Antipodal Metrics and Split Systems

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Abstract

Recall that a metric d on a finite set X is called *antipodal* if there exists a map $\sigma : X \to X$: $x \mapsto \overline{x}$ so that $d(x, \overline{x}) = d(x, y) + d(y, \overline{x})$ holds for all $x, y \in X$. Antipodal metrics canonically arise as metrics induced on specific weighted graphs, although their abundance becomes clearer in light of the fact that any finite metric space can be isometrically embedded in a more-or-less canonical way into an antipodal metric space called its *full antipodal extension*.

In this paper, we examine is some detail antipodal metrics that are, in addition, *totally split decomposable*. In particular, we give an explicit characterization of such metrics, and prove that – somewhat surprisingly – the full antipodal extension of a proper metric d on a finite set X is totally split decomposable if and only if either $\#X = 3$ or d is linear.

Keywords: antipodal metric, full antipodal extension, totally split-decomposable metric, consistent metric, split metric, weakly compatible split system, octahedral split system.

1 Introduction

Let X be a finite set and recall that a (pseudo)metric d, that is, a symmetric function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

that vanishes on the diagonal and satisfies the triangle inequality, is defined to be *antipodal* if there exists a map $\sigma : X \to X : x \mapsto \overline{x}$ so that

$$d(x, \overline{x}) = d(x, y) + d(y, \overline{x})$$

holds for all $x, y \in X$. Antipodal metrics commonly arise as metrics induced on the set of vertices of specific weighted graphs, e.g. the 1-skeletons of zonotopes (with weights attached to each class of parallel edges and not just to single edges, cf. [2, 9, 10, 11]). Yet, their abundance becomes obvious in a much more convincing way from the observation that every finite metric space can be embedded isometrically in a more or less canonical way into an antipodal metric space as follows: Given an arbitrary set X, let X^* denote the set $X \times \{+1, -1\}$. Then, given a metric $d : X \times X \to \mathbb{R}$ and a positive constant C with $2C \geq d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$, the map $d^*_C : X^* \times X^* \to \mathbb{R}$ defined by

$$d^*_C((x, \epsilon), (y, \eta)) := \begin{cases} d(x, y) & \text{if } \epsilon\eta = 1, \\ C - d(x, y) & \text{else,} \end{cases}$$

for all x, y in X and ϵ, η in $\{+1, -1\}$, is easily seen to define an antipodal metric d^*_C on X^*, while the map $X \to X^* : x \mapsto (x, +1)$ defines an isometric embedding of (X, d) into the antipodal metric space (X^*, d^*_C) (cf. [4] where this construction is called the *full antipodal extension* of a metric space).

In this paper, we will see that antipodal metrics that are in addition *totally split decomposable* (for a definition of this concept, see below or [1, 4]) have some very specific combinatorial properties that permit the identification of the "space" of all isometry classes of proper[1] antipodal metrics on a $2t$-set X with the orbit space of the dihedral group D_t on $\mathbb{R}^t_{>0}$ in case $t \neq 3$, and with the orbit space of the symmetric group S_4 on the subset of $\mathbb{R}^4_{\geq 0}$ which consists of all 4-tuples $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\geq 0}$ with $x_i = 0$ for at most one $i \in \{1, 2, 3, 4\}$ in case $t = 3$ (where the action of D_t on $\mathbb{R}^t_{>0}$ and that of S_4

[1] A metric d defined on a set X is said to be a *proper* metric if $d(x, y) \neq 0$ holds for all distinct elements x, y in X.

on $\mathbb{R}^4_{\geq 0}$ is, of course, the canonical one). The summary of our results that we want to present next will require a somewhat involved, yet unfortunately unavoidable introduction to split-decomposition theory:

We begin by recalling that a *split* $S = \{A, B\}$ of X – or, for short, an X-split – is defined to be a bipartition of X into two (non-empty) sets A, B. Given a split S and an element x in X, we denote by $S(x)$ the unique subset in S, A or B, that contains x, and by $\overline{S}(x) := X - S(x)$ the unique set in S that does not contain x. The set of all splits of X is denoted by $\mathcal{S}(X)$, any collection $\mathcal{S} \subseteq \mathcal{S}(X)$ of splits of X is called a *split system* (for X).

