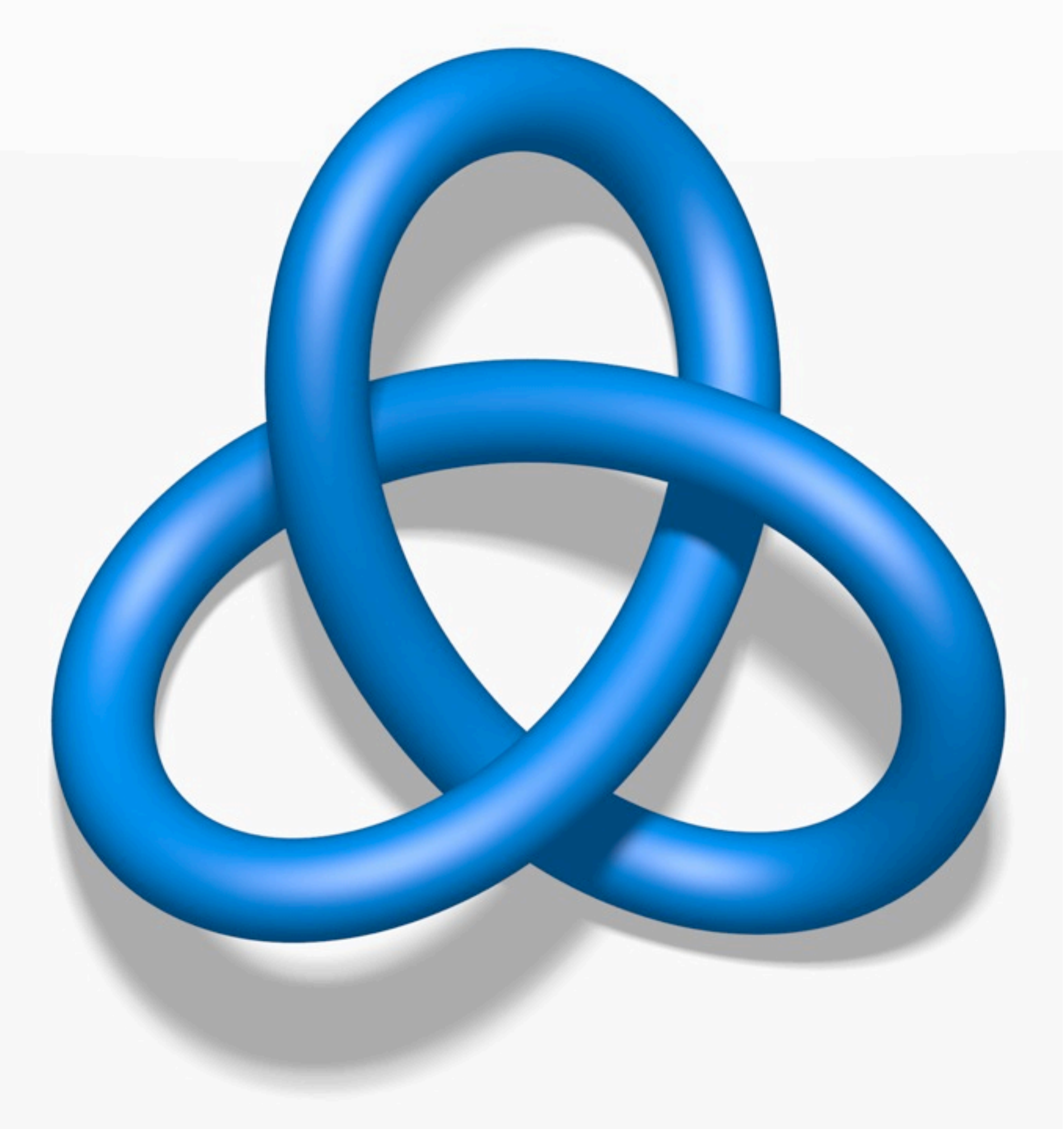
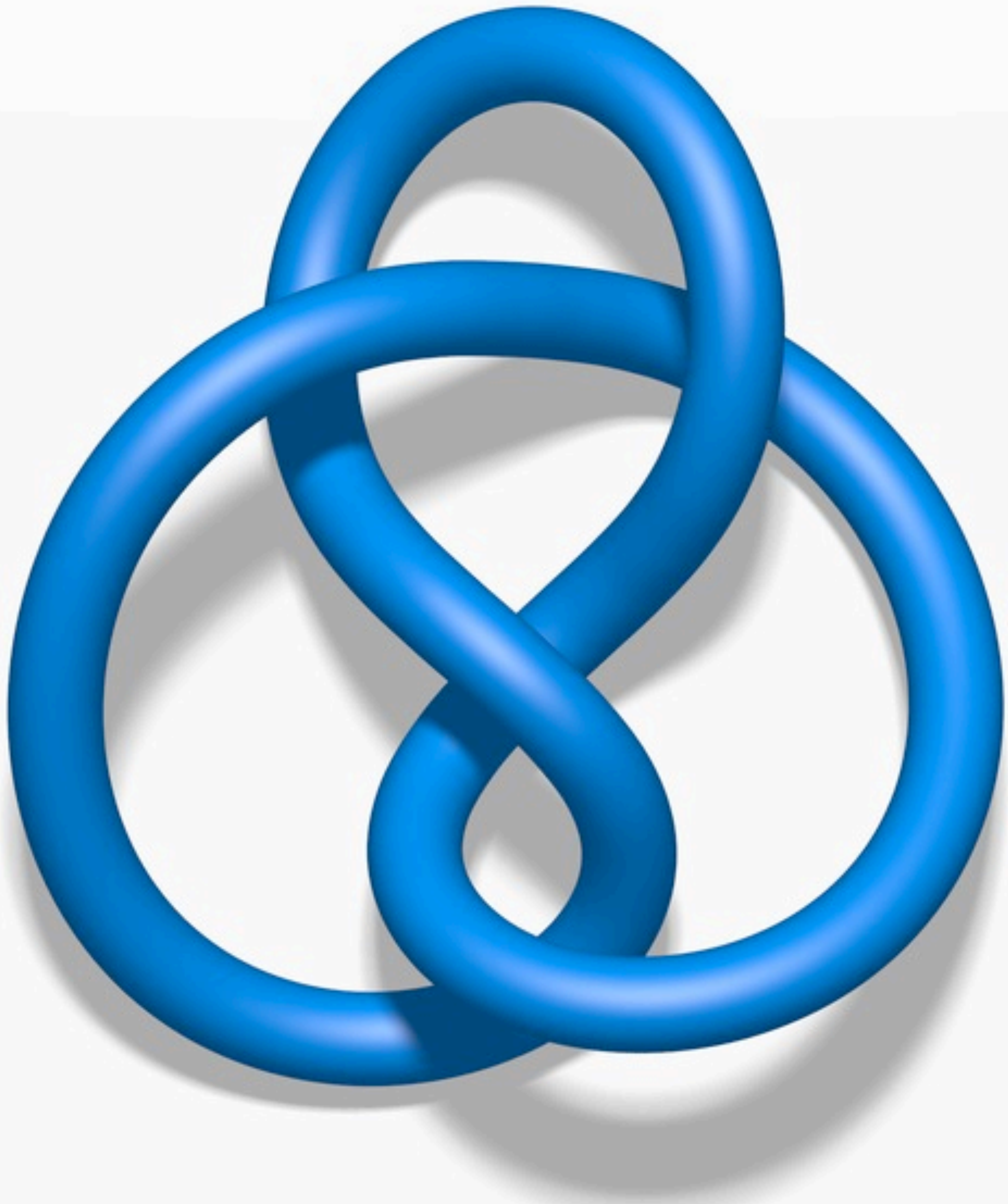
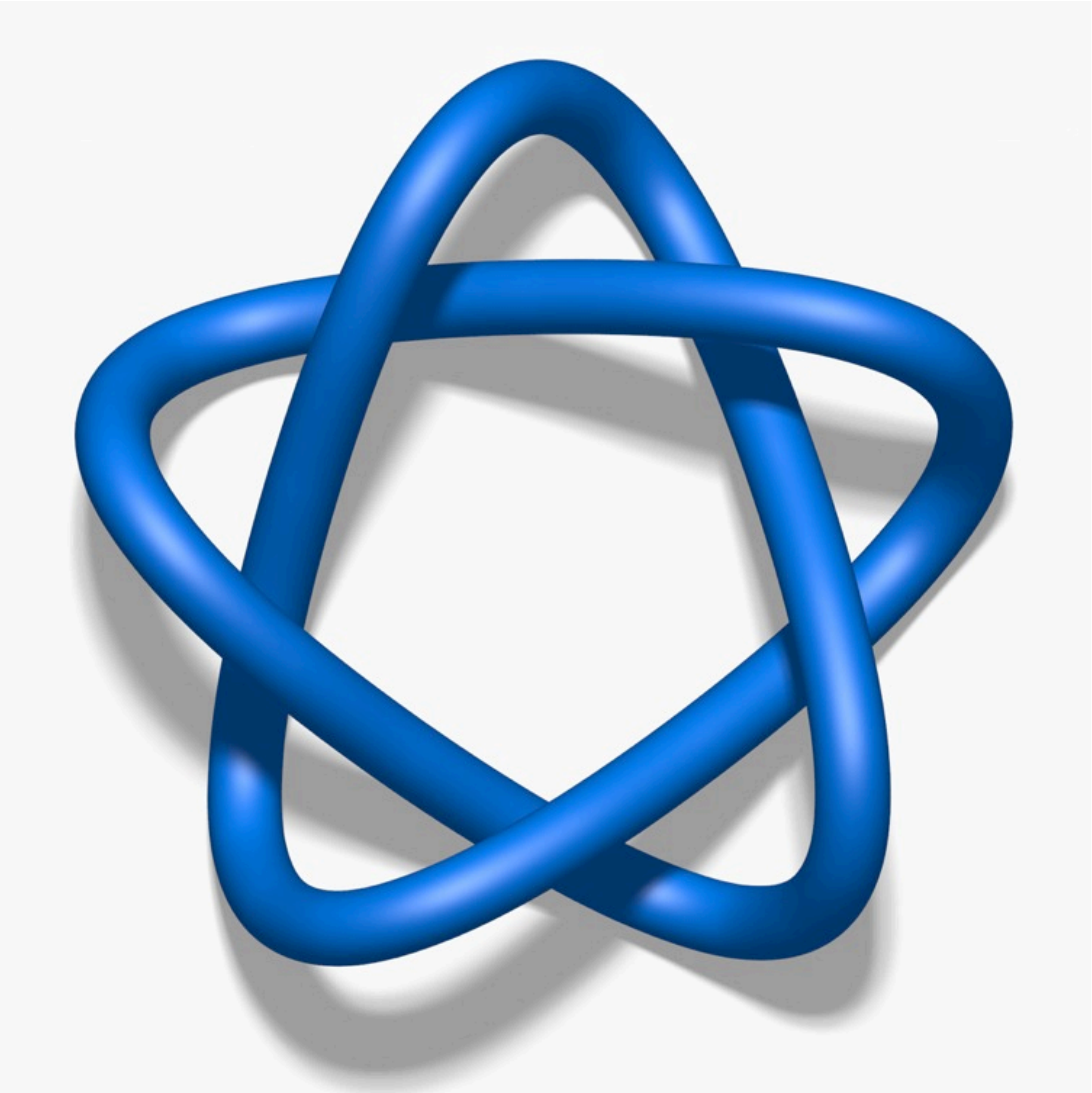
Link homology and convolution

