A note on the diamond fractal

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Abstract

We prove the uniqueness of a "Laplacian" on the compact diamond fractal. More precisely, there exists a unique (up to positive multiples) self-similar, irreducible, local and regular Dirichlet form with the normalized Hausdorff measure as reference measure. The main tool is the so called "eigenvalue test" which allows to use numerical results to set up theorems. The approach also applies to other finitely ramified, graph-directed fractals.

1 Introduction and results

The so called diamond fractal $X \subset \mathbb{R}^2$ can be constructed by a self-similar cut out procedure similar to the Cantor set. It is indicated in Figure 1.

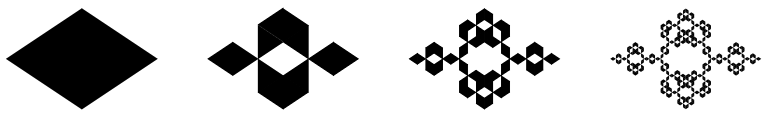

Figure 1: The first four construction steps of the diamond fractal.

For physical reasons one would like to study heat conduction on X. Therefore, Kigami, Strichartz and Walker proved the existence of a self-similar,

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local and regular Dirichlet form on X in [Kigami et al. 2001]. It is the bilinear form of a "Laplacian" on $L^2(X, \mu)$ with the normalized Hausdorff measure μ on X. Its uniqueness remained open. For Dirichlet forms self-similarity refers to a space-energy scaling similar to the space-volume scaling of the fractal. Theorem 1 will complement the results in [Kigami et al. 2001] proving uniqueness and simplifying the existence proof by means of the "eigenvalue test" suggested in [Metz 2001, Section 6]. It can also be applied to other finitely ramified, graph-directed fractals as well.

Theorem 1 *The compact diamond fractal X has a unique, irreducible, local, regular and self-similar Dirichlet form in $L^2(X, \mu)$.*

Mathematically the diamond fractal is interesting because it seems to be partly finitely and infinitely ramified, that is, some diamonds overlap in finitely many points others intersect in nontrivial line segments. Infinitely ramified fractals are much harder to analyze than finitely ramified ones [Barlow 1998]. At a second glance the diamond fractal turns out to be finitely ramified because the "infinitely ramified components" only come in two shapes, the "chevron" and the "crown" as indicated in Figure 2, no matter how often one iterates the construction rule. Hambly and Nyberg used this observation to interpret the diamond fractal as a finitely ramified graph-directed construction in the sense of Mauldin and Williams [Mauldin/Williams 1988, Hambly/Nyberg 1999]. Its initial shapes, the "diamond," the "chevron" and the "crown" are shown in Figure 2. The second line consists of their respective refinements.

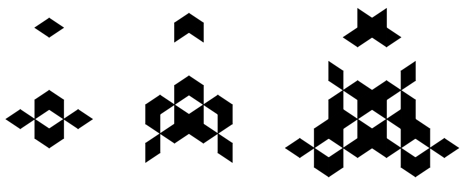

Figure 2: Diamond, chevron and crown in the first line and their unscaled refinements in the second line.

In Section 2 we define the fractal and its renormalization map on Dirichlet forms. The "eigenvalue test" of Proposition 5 will be proved in Section 3 and simultaneously applied to the diamond fractal to verify Theorem 1.

2 Preliminaries on self-similar energies

Hambly and Nyberg applied Lindstrøm's idea to graph-directed fractals showing that the existence and uniqueness in Theorem 1 is equivalent to the existence and uniqueness of the solution to the nonlinear eigenvalue problem (2) of this section [Hambly/Nyberg 1999].