In [6], we studied the exceptional split geometries arising from split systems that are *weakly compatible*, yet *incompatible*. We showed [6, Theorem 3.1] that such split systems must be either *strictly circular* or *octahedral*. These terms are defined as follows: A split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is called

- *incompatible* if $S_1(x) \cup S_2(x) \neq X$ holds for all $x \in X$ and all $S_1, S_2 \in \mathcal{S}$;
- *weakly compatible* if there exist no four points x_0, x_1, x_2, x_3 in X and three splits S_1, S_2, S_3 in $\mathcal{S}$ with "$S_i(x_0) = S_i(x_j) \iff i = j$" for all $i, j \in \{1, 2, 3\}$;
- *strictly circular* if there exists a (labeled) partition $\Pi := \{X_1, \dots, X_{2t}\}$ of X into $2t$ non-empty subsets X_i, $1 \leq i \leq 2t$, such that $\mathcal{S}$ coincides with the split system $\mathcal{S}_\Pi$ consisting of all splits $S = \{A, B\}$ with

  $$A := X_i \dot\cup \dots \dot\cup X_{i+t-1}$$

  and

  $$B := X - A = X_{i+t} \dot\cup X_{i+t+1} \dot\cup \dots \dot\cup X_{2t} \dot\cup X_1 \dot\cup \dots \dot\cup X_{i-1}$$

  for some i with $1 \leq i \leq t$; and
- *octahedral* if there exists a (labeled) partition $\Pi = \{X_1, \dots, X_6\}$ of X into six non-empty subsets $X_1, \dots, X_6$ such that $\mathcal{S}$ coincides with the split system $\widehat{\mathcal{S}_\Pi}$ consisting of $\mathcal{S}_\Pi$ together with the additional split

  $$\{X_1 \dot\cup X_3 \dot\cup X_5,\ X_2 \dot\cup X_4 \dot\cup X_6\}.$$

Curiously, these weakly compatible, yet incompatible split systems are closely related to *antipodal* split systems, that is, split systems $\mathcal{S} \subseteq \mathcal{S}(X)$ for which there exists a map $\sigma : X \to X : x \mapsto \overline{x}$ so that $S(x) \neq S(\overline{x})$ – and, therefore, $S(\overline{x}) = \overline{S}(x)$ – holds for all $S \in \mathcal{S}$ and all $x \in X$. In Section 2, it will be observed that

(I) every antipodal split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is incompatible, and

(II) a weakly compatible split system is antipodal if and only if it is incompatible (and, hence, either strictly circular or octahedral).

The study of weakly compatible split systems was motivated in part by the fact that they arise naturally in the study of finite metric spaces. In particular, a theory was developed in [1] that allows the analysis of a finite metric[2] d in terms of its associated *weighted* split systems $(\mathcal{S}(d), \alpha_d)$ which, for the convenience of the reader, we briefly review here:

An ordered pair consisting of a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ together with a map $\alpha : \mathcal{S} \to \mathbb{R}_{>0}$ is called a *(positively) weighted* split system. The weighted split system $(\mathcal{S}(d), \alpha_d)$ associated to a metric d is defined as follows: For every pair A, B of non-empty subsets of X, the *isolation index* $\alpha(A, B) = \alpha(A, B|d)$ of A, B relative to d is defined by

$$\alpha(A, B|d) := \frac{1}{2} \min_{a,a' \in A\ b,b' \in B} (\max \left\{ \begin{array}{c} d(a,b) + d(a',b') \\ d(a,b') + d(a',b) \\ d(a,a') + d(b,b') \end{array} \right\} - d(a,a') - d(b,b')).$$

The split system $\mathcal{S}(d)$ associated to d is defined by

$$\mathcal{S}(d) := \{\{A, B\} \in \mathcal{S}(X) : \alpha(A, B|d) > 0\},$$

and the value $\alpha_d(S)$ of α_d on a split $S = \{A, B\}$ in $\mathcal{S}(d)$ is defined by

$$\alpha_d(S) := \alpha(A, B|d).$$

To analyze the relationship between a metric d and its associated weighted split system $(\mathcal{S}(d), \alpha_d)$, recall also the following definitions:

[2]Originally, most of the results in [1] were established for symmetric non-negative real-valued functions defined on a finite set.

