# Mirror symmetry between orbifold curves and cusp singularities with group action

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$f$ : w. homog. polynomial $\in \mathbb{C}[x_1, \cdots, x_n]$

is called <u>invertible</u> $\Longleftrightarrow$ ① $f = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$ $\quad a_i \in \mathbb{C}^*, \quad E_{ij} \in \mathbb{Z}_{\geq 0}$

② $E = (E_{ij})$ is invertible over $\mathbb{Q}$

③ $f^T := \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}}$ and $f$ define isol. sing.

$1 \leq \dim_{\mathbb{C}} \mathrm{Jac}(f), \ \dim_{\mathbb{C}} \mathrm{Jac}(f^T) < \infty$

<u>Example</u>. · $x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{m-1} x_m^{p_m}$ $\quad (m \geq 1)$ $\quad$ chain type $\quad p_1, p_m \geq 2$

· $x_m x_1^{p_1} + x_1 x_2^{p_2} + \cdots + x_{m-1} x_m^{p_m}$ $\quad (m \geq 2)$ $\quad$ loop type

Maximal abelian symmetry group of $f$

$$G_f := \left\{ (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n \;\middle|\; \prod_{j=1}^{n} \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^{n} \lambda_j^{E_{nj}} \right\}$$

$$f(\lambda_1 x_1, \cdots, \lambda_n x_n) = \lambda f(x_1, \cdots, x_n) \quad . \quad \lambda = \prod_{j=1}^{n} \lambda_j^{E_{ij}}$$

$$G_f^{fin} := \left\{ (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n \;\middle|\; \cdots \quad \cdots \quad \cdots = 1 \right\}$$

$$1 \longrightarrow G_f^{fin} \longrightarrow G_f \longrightarrow \mathbb{C}^* \longrightarrow 1$$

$$E \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad q_i \in \mathbb{Q} \quad (\text{assumption} : q_i > 0)$$

$$g_0 := (\mathbf{e}[q_1], \cdots, \mathbf{e}[q_n]) \qquad \mathbf{e}[-] := \exp(2\pi\sqrt{-1}\,\cdot)$$

$$G_0 := \langle g_0 \rangle \subset G_f^{fin}$$

We want to consider $G$ : $\quad G_0 \subset G \subset G_f^{fin}$

$(f, G)$ : Landau - Ginzburg orbifold

$f^T$ : Berglund - Hübsch transpose of $f$

$G^T := \mathrm{Hom}(G_f^{fin}/G, \mathbb{C}^*)$ $\quad$ introduced by Berglund - Henningson

$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow \mathbb{C}^* \longrightarrow 1$ $\qquad (f, G) \longleftrightarrow (f^T, G^T)$

$$\mathcal{X}_{(f,G)} := \left[ f^{-1}(0) \setminus \{0\} \big/ \widetilde{G} \right]$$

$$\begin{array}{ccc} \underline{MF}^{\widetilde{G}}(f) & \xleftrightarrow{\text{Homological Mirror Symmetry}} & \text{"} Fuk(f^T, G^T) \text{"} \\ \Big\updownarrow \begin{array}{l}\text{semi-orth.}\\ \text{decomposition}\end{array} & & \\ D^b(Coh\, \mathbb{X}_{(f,G)}) & & \boxed{\ ?\ } \end{array}$$

From now on, Let $n=3$.

Assume $G = G_f^{fin.}$

<u>Thm</u> (Ebeling-T '10) $\mathbb{X}_{(f, G_f^{fin})} \simeq \mathbb{X}(\alpha_1, \alpha_2, \alpha_3)$ weighted projective line with (at most) 3 isotropic points of order $\alpha_1, \alpha_2, \alpha_3$

<u>Cor</u> $\underline{MF}^{\widetilde{G_f}}(f)$ has a full exceptional col.

<u>Thm</u> (Hirano-T) $\underline{MF}^{\widetilde{G_f}}(f)$ has a full strongly exceptional col.

Moreover, $\exists A$ : f.d. alg. of gl.dim $\leq 3$ s.t. $\underline{MF}^{\widehat{G_f}}(f) \simeq D^b(A)$

$T_{\gamma_1,\gamma_2,\gamma_3}$ : $x_1^{\gamma_1} + x_2^{\gamma_2} + x_3^{\gamma_3} - a x_1 x_2 x_3 \quad a \in \mathbb{C}^*$

<u>Thm</u> (Ebeling-T) ① Gorenstein parameter of $f^T$ is negative

$\left( := \sum_i \deg x_i - \deg f^T \right)$

$\Rightarrow f^T - x_1x_2x_3 = T_{\gamma_1,\gamma_2,\gamma_3}$ for $(\gamma_1, \gamma_2, \gamma_3)$ after a suitable holomorphic change of coordinates

② G. parameter $=0 \Rightarrow f^T - a x_1x_2x_3 = T_{\gamma_1,\gamma_2,\gamma_3}$ for some $(\gamma_1, \gamma_2, \gamma_3)$

after a holomorphic change of coordinates

③ G. parameter $>0 \Rightarrow f^T - x_1x_2x_3 = T_{\gamma_1,\gamma_2,\gamma_3}$ after a polynomial change of coordinates and a deformation

<u>Example</u>. $f^T = x_1^2 + x_3x_2^2 + x_2x_3^2$ $D_4$ singularity

$x_1^2 + x_3x_2^2 + x_2x_3^2 - x_1x_2x_3 \qquad (x_1, x_2, x_3) \mapsto (x_1+x_2+x_3, x_2, x_3)$

$\rightsquigarrow x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1 \qquad (x_1, x_2, x_3) \mapsto (x_1+2, x_2+2, x_3+2)$

$\longrightarrow x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 + 4x_1 + 4x_2 + 4x_3 + 12$