Let us define the diamond fractal in a geometric graph-directed way. Think of the diamond, the chevron and the crown as subsets of $\mathbb{R}^2$ contained in the interior of disjoint compact circles C_d, C_{ch}, C_{cr}. Four similitudes $\psi_1, \ldots, \psi_4$ define the refined diamond. Two similitudes $\psi_1, \psi_2 : C_d \to C_d$ place scaled diamonds at the positions indicated in the bottom line of Figure 2 and two similitudes $\psi_3, \psi_4 : C_{ch} \to C_d$ define the scaled chevrons. The refinement of the chevron requires six maps $\psi_5, \ldots \psi_{10}$. Two maps from C_d to C_{ch} for the diamonds, three maps from C_{ch} to C_{ch} for the chevrons and one from C_{cr} to C_{ch} for the crown. Finally, $\psi_{12}, \ldots, \psi_{22}$ acting on suitable circles and mapping into C_{cr} define the refined versions of the crown. The graph with vertices instead of circles and directed edges representing the similitudes gave rise to the above terminology. Let M be the union of the diamond, the chevron and the crown and denote the domain of ψ_i by C_i for $1 \leq i \leq 22$. For $\Psi(M) := \bigcup_{i=1}^{22} \psi_i(M \cap C_i)$ the limiting object is $F := \bigcap_{n=1}^{\infty} \Psi^n(M)$ and the diamond fractal reappears as $X = F \cap C_d$.

Each initial shape corresponds to an electrical resistor network with vertices placed at the (finitely many) possible intersection points with other shapes as indicated in Figure 3. Collect the vertices $v_1, \ldots, v_8$ in a set V_0. Define a conductance, the inverse of the resistance, to be a function $c_0 : V_0^2 \to \mathbb{R}_+$ which is symmetric and zero on all edges $\{x, y\} \subset V_0^2$ not indicated in the second line of Figure 3. Furthermore, it has to have non-negative, real values $c_1, \ldots c_6$ on the remaining edges numbered as in the second line of Figure 3. The conductances c_3 and c_5 are arranged symmetrically to respect the obvious symmetries of the crown. Let us denote the graph in the second line of Figure 3 by Γ_0.

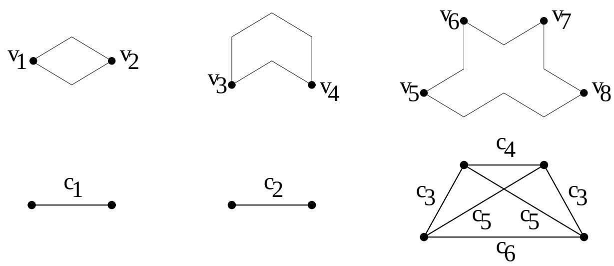

Figure 3: The initial shapes and their vertices in the first line. Their corresponding networks are shown in the second line.

The following constructions on Dirichlet forms can be found in [Hambly/Nyberg 1999, Sect. 4]. On V_0 we define a discrete Dirichlet form, in the sense of [Fukushima et al. 1994], on functions $f, g : V_0 \to \mathbb{R}$ by

$$\mathcal{E}_0(f, g) := \frac{1}{2} \sum_{x,y \in V_0} (f(y) - f(x))(g(y) - g(x))c_0(x, y). \tag{1}$$

By definition c_0 assumes at most six different values $c_1(\mathcal{E}_0), \ldots, c_6(\mathcal{E}_0) \in \mathbb{R}_+$ on the edges of Γ_0. The quadratic form $\mathcal{E}_0(f) := \mathcal{E}_0(f, f)$ can equivalently be used to define $\mathcal{E}$. The quadratic form is positive semidefinite and physically the total energy dissipation of the voltage f on the electrical resistor network (V_0, c_0), [Doyle/Snell 1984]. A sequence of refined vertex sets is given by $V_n := \Psi^n(V_0)$, $n \in \mathbb{N}$. Similar to the map Ψ on sets we define the coupling map Ψ on Dirichlet forms. For $f : V_1 \to \mathbb{R}$ set

$$\mathcal{E}_1(f) := \Psi(\mathcal{E}_0)(f) := \sum_{i=1}^{22} \mathcal{E}_0(f \circ \psi_i),$$

where $\psi_i(M) := \psi_i(M \cap C_i)$ for all i. By polarization this defines a Dirichlet form $\mathcal{E}_0$ on V_1 with a conductance $c_{\mathcal{E}_1}$ on V_1. Next we define the trace $\Phi(\mathcal{E}_1)$ of the network $(V_1, c_{\mathcal{E}_1})$ on V_0, [Fukushima et al. 1994, Section 6.2]. For $f : V_0 \to \mathbb{R}$ set

$$\Phi(\mathcal{E}_1)(f) := \inf\{\mathcal{E}_1(g) | g : V_1 \to \mathbb{R}, g|_{V_0} = f\}.$$