- to every split $S \in \mathcal{S}(X)$, one associates a (pseudo)metric δ_S – also called the *split metric* (or *cut metric*) associated to S – which is defined by

$$\delta_S : X \times X \to \{0,1\} : \delta_S(x,y) := \begin{cases} 1 & \text{if } S(x) \neq S(y), \\ 0 & \text{else,} \end{cases}$$

- a metric is called a *Hamming metric* (cf. [12, p.2048]) if it is a positive linear combination of such split metrics.

Thus, a metric d is a Hamming metric if and only if it is of the form

$$d = d_{\mathcal{S},\alpha} := \sum_{S \in \mathcal{S}} \alpha(S)\, \delta_S$$

for some arbitrary weighted split system $(\mathcal{S}, \alpha)$. Using this notation, the following facts were established in [1]:

(1) For every given metric d defined on X, the following holds:

  (a) The split system $\mathcal{S}(d)$ is always weakly compatible.

  (b) For $\alpha := \alpha_d$, the inequality

  $$d_{\mathcal{S}(d),\alpha}(x,y) \leq d(x,y) \tag{1}$$

  holds for all $x, y \in X$. More precisely, the *split prime residue* $d_0 := d - d_{\mathcal{S}(d),\alpha}$ of d is always a (pseudo)metric.

  (c) The split prime residue d_0 of d vanishes or, equivalently, equality holds in Inequality (1) for all x, y in X if and only if

  $$\alpha(\{x,y\},\{u,v\}|d) \leq \alpha(\{x,t\},\{u,v\}|d) + \alpha(\{x,y\},\{u,t\}|d)$$

  holds for all x, y, u, v, t in X (or, equivalently, for all x, y, u, v, t in X with $\#\{x,y,u,v,t\} = 5$), in which case d is called *totally split decomposable.*

(2) Given an arbitrary weighted split system $(\mathcal{S}, \beta)$, one has

$$(\mathcal{S}, \beta) = (\mathcal{S}(d), \alpha_d)$$

for some metric d on X if and only if $\mathcal{S}$ is weakly compatible if and only if one has $(\mathcal{S}, \beta) = (\mathcal{S}(d), \alpha_d)$ for some totally split-decomposable metric d, in which case this totally split-decomposable metric d must necessarily coincide with $d_{\mathcal{S},\beta}$.[3]

Thus, totally split-decomposable metrics form a particular class of Hamming metrics — more precisely, they form the class of those Hamming metrics d that are of the form $d = d_{\mathcal{S},\alpha}$ for some weighted split system $(\mathcal{S}, \alpha)$ with $\mathcal{S}$ a weakly compatible split system. Moreover, though such a metric may have other representations as a positive linear combination of split metrics, its representation as a positive linear combination of weakly compatible split metrics (i.e. split metrics whose associated splits form a weakly compatible split system) is necessarily unique.

Since these facts were discovered, a number of further remarkable features regarding metrics, split systems, and the relationship between both have come to light (cf. [5, 6, 7, 8, 13]). In this note, we augment these investigations by applying the machinery developed in [1] to the analysis of antipodal metrics and antipodal split systems.

In particular, we show in Section 2 that

(i) split systems associated to antipodal metrics are themselves always antipodal and, therefore, they are either strictly circular or octahedral split systems (Corollary 2),

(ii) conversely, the Hamming metric $d_{\mathcal{S},\alpha}$ associated to an arbitrary weighted split system $(\mathcal{S}, \alpha)$ is antipodal if and only if the underlying split system $\mathcal{S}$ itself is antipodal (Theorem 2) and, hence,

(iii) the Hamming metric $d_{\mathcal{S},\alpha}$ associated to a weakly compatible split system $(\mathcal{S}, \alpha)$ is antipodal if and only if $\mathcal{S}$ is incompatible (Corollary 1).

Our main result is established in Section 3. It provides a complete and absolutely explicit description of all totally split-decomposable antipodal metrics: For any $t \geq 1$, let $X^{(t)}$ denote the set of cardinality $2t$ consisting of all