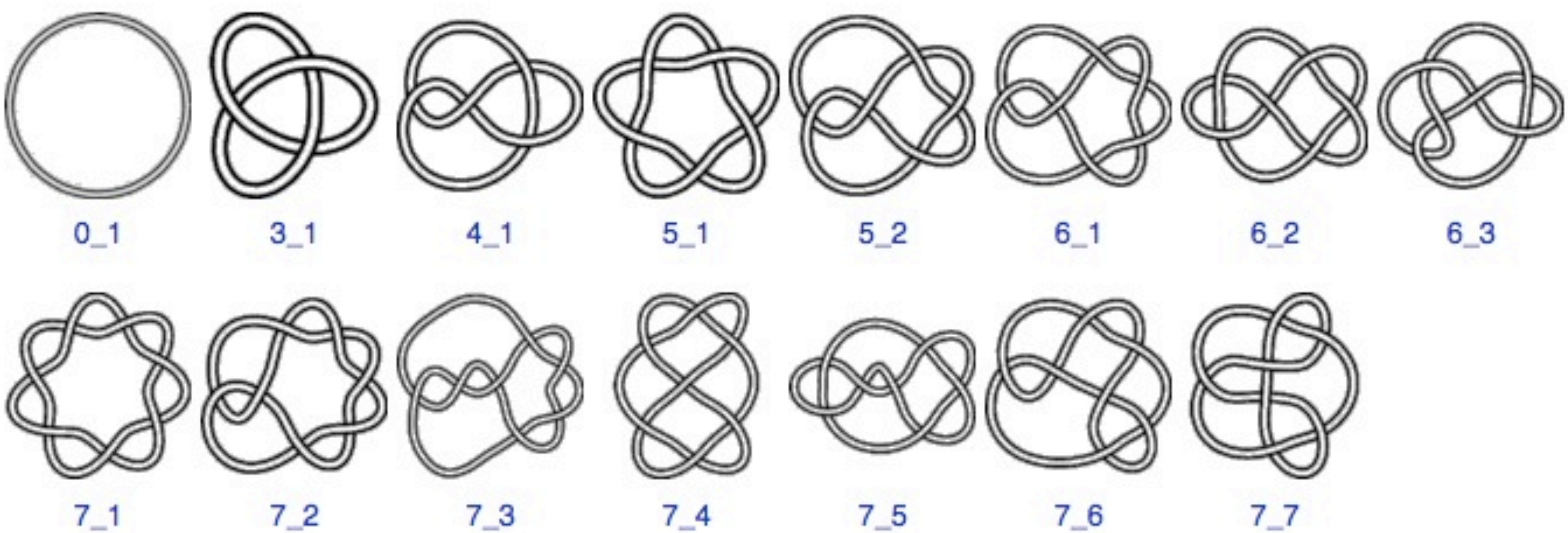
joint with Nils Carqueville

What is a knot?