According to the Dirichlet principle, [Metz 2001], this solves the $\mathcal{E}_1$-Dirichlet problem on the "open" set $V_1 \setminus V_0$ with "boundary" data f. Finally, the renormalization map is defined to be $\Lambda := \Phi \circ \Psi$.

Now we are able to state the eigenvalue problem which reformulates the existence and uniqueness problem of Theorem 1. Let $\mathbb{D}$ denote the set of all Dirichlet forms on V_0 defined by conductances as in (1). Since every Dirichlet form on V_0 is uniquely defined by a set of conductances $c_1, \ldots, c_6 \in \mathbb{R}_+$ we can think of $\mathbb{D}$ as $\mathbb{R}_+^6$. We endow the real vector space $\mathbb{D} - \mathbb{D}$ with a norm $\|\cdot\|$. Each $\mathcal{E} \in \mathbb{D}$ defines a graph $\Gamma(\mathcal{E})$ with vertices V_0 and an edge set characterized by positive conductances. The interior $\mathbb{D}^\circ$ of $\mathbb{D}$ consists of those $\mathcal{E} \in \mathbb{D}$ with $\Gamma(\mathcal{E}) = \Gamma_0$. Denote the positive semidefinite forms in $\mathbb{D} - \mathbb{D}$ by $\mathbb{P}$. Equation (1) shows that $\mathbb{P}^\circ$ consists of (quadratic) forms vanishing only on functions which are constant on the connected components of Γ_0. Consequently the boundary $\partial\mathbb{P}$ of $\mathbb{P}$ is the set of (quadratic) forms annihilating a strictly bigger set of functions. Hence $\mathbb{D} \cap \mathbb{P}^\circ$ is the set of Dirichlet forms whose graphs have the connected components of Γ_0 but their conductances need not all be positive. Let us slightly abuse terminology and call such Dirichlet forms irreducible, whereas, those in $\partial\mathbb{P}$ will be termed reducible. We want to solve the following finite dimensional eigenvalue problem

(2) $$\Lambda(\mathcal{E}) = \gamma\mathcal{E} \text{ for some } \gamma > 0 \text{ and } \mathcal{E} \in \mathbb{D} \cap \mathbb{P}^\circ.$$

Its solution is named a (discrete) self-similar energy.

From [Metz 1995, Thm. 2.2] we know that: Λ maps $\mathbb{P}$ and $\mathbb{D}$ into itself, is continuous on $\mathbb{D}$, is positively homogeneous, and superadditive in the sense that $\Lambda(\mathcal{A}+\mathcal{B})(f) \geq \Lambda(\mathcal{A})(f)+\Lambda(\mathcal{B})(f)$ for all $f : V_0 \to \mathbb{R}$ and all $\mathcal{A}, \mathcal{B} \in \mathbb{P}$. This will force Λ to be non-expansive with respect to Hilbert's projective metric h on $\mathbb{P}$.

Hilbert's metric is studied in (see [Nussbaum 1988, Metz 1995]). Take $\mathcal{A} \in \mathbb{P}$ and denote the set of functions with vanishing $\mathcal{A}$-energy by $\ker \mathcal{A}$. For $\mathcal{B} \in \mathbb{P}$ with $\ker \mathcal{B} = \ker \mathcal{A}$ consider

$$\begin{aligned}
\inf(\mathcal{A}/\mathcal{B}) &:= \sup\{\alpha > 0 | \alpha\mathcal{B}(f) \leq \mathcal{A}(f)\, \forall\, f \notin \ker \mathcal{A}\}, \\
\sup(\mathcal{A}/\mathcal{B}) &:= \inf\{\alpha > 0 | \mathcal{A}(f) \leq \alpha\mathcal{B}(f)\, \forall\, f \notin \ker \mathcal{A}\}, \\
h(\mathcal{A}, \mathcal{B}) &:= \ln \frac{\sup(\mathcal{A}/\mathcal{B})}{\inf(\mathcal{A}/\mathcal{B})}.
\end{aligned}$$