[3]Actually, using the terminology of [1], the metrics d with $(\mathcal{S}, \beta) = (\mathcal{S}(d), \alpha_d)$ for some weighted, weakly compatible split system $(\mathcal{S}, \beta)$ are exactly the metrics whose *split decomposable* part coincides with $d_{\mathcal{S},\beta}$ and which, therefore, allow a *coherent decomposition* of the form $d = d_0 + d_{\mathcal{S},\beta}$ with d_0 their split prime residue as defined above. Consequently, they are exactly the metrics of the form $d = d_0 + d_{\mathcal{S},\beta}$ where d_0 is any *split prime* metric defined on X with the property that every map $f : X \to \mathbb{R}$ with $f(x) + f(y) \geq d_0(x, y) + d_{\mathcal{S},\beta}(x, y)$ for all $x, y \in X$ is of the form $f = f_0 + f_1$ with $f_0(x) + f_0(y) \geq d_0(x, y)$ and $f_1(x) + f_1(y) \geq d_{\mathcal{S},\beta}(x, y)$, again for all $x, y \in X$.

maps x from the set $\{1, \dots, t\}$ into the set $\{+1, -1\}$ with $x(j) \in \{x(i), x(k)\}$ for all $i, j, k \in \{1, \dots, t\}$ with $i < j < k$. For all constants $c_1, \dots, c_t \geq 0$, let $d_{c_1,\dots,c_t}$ denote the metric defined on $X^{(t)}$ by

$$d_{c_1,\dots,c_t}(x, y) := \sum_{i \in \{1,\dots,t\}} c_i \, |x(i) - y(i)|$$

and, for all constants $c_1, \dots, c_t, c_{t+1} \geq 0$, let $\widehat{d}_{c_1,\dots,c_t,c_{t+1}}$ denote the metric defined on $X^{(t)}$ by

$$\widehat{d}_{c_1,\dots,c_t,c_{t+1}}(x, y) := d_{c_1,\dots,c_t}(x, y) + c_{t+1} \, |\Pi_{i=1}^t x(i) - \Pi_{i=1}^t y(i)|,$$

so one has $\widehat{d}_{c_1,\dots,c_t,0} = d_{c_1,\dots,c_t}$ for all $c_1, \dots, c_t \geq 0$. Then, the following holds:

Theorem 1 *A proper metric d defined on a set X with $\#X \neq 6$ is totally split decomposable and antipodal if and only if it is isometric to a metric of the form $d_{c_1,\dots,c_t}$ for some positive constants $c_1, \dots, c_t$ in which case the parameters $c_1, \dots, c_t$ are determined uniquely by d up to cyclic or anticyclic reordering.*

In contrast, if X has cardinality 6, then d is isometric either to a metric of the form d_{c_1,c_2,c_3} for some positive constants c_1, c_2, c_3 or to a metric of the form $\widehat{d}_{c_1,c_2,c_3,c_4}$ for some positive constants c_1, c_2, c_3, c_4 — so it is always isometric to a metric of the form $\widehat{d}_{c_1,c_2,c_3,c_4}$ for some non-negative constants c_1, c_2, c_3, c_4, of which at most one may vanish, in which case the parameters c_1, c_2, c_3, c_4 are determined uniquely by d up to (arbitrary) permutation.

Remark 1

(i) The metrics of the form $d_{c_1,\dots,c_t}$ are easily seen to be graph metrics defined on the set of vertices of the 1-skeleton of a two-dimensional zonotope with $2t$ vertices relative to a weighting of the t pairs of parallel edges by the weights $2c_1, \dots, 2c_t$ – the factor 2 taking account of the fact that $|x(i) - y(i)|$ is either 2 or 0 (and not 1 or 0), for all x, y in $X^{(t)}$.

(ii) It can be checked easily that the split system consisting of all splits of the form $\{A_i, B_i\}$ with $A_i := \{x \in X^{(t)} : x(i) = +1\}$ and $B_i := X^{(t)} - A_i = \{x \in X^{(t)} : x(i) = -1\}$, $i = 1, \dots, t$, and the split $\{A_0, B_0\}$ defined by $A_0 := \{x \in X^{(t)} : \Pi_{i=1}^t x(i) = +1\}$ and $B_0 := X^{(t)} - A_0 = \{x \in X^{(t)} : \Pi_{i=1}^t x(i) = -1\}$ is antipodal if and only if t is odd, and that it is weakly compatible if and only if $t = 3$ holds. Thus, such split systems can give rise to antipodal Hamming metrics for all odd t, but only for $t = 3$ to antipodal and weakly compatible split systems.

Combining the above observations with results from [6], a number of further, rather explicit characterizations of octahedral split systems in terms of metrics are derived in Section 4, and in Section 5 we consider totally split-decomposable antipodal Hamming metrics.