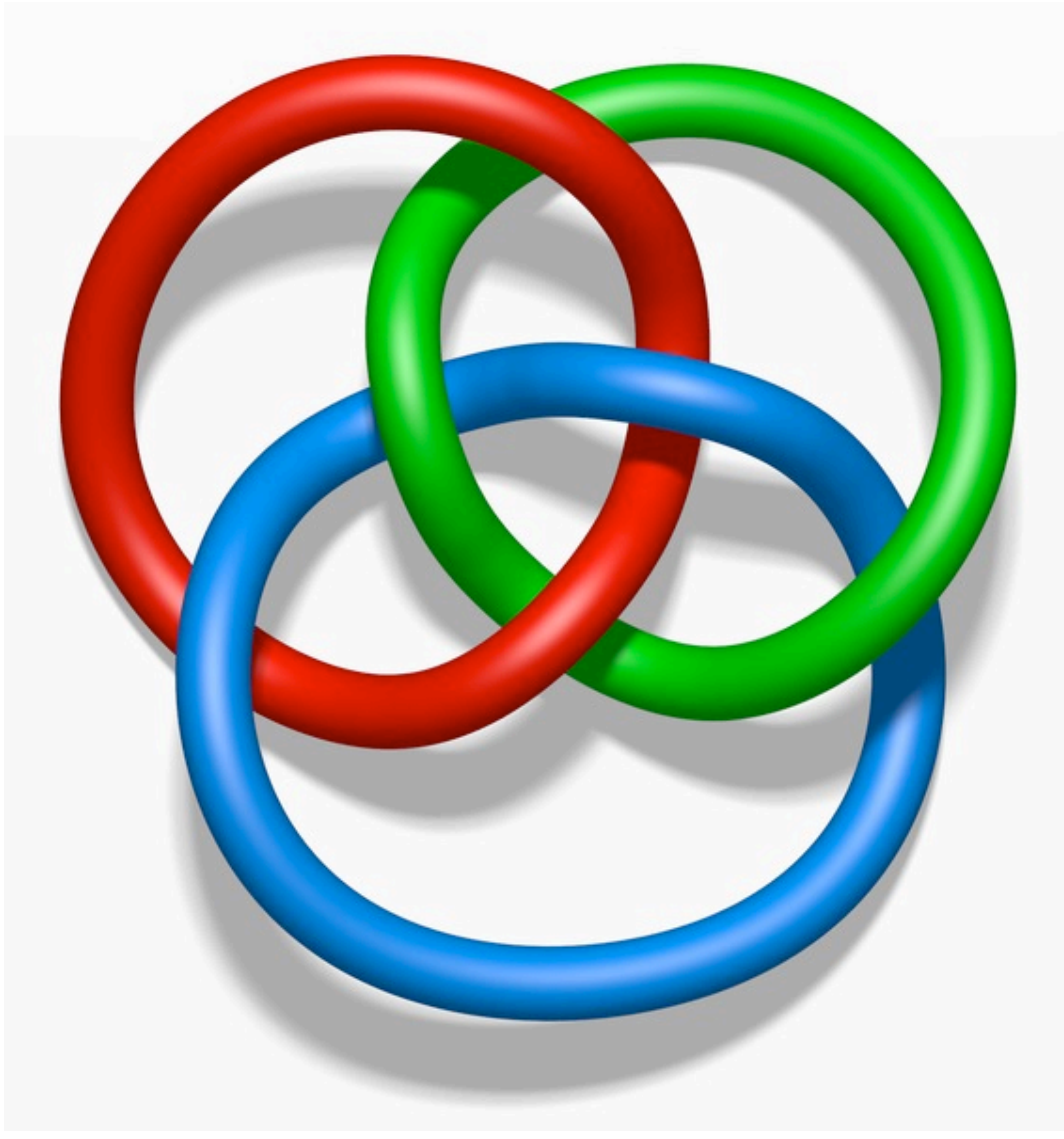


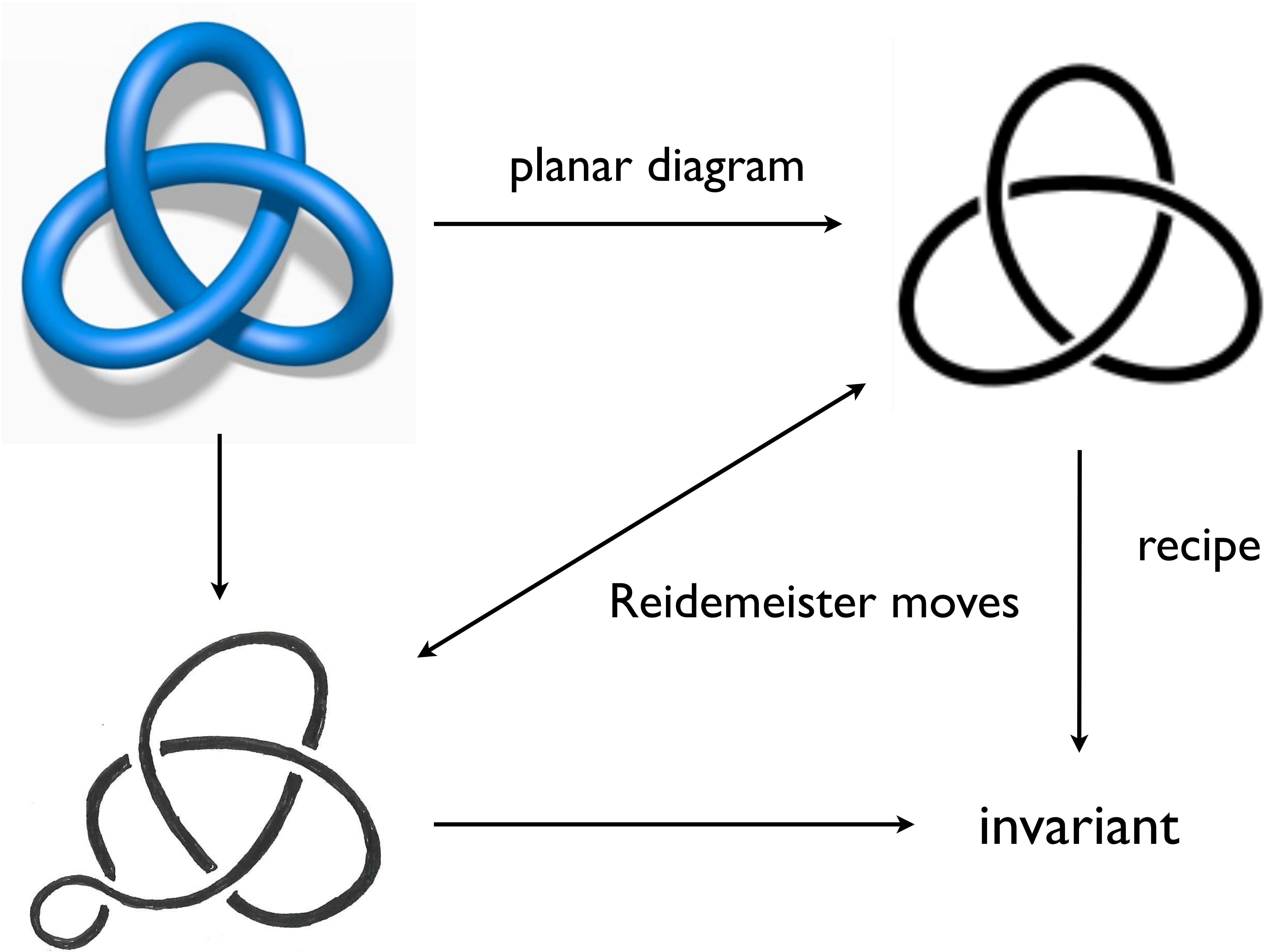




images courtesy of the Knot Atlas

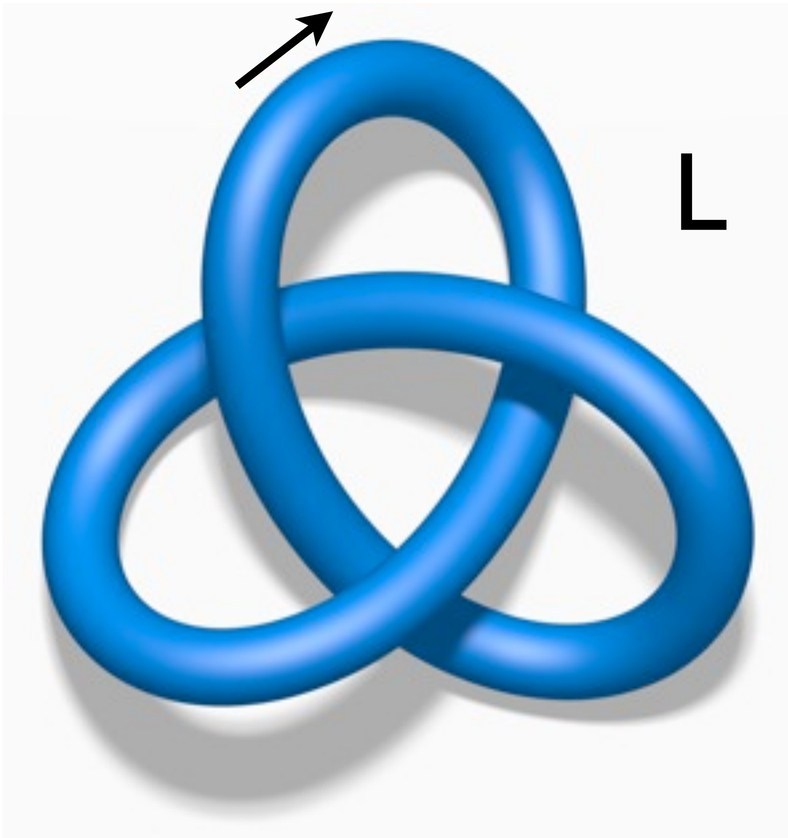
What is a link?