The function h measures the "angle" between rays because $h(\alpha\mathcal{A}, \beta\mathcal{B}) = h(\mathcal{A}, \mathcal{B})$ for all $\alpha, \beta > 0$. Projected to the unit sphere S centered at 0 in $(\mathbb{D} - \mathbb{D}, \|\cdot\|)$ we get a complete metric space. On every set of forms with equal kernels (except $\{0\}$) and restricted to S, h and $\|\cdot\|$ define locally the same topology. Finally, h-balls $B_r(\mathcal{C})$ centered at $\mathcal{C} \in \mathbb{P} \setminus \{0\}$ with radius $r > 0$ are convex (but not necessarily strictly convex).

By definition $\mathcal{A}(f) - \inf(\mathcal{A}/\mathcal{B})\mathcal{B}(f) \geq 0$ for every $f : V_0 \to \mathbb{R}$. Thus the superlinearity of Λ implies

$$\inf(\mathcal{A}/\mathcal{B}) \cdot \Lambda(\mathcal{B})(f) = \Lambda(\inf(\mathcal{A}/\mathcal{B}) \cdot \mathcal{B})(f) \leq \Lambda(\mathcal{A})(f). \tag{3}$$

Consequently $\inf(\Lambda(\mathcal{A})/\Lambda(\mathcal{B})) \geq \inf(\mathcal{A}/\mathcal{B})$. The reversed inequality holds for $\sup(\Lambda(\mathcal{A})/\Lambda(\mathcal{B}))$ and $\sup(\mathcal{A}/\mathcal{B})$. Especially, Λ is h-non-expansive.

3 The eigenvalue test

We will analyze the existence and uniqueness problem graphically in terms of the action of Λ on $\mathbb{D}$-parts. This allows to reduce the problem to a view inequalities in Proposition 5. They will be verified with the help of numerical results in [Kigami et al. 2001]. We will prove the eigenvalue test as suggested in [Metz 2001, Section 6]. For the readers convenience we apply the arguments simultaneously to the diamond fractal.

Since Λ is positively homogeneous we can normalize it to a function $\tilde{\Lambda}$ staying on the set N of all irreducible Dirichlet forms whose conductances sum up to 1. The set N is $\|\cdot\|$-bounded and convex.

Proposition 2 ([Nussbaum 1988, Thm. 4.1]) *Let $\mathcal{A} \in \mathbb{D}^\circ \cap N$ and assume that its normalized renormalization orbit $(\tilde{\Lambda}^n(\mathcal{A}))_n$ is h-bounded. Then (2) has a solution.*

Proof: Let ω be the set of accumulation points of the orbit. Since the orbit is bounded there exists a closed h-ball $B_r(\mathcal{C})$ with center $\mathcal{C} \in \mathbb{P}^\circ$, containing ω and, consequently, $\omega \subset \bigcap_{\mathcal{B}\in\omega} B_{2r}(\mathcal{B}) =: D$. Since h-balls are convex, D is compact and convex. Furthermore, $\Lambda(D) \subset D$ because Λ does not expand h-distances. Brouwer's fixed point theorem thus provides a $\tilde{\Lambda}$-fixed point. This solves (2). $\square$

To apply the above theorem we intend to prove that the orbit moves away from $\mathbb{D} \cap \partial\mathbb{P}$. To this end we have to get an idea where the orbit could possibly touch $\mathbb{D} \cap \partial\mathbb{P}$.

