!)

Then $(\Lambda F, \delta_+ + \delta_-)$ is a m.f. of $W := \sum_i a_i b_i$, called:

$\{\underline{a}, \underline{b}\} := (\Lambda F, \delta_+ + \delta_-) = \{a_1, b_1\} \otimes \dots \otimes \{a_n, b_n\}$

• $(\Lambda F, \delta_+)$ is the Koszul complex

$$\begin{tikzcd} \Lambda^n F \arrow[r] & \cdots \arrow[r, "\delta_+"] & \Lambda^1 F \arrow[r, "\delta_+"] & \Lambda^0 F \arrow[d, "\rho : \text{ quasi-iso if } \underline{b} \text{ regular}"] \\ & & & S/\underline{b}S = R/\underline{b}R \quad (W \in \underline{b}S) \end{tikzcd}$$

Assume $\underline{b}$ is regular:

$$\begin{tikzcd} \rho : \{\underline{a}, \underline{b}\} \arrow[r] \arrow[d] & S/\underline{b}S \text{ commutes with differentials} \\ C := \mathrm{Coker}\big(\underbrace{\{\underline{a},\underline{b}\}^1}_{\Lambda^{odd}} \xrightarrow{d} \underbrace{\{\underline{a},\underline{b}\}^0}_{\Lambda^{even}}\big) \arrow[r, "\pi"] & \overset{\shortparallel}{R/\underline{b}R} \end{tikzcd}$$

$$\boxed{\underline{\text{Thm}} \text{ (Eisenbud)} \quad C = (R/\underline{b}R)^{CM}}$$

Thus:
$$\mathrm{hmf}(W) \simeq \underline{\mathrm{CM}}(R) \simeq D^b_{sg}(R)$$
$$\{\underline{a}, \underline{b}\} \longleftrightarrow C \longleftrightarrow R/\underline{b}R$$

<u>Examples</u>: • $W = w_1 x_1 + \cdots + w_n x_n$, $\{\underline{w}, \underline{x}\}$ "stabilizes" $\mathbb{C}$

• $S = \mathbb{C}[[x_1, x_2, x_3, x_4]]$ $\quad \overline{W} = W(x_3, x_4) - W(x_1, x_2)$

$\overline{W} = a_1(x_3 - x_1) + a_2(x_4 - x_2)$

$\{\underline{a}, x_3 - x_1, x_4 - x_2\}$ stabilizes $\mathbb{C}[[\underline{x}]]/(x_3 - x_1, x_4 - x_2)$

The End!