planar diagram
Reidemeister moves
recipe
invariant

Khovanov-Rozansky link homology

Fix an integer $N \geq 2$



Khovanov-Rozansky '04

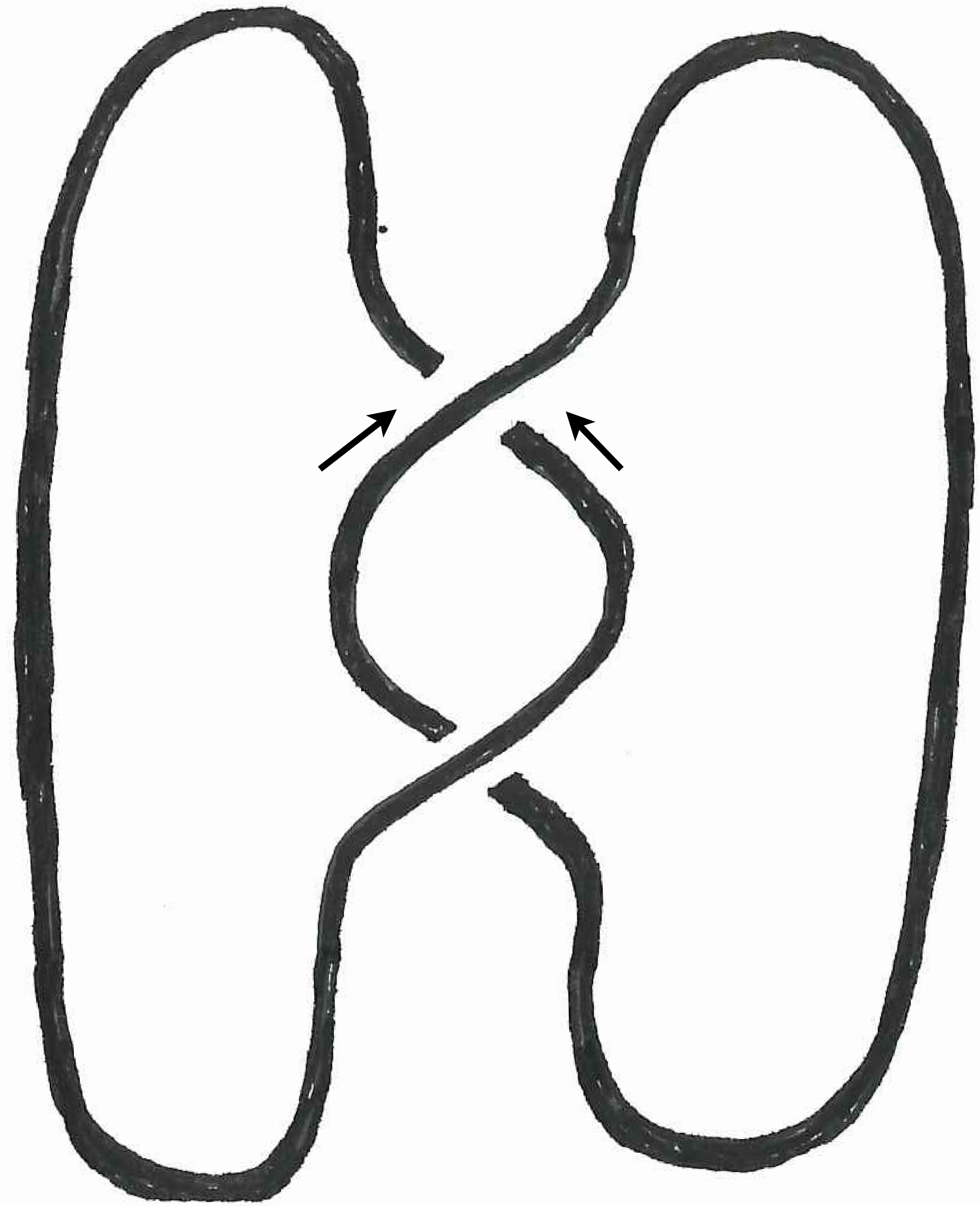
$$\mathrm{KR}_N(L) = \oplus_{i,j\in\mathbb{Z}} H^{i,j}$$

$$\dim_{\mathbb{Q}} H^{i,j} < \infty$$

- Euler characteristic is a specialisation of HOMFLY

$$\chi(\mathrm{KR}_N(L)) := \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}}(H^{i,j})$$

- KR homology is functorial
- Constructed using matrix factorisations



$L =$ Hopf link

$\mathrm{KR}_N(L)$

$\mathcal{T}$ a triangulated category (MFs)

$\mathcal{T}$ $\mathcal{T}$

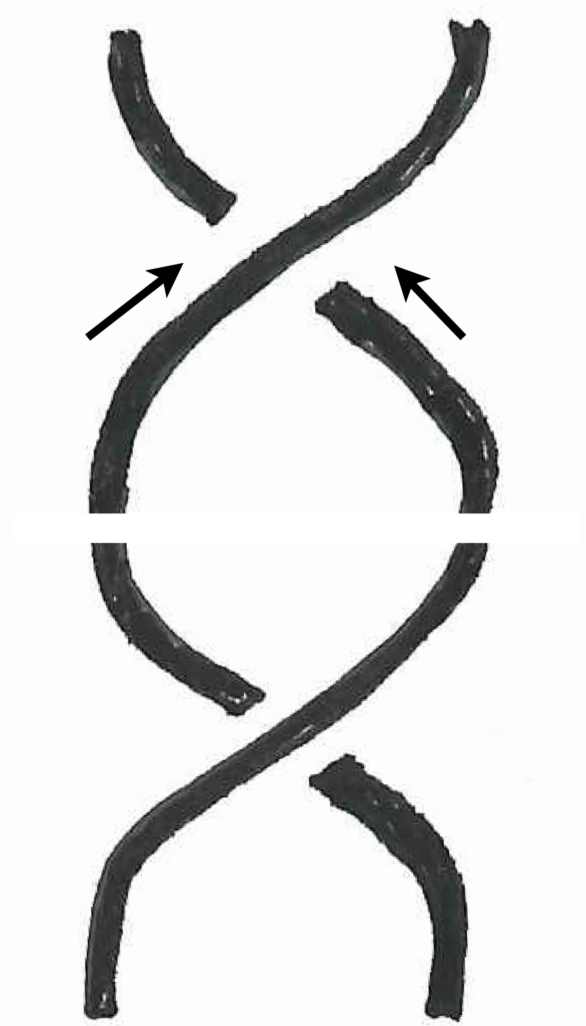
$$\mathcal{T}^{\otimes 2}$$

$$F \uparrow$$

$$\mathcal{T}^{\otimes 2}$$

$$\mathrm{KR}_N(L) = \mathrm{HH}_*(F)$$

$\mathcal{T}$ a triangulated category (MFs)