Two elements of $\mathbb{P} \setminus \{0\}$ have a finite h-distance if and only if their kernels coincide. The equivalence classes of finite h-distance are called $\mathbb{P}$-parts. Thus Λ acts on $\mathbb{P}$-parts due to its h-non-expansiveness. According to the minimum principle, [Metz 2001, Prop. 2.3(a)], two Dirichlet forms are in the same $\mathbb{P}$-part if and only if they have the same set of connected components.

Two Dirichlet forms define the same graph when their conductances have the same zeros. The cone $\mathbb{D} \setminus \{0\}$ splits into corresponding equivalence classes called $\mathbb{D}$-parts. Since there are at most 6 positive conductances, we have $2^6 - 1$ different $\mathbb{D}$-parts. The above characterization by connected components shows that the intersection of every $\mathbb{P}$-part with $\mathbb{D}$ splits into finitely many $\mathbb{D}$-parts. For $\mathcal{B} \in \mathbb{D}$ the graph $\Gamma(\mathcal{B}_1)$ is defined to have the vertex set V_1 and an edge set characterized by positive conductances. We say that a set $P \subset \mathbb{P}$ is Λ-invariant when $\Lambda(P) \subset P$, otherwise it is called Λ-variant. The strong minimum principle, [Metz 2001, Prop. 2.3(b)], along a connecting path can be used to prove

Lemma 3 ([Metz 2001, Rem. 7(b)]) *For $\mathcal{B} \in \mathbb{D}$ the conductance of $\Lambda(\mathcal{B})$ between two different points $x, y \in V_0$ is strictly positive if and only if there exists a path in $\Gamma(\mathcal{B}_1)$ connecting x to y while hitting V_0 at most in x and y.*

Because of Lemma 3, Λ acts on $\mathbb{D}$-parts. We index the $\mathbb{D}$-parts by a representative conductance $(c_1, \ldots, c_6) \in \mathbb{R}^6_+$, where "$*$" indicates an element of $\mathbb{R}_+$. The action of Λ on these parts can be decided graphically with the help of Lemma 3.

$$\begin{aligned} \Lambda: \quad & \mathbb{D}_{(0,*,*,*,*,*)} \mapsto \{0\} \mapsto \{0\}, \\ & \mathbb{D}_{(*,0,*,*,*,*)} \mapsto \mathbb{D}_{(0,*,*,*,*,*)}, \\ & \mathbb{D}_{(1,1,0,0,0,0)} \mapsto \mathbb{D}_{(1,1,0,0,0,0)}. \end{aligned}$$

This shows that $\mathbb{D}_{(1,1,0,0,0,0)}$ is Λ-invariant and $\mathbb{D}_{(0,0,*,*,*,*)}$ is Λ^k-variant for all $k \in \mathbb{N} \setminus \{0\}$. So we only have to deal with $\mathbb{D}_{(1,1,*,*,*,*)}$.

$$\Lambda : \quad \{\mathbb{D}_{(1,1,1,1,1,1)}, \mathbb{D}_{(1,1,0,0,1,0)}, \mathbb{D}_{(1,1,0,0,0,1)}\} \mapsto \{\mathbb{D}_{(1,1,1,1,1,1)}\}.$$

This time only $\mathbb{D}^\circ = \mathbb{D}_{(1,1,1,1,1,1)}$ is Λ-invariant. The remaining cases are:

$$\begin{aligned} \Lambda : \quad & \mathbb{D}_{(1,1,1,1,0,0)} \mapsto \mathbb{D}_{(1,1,1,1,1,1)}, \\ & \mathbb{D}_{(1,1,1,0,0,0)} \mapsto \mathbb{D}_{(1,1,1,0,0,0)}, \\ & \mathbb{D}_{(1,1,0,1,0,0)} \mapsto \mathbb{D}_{(1,1,0,0,0,1)}. \end{aligned}$$

Here only $\mathbb{D}_{(1,1,1,0,0,0)}$ is Λ-invariant. In total there are two Λ-invariant $\mathbb{D}$-parts in $\mathbb{D} \cap \partial\mathbb{P}$, namely, $\mathbb{D}_2 := \mathbb{D}_{(1,1,0,0,0,0)}$ and $\mathbb{D}_3 := \mathbb{D}_{(1,1,1,0,0,0)}$.