Finally, we combine our results in the last section to obtain the following surprising characterization of *linear* metric spaces (that is, metric spaces that are isometric to some subspace of the real line): A metric space (X, d) is linear if and only if the associated antipodal metric spaces (X^*, d^*_C) are totally split decomposable for all or, equivalently, for at least one C with

$$2C \geq \max(d(x,y) + d(u,v), d(x,u) + d(y,v), d(x,v) + d(y,u)) +$$

$$+ \min(d(x,y) + d(u,v), d(x,u) + d(y,v), d(x,v) + d(y,u))$$

for all $x, y, u, v \in X$.

2 Antipodal Split Systems

We begin this section by proving Assertions (I) and (II) stated in the introduction. Suppose that $\mathcal{S} \subseteq \mathcal{S}(X)$ is an antipodal split system with respect to a map $\sigma : X \to X$: $x \mapsto \overline{x}$. Then, for any distinct pair of splits S_1, S_2 in $\mathcal{S}$, we see that $S_1(x) \cup S_2(x) \neq X$ must clearly hold for every $x \in X$, since $\overline{x} \notin S_1(x) \cup S_2(x)$. This proves (I).

To see that (II) holds, assume that $\mathcal{S}$ is a weakly compatible, yet incompatible split system. In [6, Lemma 2.1], we showed that in this case $\bigcap_{S \in \mathcal{S}} \overline{S}(x) \neq \emptyset$ holds for every $x \in X$. So, we can define the required map

$$\sigma : X \to X : x \mapsto \overline{x}$$

by choosing, for every x in X, an arbitrary element $\overline{x}$ in $\bigcap_{S \in \mathcal{S}} \overline{S}(x)$ as its $\sigma-$image $\sigma(x)$, in which case $S(x) \neq S(\overline{x})$ clearly holds for all $S \in \mathcal{S}$ and all $x \in X$. Thus $\mathcal{S}$ is antipodal and (II) holds in view of (I).

Now let $(\mathcal{S}, \alpha)$ be an arbitrary weighted split system, and consider the metric

$$d_{\mathcal{S},\alpha} := \sum_{S \in \mathcal{S}} \alpha(S)\, \delta_S$$

defined in the introduction. In case $\alpha(S) = 1$ for all $S \in \mathcal{S}$, we will also write $d_{\mathcal{S}}$ instead of $d_{\mathcal{S},\alpha}$.

Clearly, we have

$$d_{\mathcal{S},\alpha}(x, y) = \sum_{S(x) \neq S(y)} \alpha(S)$$

for all $x, y \in X$ and, hence, we have

$$d_{\mathcal{S},\alpha}(x, y) + d_{\mathcal{S},\alpha}(y, z) = d_{\mathcal{S},\alpha}(x, z) \tag{2}$$

for some $x, y, z \in X$ if and only if there is no split S in $\mathcal{S}$ with $S(x) = S(z) \neq S(y)$, i.e. if and only if y is contained in $\bigcap_{S(x)=S(z)} S(x)$. Thus, Equation (2) holds for some fixed $x, z \in X$ and *all* $y \in X$ if and only if there is no $S \in \mathcal{S}$ with $S(x) = S(z)$, i.e. if and only if

$$d_{\mathcal{S},\alpha}(x, z) = \sum_{S \in \mathcal{S}} \alpha(S)$$

holds.

Clearly, this implies

Theorem 2 *A Hamming metric $d_{\mathcal{S},\alpha}$ on a weighted split system $(\mathcal{S}, \alpha)$ is antipodal relative to some map $\sigma : X \to X : x \mapsto \overline{x}$ if and only if*

$$d_{\mathcal{S},\alpha}(x, \overline{x}) = \sum_{S \in \mathcal{S}} \alpha(S)$$

holds for all $x \in X$ if and only if $\mathcal{S}$ is antipodal with respect to the map σ.