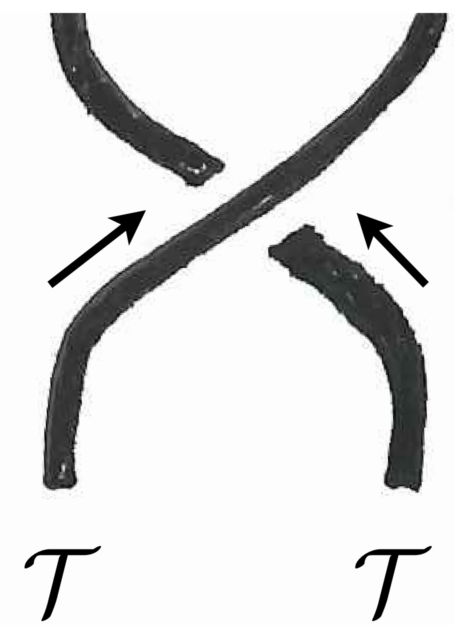


$\mathcal{T}$ $\mathcal{T}$

$$\begin{array}{c} \mathcal{T}^{\otimes 2} \\ c \uparrow \\ \mathcal{T}^{\otimes 2} \\ c \uparrow \\ \mathcal{T}^{\otimes 2} \end{array}$$

$$F := c^2$$

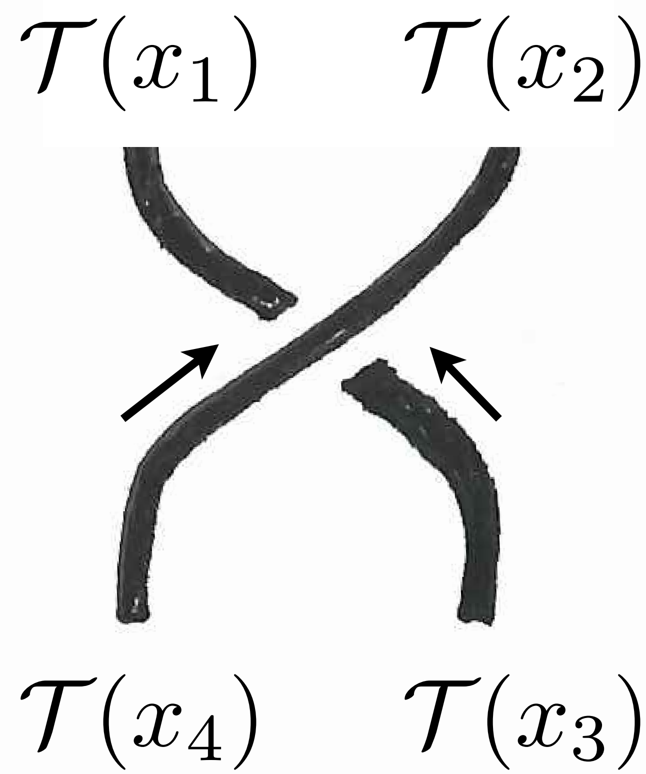
$$\mathrm{KR}_N(L) = \mathrm{HH}_*(F)$$



$$\begin{array}{c} \mathcal{T}^{\otimes 2} \\ \uparrow c \\ \mathcal{T}^{\otimes 2} \end{array}$$

$$\mathcal{T} = \mathcal{T}(x) = \mathbb{K}^b(\mathrm{hmf}(\mathbb{Q}[x], x^{N+1}))$$

$$\mathcal{T}^{\otimes 2} = \mathcal{T}(x,y) = \mathbb{K}^b(\mathrm{hmf}(\mathbb{Q}[x,y], x^{N+1} + y^{N+1}))$$



$$\mathbb{K}^b(\mathrm{hmf}(x_1^{N+1} + x_2^{N+1}))$$

$$\uparrow c$$

$$\mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1}))$$

$$\mathcal{T} = \mathcal{T}(x) = \mathbb{K}^b(\mathrm{hmf}(\mathbb{Q}[x], x^{N+1}))$$

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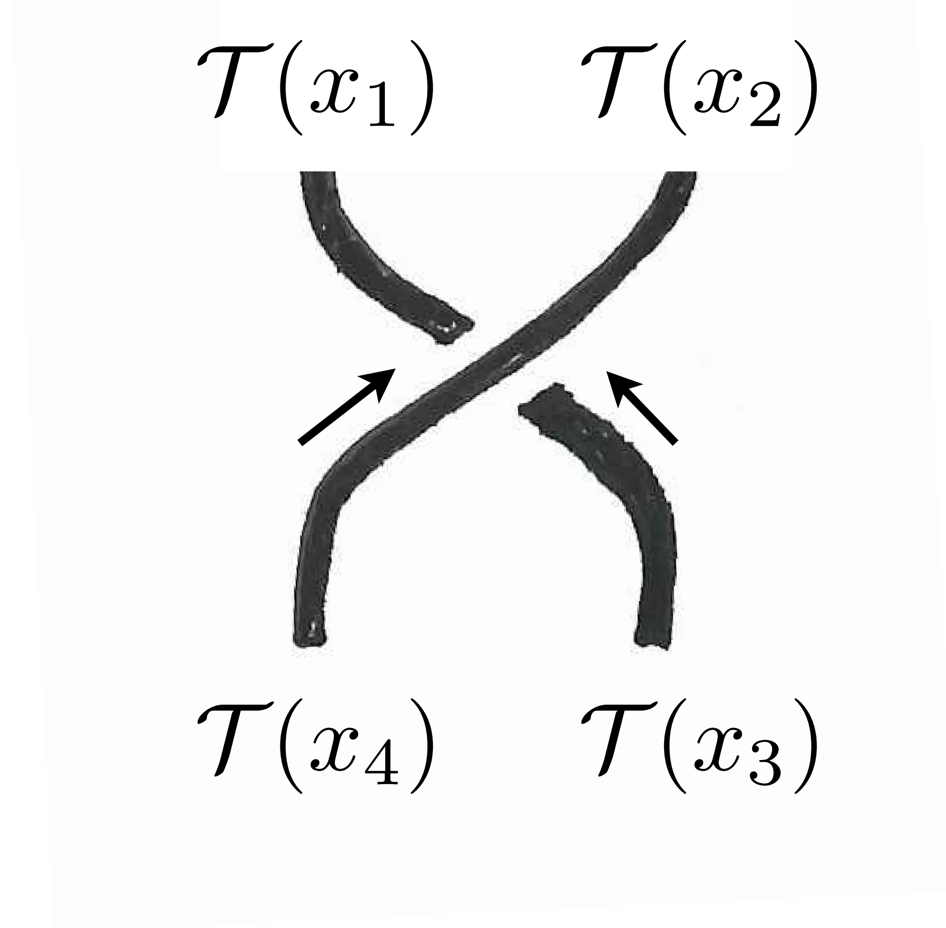


$$\mathbb{K}^b(\mathrm{hmf}(x_1^{N+1} + x_2^{N+1}))$$

$$c \Big\uparrow$$

$$\mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1}))$$

$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$

we will define $\Delta, X \in \mathrm{hmf}(W)$

$$(0 \longrightarrow X \longrightarrow \Delta \longrightarrow 0) \in \mathbb{K}^b(\mathrm{hmf}(W))$$

$c = \Phi_{X \longrightarrow \Delta}$ is an integral transform

$$\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, x_2, x_3, x_4]$$

$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$

$$\mathbb{D}^b_{sg}(\mathbb{Q}[x_1, x_2, x_3, x_4]/W) \xrightarrow{\cong} \mathrm{hmf}(W)$$

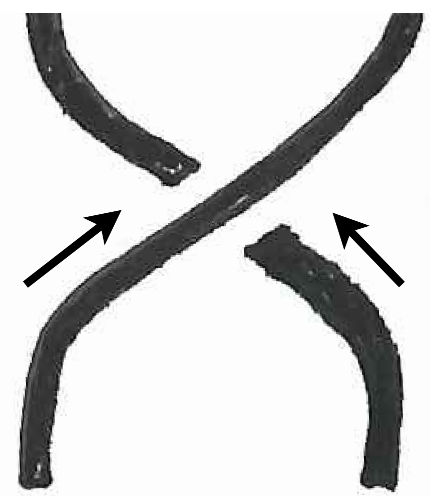
$$\mathbb{Q}[\mathbf{x}]/(x_1 - x_4, x_2 - x_3) \longleftrightarrow \Delta = \Delta_{x_1 = x_4, x_2 = x_3}$$

Diagonal

$$\uparrow \text{canonical}$$

$$\mathbb{Q}[\mathbf{x}]/(x_1 + x_2 - x_3 - x_4, x_1 x_2 - x_3 x_4) \longleftrightarrow X$$