The Λ-invariance of $\mathbb{D}^\circ$ shows that $h(\Lambda(\mathcal{A}), \mathcal{A}) < \infty$ for every $\mathcal{A} \in \mathbb{D}^\circ$.

Lemma 4 *Suppose there exists an* $\mathcal{A} \in \mathbb{D}^\circ$ *with* $h(\Lambda(\mathcal{A}), \mathcal{A}) < \infty$ *but* $(\tilde{\Lambda}^n(\mathcal{A}))_n$ *is h-unbounded. Then the following statements hold:*

(i) There exists a $k \in \mathbb{N}$ *and a* Λ^k*-invariant* $\mathbb{D}$*-part* $\mathbb{D}_i \subset \partial\mathbb{P}$ *containing an accumulation point* $\mathcal{B}$ *of* $(\tilde{\Lambda}^n(\mathcal{A}))_n$.

(ii) For every $\mathcal{B}$ *in (a) and every* $n \in \mathbb{N}$,

$$\begin{aligned} &\inf(\Lambda^{(n+1)k}(\mathcal{A})/\Lambda^{nk}(\mathcal{A})) \le \inf(\Lambda^k(\mathcal{B})/\mathcal{B}), \\ &\sup(\Lambda^k(\mathcal{B})/\mathcal{B}) \le \sup(\Lambda^{(n+1)k}(\mathcal{A})/\Lambda^{nk}(\mathcal{A})). \end{aligned}$$

Proof: (ii): Use the continuity of Λ on $\mathbb{D}$ and (3) for inf and sup.
(i): Let $\mathcal{B}$ be as above. Then the continuity of Λ and (ii) imply

$$h(\Lambda(\mathcal{B}), \mathcal{B}) \le \inf_{n \in \mathbb{N}} h(\Lambda^{n+1}(\mathcal{A}), \Lambda^n(\mathcal{A})) \le h(\Lambda(\mathcal{A}), \mathcal{A}) < \infty.$$

Consequently the $\mathbb{P}$-part $\mathbb{P}_{\mathcal{B}}$ of $\mathcal{B}$ is Λ-invariant. So a new renormalization orbit started in $\mathcal{B}$ has to visit some $\mathbb{D}$-part in $\mathbb{P}_{\mathcal{B}}$, say $\mathbb{D}_i$, twice. Hence there exists a $k \in \mathbb{N} \setminus \{0\}$ such that this part is Λ^k-invariant. On the other hand the set of accumulation points of the initial orbit is Λ-invariant. □

Since we are free to choose $\mathcal{A} \in \mathbb{D}^\circ$ in Lemma 4(i) we actually proved

Proposition 5 (eigenvalue test, [Metz 2001, Rem. 27]) *Suppose* $\mathbb{D}^\circ$ *and* $\mathbb{D}_i \subset \partial\mathbb{P}$ *are* Λ*-invariant and there exist* $\mathcal{A}' \in \mathbb{D}^\circ$, $\mathcal{B}' \in \mathbb{D}_i$ *such that* $\inf(\Lambda(\mathcal{A}')/\mathcal{A}')$ *is strictly bigger than* $\sup(\Lambda(\mathcal{B}')/\mathcal{B}')$. *Then* $\mathbb{D}_i$ *contains no accumulation point of* $(\tilde{\Lambda}_n(\mathcal{A}))_n$.

Suppose we can find a Λ-eigenform in each $\mathbb{D}_i$ with eigenvalue γ_i. Then $\inf(\Lambda(\mathcal{B}), \mathcal{B}) = \sup(\Lambda(\mathcal{B})/\mathcal{B}) = \gamma_i$ and we have to find an $\mathcal{A}' \in \mathbb{D}^\circ$ such that

$$\inf(\Lambda(\mathcal{A}'), \mathcal{A}') > \max\{\gamma_2, \gamma_3\}. \tag{4}$$

This would show that $\mathbb{D}_2 \cup \mathbb{D}_3$ contains no accumulation point of $(\tilde{\Lambda}_n(\mathcal{A}))_n$. Hence Proposition 2 guarantees the existence of a Λ-eigenform in $\mathbb{P} \cap \mathbb{D}^\circ$. The uniqueness will follow from the next proposition.