As consequences, we note the following

Corollary 1 *Suppose that X is a finite set, and that d is a totally split-decomposable metric defined on X. Then d is antipodal if and only if the split system $\mathcal{S}(d)$ is antipodal.*

Proof: This follows from Statements (1-a) and (2) in the introduction, together with Theorem 2. ∎

Corollary 2 *If an arbitrary metric d is antipodal, then $\mathcal{S}(d)$ is antipodal and, hence, it is either empty, strictly circular, or octahedral.*

Proof: This follows immediately from combining Corollary 1 with (i) the fact that $d_0 = d - d_{\mathcal{S}(d),\alpha_d}$ is always a metric (cf. Statement (1-b)) and (ii) the obvious fact that a sum $d = d_1 + d_2$ of two metrics d_1 and d_2 is antipodal relative to some map $\sigma : X \to X$ if and only if both, d_1 and d_2, are antipodal with respect to that map. ∎

3 Proof of Theorem 1

Suppose that d is antipodal. Then $\mathcal{S} := \mathcal{S}(d)$ must be either strictly circular or octahedral, by Corollary 2. Using the notation in the introduction, we can therefore assume that there exists an integer t and a (labeled) partition $\Pi := \{X_1, \ldots, X_{2t}\}$ of X into $2t$ nonempty subsets X_i, $1 \leq i \leq 2t$ such that $\mathcal{S}$ either coincides with the split system $\mathcal{S}_\Pi := \{S_i : 1 \leq i \leq t\}$ with

$$S_i := \{X_i \dot\cup \ldots \dot\cup X_{i+t-1}\,,\; X_{i+t} \dot\cup X_{i+t+1} \dot\cup \ldots \dot\cup X_{2t} \dot\cup X_1 \dot\cup \ldots \dot\cup X_{i-1}\}$$

for $1 \leq i \leq t$, or one has $t = 3$ and $\mathcal{S}$ coincides with the split system

$$\widehat{\mathcal{S}_\Pi} = \mathcal{S}_\Pi \cup \{\widehat{S} := \{X_1 \dot\cup X_3 \dot\cup X_5,\; X_2 \dot\cup X_4 \dot\cup X_6\}\}.$$

Our assumption that d is a proper metric implies immediately that $\#X_i = 1$ must hold for all $i = 1, \ldots, 2t$. Hence, we must have $\#X = 2t$, and we may label the elements in X as $x_1, \ldots, x_{2t}$ so that $X_i = \{x_i\}$ holds for all $i = 1 \ldots, 2t$.

To identify X with $X^{(t)}$, we proceed as follows: For every $t \geq 1$, we associate to each $x_i \in X$ the map

$$x_i : \{1, \ldots, t\} \to \{+1, -1\} : k \mapsto x_i(k)$$

defined, for all $k = 1, \ldots, t$ and $i = 1, \ldots, 2t$, by

$$x_i(k) := +1 \Leftrightarrow S_k(x_i) = S_k(x_k) \text{ and } x_i(k) := -1 \Leftrightarrow S_k(x_i) \neq S_k(x_k).$$

Clearly, the following table results

	1	2	3	…	k	$k+1$	…	$t-2$	$t-1$	t
x_1	$+$	$-$	$-$	…	$-$	$-$	…	$-$	$-$	$-$
x_2	$+$	$+$	$-$		…	$-$	$-$	…	$-$	$-$
x_3	$+$	$+$	$+$	…	$-$	$-$	…	$-$	$-$	$-$
$\vdots$										
x_k	$+$	$+$	$+$	…	$+$	$-$	…	$-$	$-$	$-$
$\vdots$										
x_{t-1}	$+$	$+$	$+$	…	$+$	$+$	…	$+$	$+$	$-$
x_t	$+$	$+$	$+$	…	$+$	$+$	…	$+$	$+$	$+$
x_{t+1}	$-$	$+$	$+$	…	$+$	$+$	…	$+$	$+$	$+$
x_{t+2}	$-$	$-$	$+$	…	$+$	$+$	…	$+$	$+$	$+$
$\vdots$										
x_{t+k}	$-$	$-$	$-$	…	$-$	$+$	…	$+$	$+$	$+$
$\vdots$										
x_{2t-2}	$-$	$-$	$-$	…	$-$	$-$	…	$-$	$+$	$+$
x_{2t-1}	$-$	$-$	$-$	…	$-$	$-$	…	$-$	$-$	$+$
x_{2t}	$-$	$-$	$-$	…	$-$	$-$	…	$-$	$-$	$-$

where $+$ is standing for $+1$ and $-$ is standing for -1.