Soergel bimodule



$$\mathbb{K}^b(\mathrm{hmf}(x_1^{N+1}+x_2^{N+1}))$$

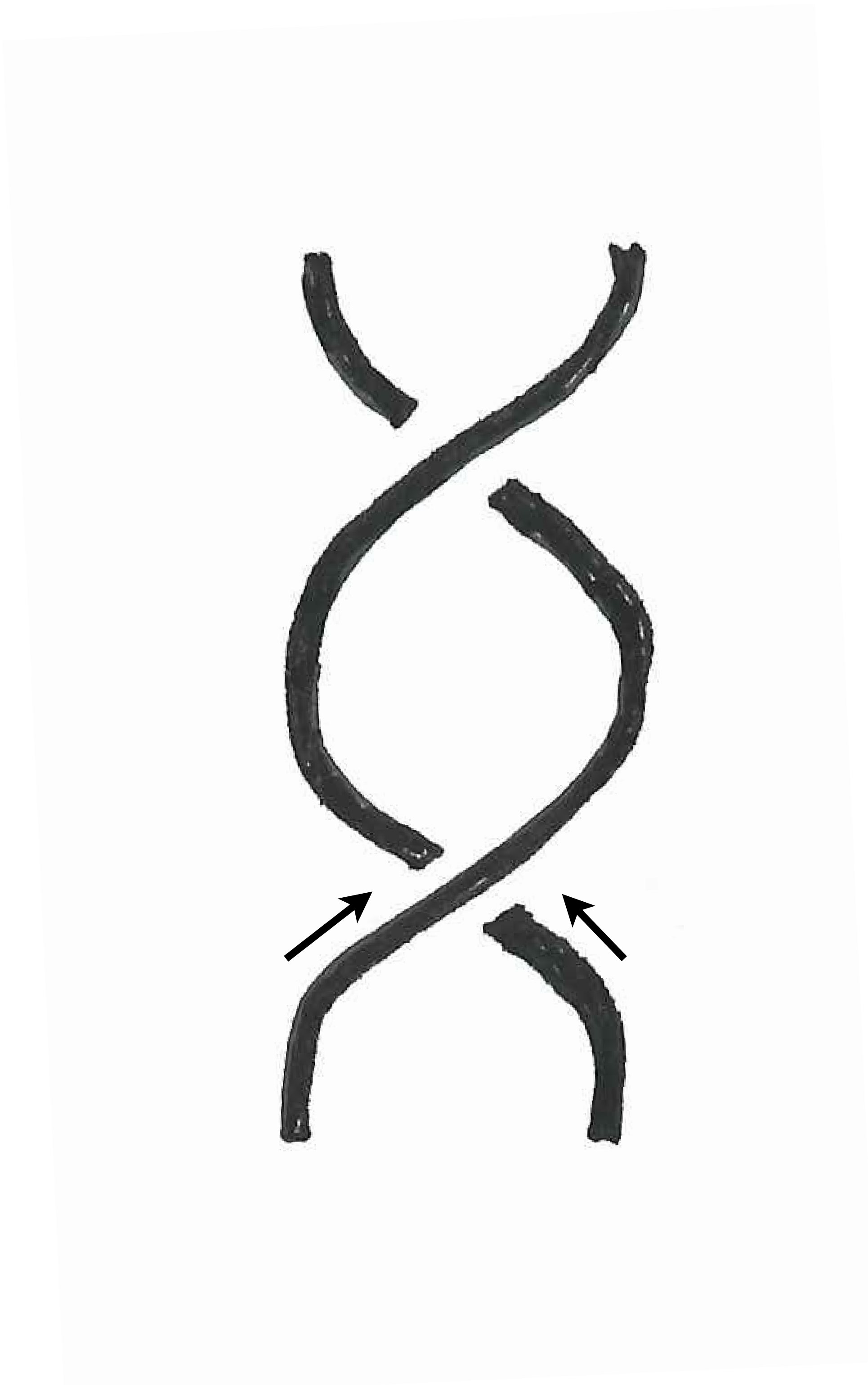
$$\uparrow c$$

$$\mathbb{K}^b(\mathrm{hmf}(x_3^{N+1}+x_4^{N+1}))$$

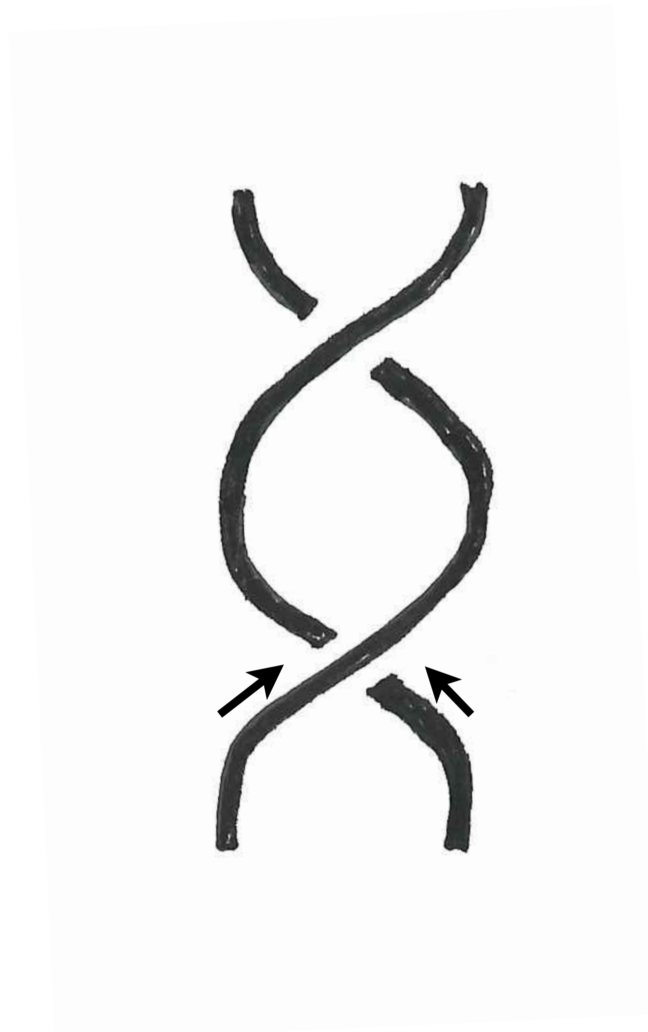
$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$

$$(0 \longrightarrow X \longrightarrow \Delta \longrightarrow 0) \in \mathbb{K}^b(\mathrm{hmf}(W))$$

$c = \Phi_{X \longrightarrow \Delta}$ is an integral transform



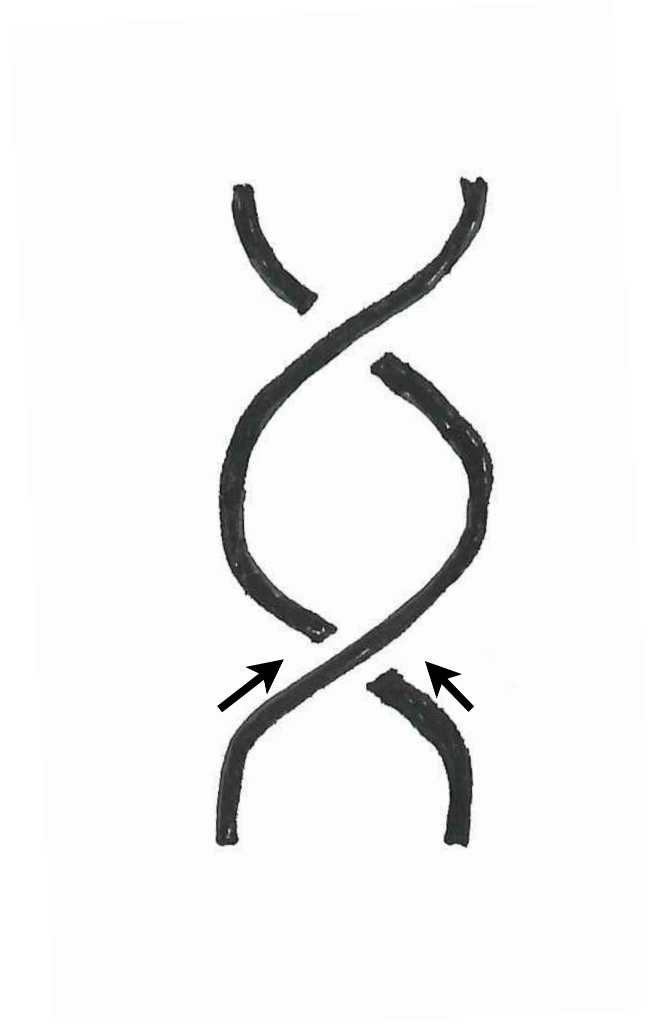
$$
\begin{array}{ccc}
\mathbb{K}^b(\mathrm{hmf}(x_5^{N+1} + x_6^{N+1})) & & \\
\uparrow c & & \\
\mathbb{K}^b(\mathrm{hmf}(x_1^{N+1} + x_2^{N+1})) & & F \\
\uparrow c & & \\
\mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1})) & &
\end{array}
$$