Proposition 6 ([Sabot 1997, Lem. 5.13]) *Suppose there are multiple eigenforms of $\tilde{\Lambda}$ in $N \cap \mathbb{D} \cap \mathbb{P}^\circ$. Then there is a sequence of eigenforms in $N \cap \mathbb{D} \cap \mathbb{P}^\circ$ approaching $\partial\mathbb{P} \cap \mathbb{D}$.*

According to Sabot's proposition above non-unique eigenforms imply the existence of a sequence of eigenforms approaching $\mathbb{D} \cap \partial\mathbb{P}$. This contradicts (4) because of Lemma 4(ii). Thus the solution is also unique. This completes the proof of Theorem 1.

Let us calculate γ_3. Using the formulas for the effective conductance of resistors in parallel or in line we have to solve

$$\gamma_3 \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} [2c_1^{-1} + (2c_2)^{-1}]^{-1} \\ [2c_1^{-1} + c_2^{-1} + 2(c_2 + c_3)^{-1}]^{-1} \\ [2c_1^{-1} + (c_2 + c_3)^{-1} + c_3^{-1} + (2c_3)^{-1}]^{-1} \end{pmatrix}. \tag{5}$$

Solving for $p := c_1/c_2$, $q := c_3/c_2$ and finally for γ_3 we get

$$(c_1, c_2, c_3)(\mathcal{F}_3) := (2, 1, 1) \text{ and } \gamma_3 = \frac{1}{3}.$$

The eigenvalue equation for γ_2 can be deduced from (5) in setting $c_3 := 0$. We solve for p and then for γ_2 to find

$$(c_1, c_2)(\mathcal{F}_2) := (1 + \sqrt{5}, 1) \text{ and } \gamma_2 = \frac{2}{5 + \sqrt{5}}.$$

The eigenvalue γ_i is unique in $\mathbb{D}_i$ by [Metz 1995, Prop. 4.2]. So (4) tells us that we have to find an $\mathcal{A}' \in \mathbb{D}^\circ$ with $\inf(\Lambda(\mathcal{A}'), \mathcal{A}') > \frac{1}{3}$. The eigenform and the eigenvalue in $\mathbb{D}^\circ$ is approximated numerically in [Kigami et al. 2001, Eq. 2.17]. Their numerical eigenvalue is bigger than 0.38 which indicates that there is a chance to prove (4). The approximate eigenform suggests an $\mathcal{A}' \in \mathbb{D}^\circ$ with

$$(c_1, \ldots, c_6)(\mathcal{A}') := \frac{1}{10}(10, 9, 5, 5, 4, 3).$$

Inserting this into the Schur-complement formula for Λ in [Metz 1995, Thm. 2.2(g)] and evaluating on the diamond, the chevron and the crown separately we derive

i	exact $c_i(\Lambda(\mathcal{A}'))$	$c_i(\Lambda(\mathcal{A}')) \geq \cdot$	$\cdot \geq c_i(\mathcal{A}')/3$
1	$\frac{90}{23}$	3.91	3.34
2	$\frac{1255}{366}$	3.42	3.00
3	$\frac{155538215}{77052516}$	2.01	1.67
4	$\frac{141829465}{77052516}$	1.84	1.67
5	$\frac{108671915}{77052516}$	1.41	1.34
6	$\frac{94652065}{77052516}$	1.22	1.00

The difference between the third and the fourth column of the table is positive in each entry. Thus $\Lambda(\mathcal{A}')(f) > \mathcal{A}'(f)/3$ for all $f : V_0 \to \mathbb{R}$. Thus (4) is true and Theorem 1 is proved.

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