$$\begin{array}{c} \mathbb{K}^b(\mathrm{hmf}(x_5^{N+1} + x_6^{N+1})) \\ \uparrow c \\ \mathbb{K}^b(\mathrm{hmf}(x_1^{N+1} + x_2^{N+1})) \\ \uparrow c \\ \mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1})) \end{array} \qquad F: \mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1})) \to \mathbb{K}^b(\mathrm{hmf}(x_5^{N+1} + x_6^{N+1}))$$

$$F = c^2 = \Phi_{X \to \Delta} \circ \Phi_{X \to \Delta} = \Phi_K$$

$K := (X \to \Delta) \otimes (X \to \Delta)$ is the convolution



$$\mathbb{K}^b(\mathrm{hmf}(x_5^{N+1} + x_6^{N+1}))$$

$$\uparrow \quad F = \Phi_K, \quad K = (X \to \Delta) \otimes (X \to \Delta)$$

$$\mathbb{K}^b(\mathrm{hmf}(x_3^{N+1} + x_4^{N+1}))$$

$K \otimes \Delta_{x_5 = x_4} \otimes \Delta_{x_6 = x_3}$ is a complex over $\mathbb{Q}[x_1, \dots, x_6]$

Define: $\mathrm{KR}_N(L) := H^*(K \otimes \Delta_{x_5 = x_4} \otimes \Delta_{x_6 = x_3})$

Theorem (KR): $\mathrm{KR}_N(L)$ is f.d. over $\mathbb{Q}$ and is an invariant of links.

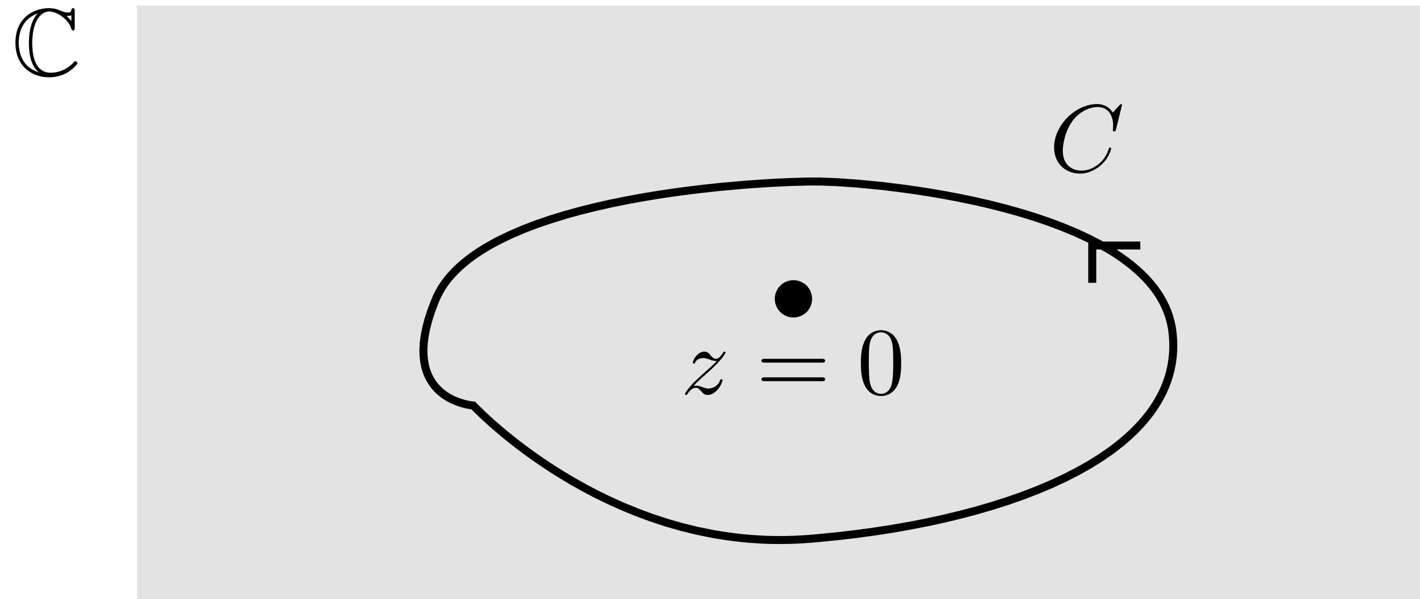
Computing Khovanov-Rozansky link homology

The problem

- $K \otimes \Delta_{x_5=x_4} \otimes \Delta_{x_6=x_3}$ is a cpx over $\mathbb{Q}[x_1, \ldots, x_6]$
- The computer will not enjoy computing H^*
- Monomials x_i^N act null-homotopically, try to exploit this...

A toy example

- (X, D) a $\mathbb{Z}/2$-graded complex of finite free $\mathbb{C}[z]$-modules.
- Suppose $\lambda D + D\lambda = z^n \cdot 1_X$, for some homotopy λ.
- Then on $U = \mathbb{C} \setminus \{0\}$,
$$\frac{\lambda}{z^n} D + D \frac{\lambda}{z^n} = 1_X,$$
so X is contractible on U, $H^*(X)$ supported at $\{0\}$.



Contracting homotopy : meromorphic function

$$H^*(X) \qquad \mathrm{Res}_{z=0}(f)$$

$$\oint_C \frac{\lambda}{z^n} dX \qquad \frac{1}{2\pi i} \oint_C f(z) dz$$

Theorem (M-Dyckerhoff)

$$\chi(H^*X) = \frac{1}{2\pi i}\oint_C \mathrm{str}\left(\frac{\lambda\, dX}{z^n}\right) dz \quad dX := \partial_z(D)dz$$

To make sense of integrating without the supertrace:

$\nabla^0 : \mathbb{C}[z] \longrightarrow \mathbb{C}[z] \otimes_{\mathbb{C}[z^n]} \Omega^1_{\mathbb{C}[z^n]/\mathbb{C}}$ a connection

horizontal sections $\mathbb{C}[z]/z^n$

Lipman: $\displaystyle \frac{1}{2\pi i}\oint_C \frac{g(z)}{z^n} dz = \mathrm{tr}\Bigg(g[z, \nabla^0] \text{ on } \mathbb{C}[z]/z^n \Bigg)$

$$\frac{1}{2\pi i}\oint_C \frac{g(z)}{z^n}dz = \mathrm{tr}\left(\; g[z,\nabla^0] \text{ on } \mathbb{C}[z]/z^n \; \right)$$

$$\nabla^0 : X \longrightarrow X \otimes_{\mathbb{C}[z^n]} \Omega^1_{\mathbb{C}[z^n]/\mathbb{C}}$$

$$\oint_C \frac{\lambda}{z^n}dX := \lambda[D,\nabla^0] \text{ is an idempotent on } X \otimes \mathbb{C}[z]/z^n$$

$[D,\nabla^0]$ is the Atiyah class

Theorem: $\mathrm{Im}(\lambda[D,\nabla^0]) \cong H^* X$

Computations

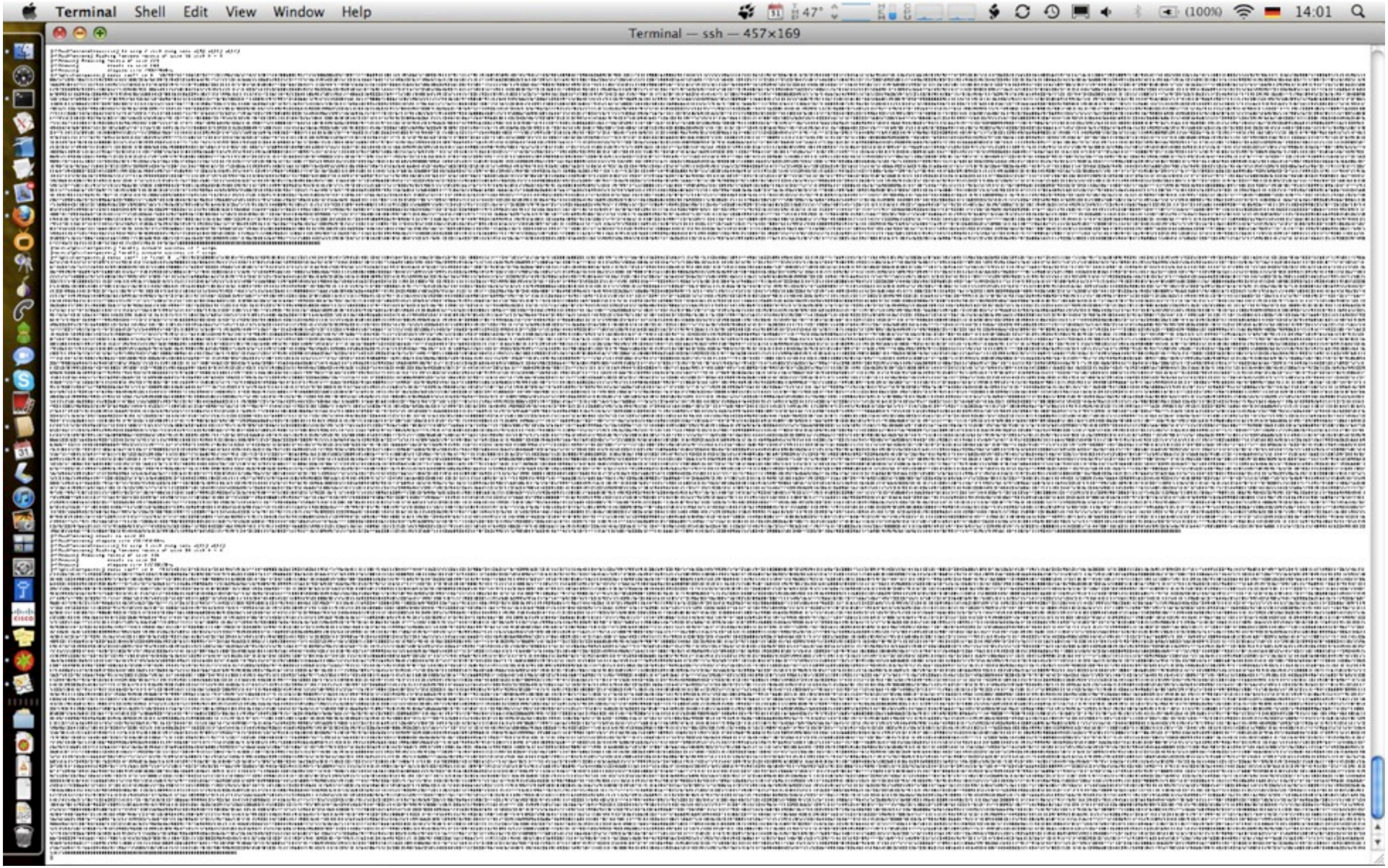
- Using Singular to split idempotents
- Poincare polynomial

$$\mathrm{KR}_N(L, q, t) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}} H^{i,j}$$

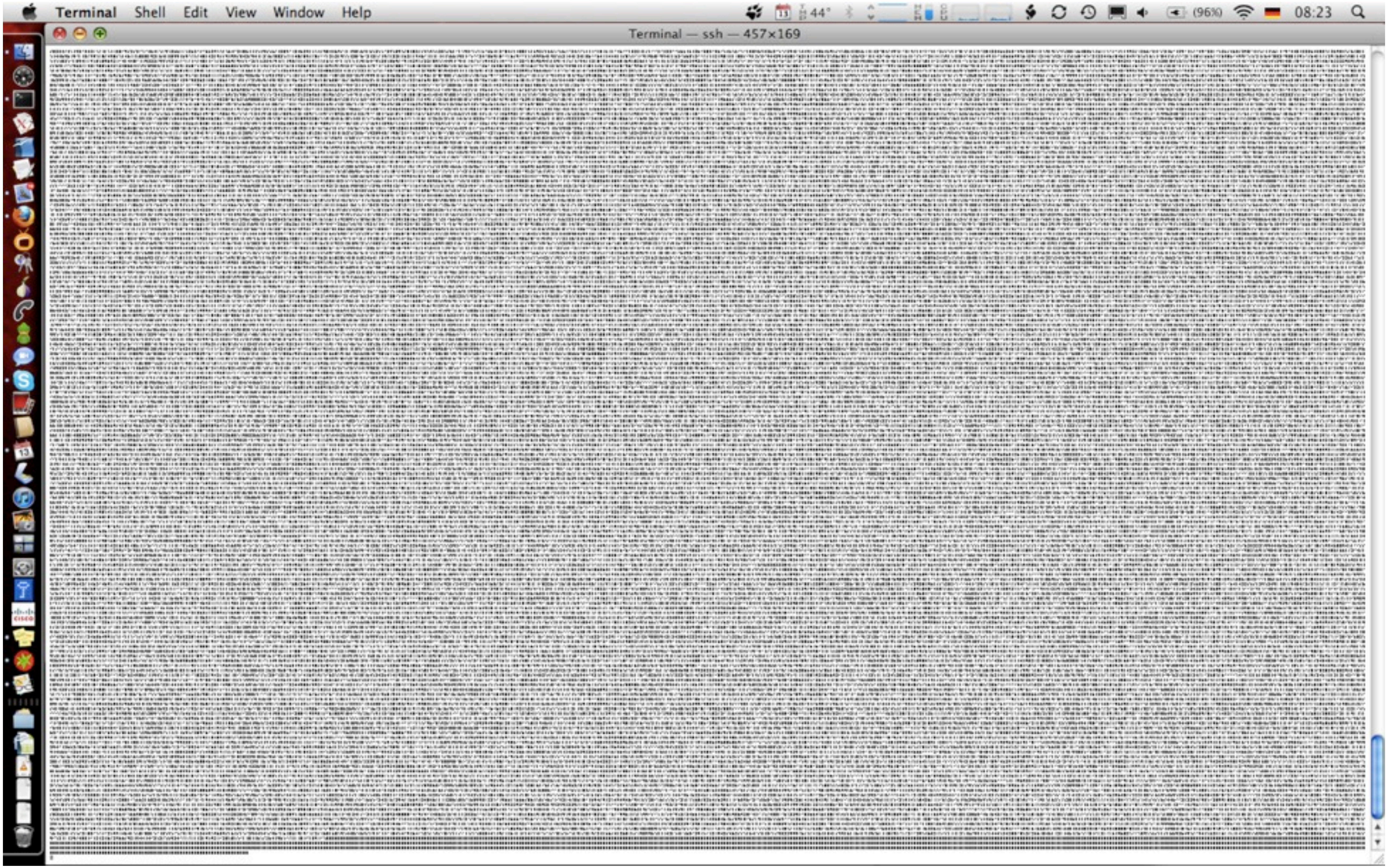
$$q^{-10}t^{-3}(q^{20} + 2q^{18}t + q^{18} + 2q^{16}t + 2q^{14}t^3 + q^{14}t$$
$$+7q^{12}t^3 + 2q^{12}t^2 + q^{12}t + 2q^{10}t^4 + 9q^{10}t^3 + 2q^{10}$$
$$+t^2 + q^8t^5 + 2q^8t^4 + 7q^8t^3 + q^6t^5 + 2q^6t^3$$
$$+2q^4t^5 + q^2t^6 + 2q^2t^5 + t^6)$$

N=3

N = 4 Borromean rings



N = 4 Borromean rings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N = 4 L6n1