Note that we have $\delta_{S_k}(x_i, x_j) = 1$ for $i, j = 1, \dots, 2t$ and $k = 1, \dots, t$ if and only if $x_i(k) \neq x_j(k)$ or – equivalently – $|x_i(k) - x_j(k)| = 2$, and we have $\delta_{S_k}(x_i, x_j) = |x_i(k) - x_j(k)| = 0$ otherwise. Hence, we have $\delta_{S_k}(x_i, x_j) = |x_i(k) - x_j(k)|/2$ for all $i, j = 1, \dots, 2t$ and $k = 1, \dots, t$.

Moreover, we have either $\mathcal{S} = \{S_1, S_2, S_3\}$ or $\mathcal{S} = \{S_1, S_2, S_3, \widehat{S}\}$. In case $t = 3$ and, in the second case, we have $\delta_{\widehat{S}}(x_i, x_j) = 1$ for $i, j = 1, \dots, 6$ if and only if $\Pi_{k=1}^3 x_i(k) \neq \Pi_{k=1}^3 x_j(k)$ or – equivalently – $|\Pi_{k=1}^3 x_i(k) - \Pi_{k=1}^3 x_j(k)| = 2$, and we have $\delta_{\widehat{S}}(x_i, x_j) = |\Pi_{k=1}^3 x_i(k) - \Pi_{k=1}^3 x_j(k)| = 0$ otherwise. Hence, we have $\delta_{\widehat{S}}(x_i, x_j) = |\Pi_{k=1}^3 x_i(k) - \Pi_{k=1}^3 x_j(k)|/2$ for all $i, j = 1, \dots, 6$.

In other words, putting $x_i[S_k] := x_i(k)$ for $i = 1, \dots, 2t$ and $k = 1, \dots, t$ in any case and, in case $t = 3$, also $x_i[\widehat{S}] := \Pi_{k=1}^3 x_i(k)$ for $i = 1, \dots, 6$, we see that $\delta_S(x_i, x_j) = |x_i[S] - x_j[S]|/2$ holds for all $S \in \mathcal{S}$ and all $i, j = 1, \dots, 2t$.

Now, put $c_S := \alpha_d(S)/2$ for all S in $\mathcal{S}$ and put $c_i := c_{S_i}$ for $i = 1, \dots, t$ in any case and, in case $t = 3$, put $c_{\widehat{S}} := \alpha_d(\widehat{S})/2$ and $c_4 := c_{\widehat{S}}$ in case $\widehat{S} \in \mathcal{S}$ and $c_{\widehat{S}} = c_4 := 0$ otherwise.

In view of our assumption that d is totally split decomposable, we then have

$$d(x,y) = \sum_{S\in\mathcal{S}} \alpha_d(S)\,\delta_S(x,y)$$

for all $x, y \in X$. Consequently, we have

$$d(x_i,x_j) = \sum_{S\in\mathcal{S}} \alpha_d(S)\,\delta_S(x_i,x_j) = \sum_{S\in\mathcal{S}} c_S\,|x_i[S]-x_j[S]|.$$

This shows that the map $X \to X^{(t)} : x_i \mapsto x_i(\cdot)$ induces indeed the required isometry between d and $d_{c_1,\dots,c_t}$ in case $t \neq 3$ while, in case $t = 3$, it induces the required isometry between d and $\widehat{d}_{c_1,c_2,c_3,c_4}$.

The remaining assertions regarding the uniqueness of the parameters $c_1, \dots, c_t$ or c_1, c_2, c_3, c_4 now follow easily. If $t \neq 3$ holds, then the circular sequence $S_1, \dots, S_t$ of the splits in $\mathcal{S}(d)$ is easily seen to be uniquely determined – up to cyclic or anticyclic reordering – by $\mathcal{S}(d)$ and thus also by d. Therefore, the parameters $c_1 = \alpha_d(S_1)/2, \dots, c_t = \alpha_d(S_t)/2$ are also uniquely determined. If $t = 3$ holds, the combinatorial symmetry group of the octahedral split system $\widehat{\mathcal{S}_\Pi}$ is the full symmetric group on $\widehat{\mathcal{S}_\Pi}$. Thus, the parameters c_1, c_2, c_3, c_4 are also determined uniquely up to arbitrary permutation if $t = 3$ holds. This is the case regardless whether $\mathcal{S}(d)$ is strictly circular or octahedral because *every* proper subset of $\widehat{\mathcal{S}_\Pi}$ is strictly circular and $\widehat{\mathcal{S}_\Pi}$ is the only octahedral extension of every proper subset of $\widehat{\mathcal{S}_\Pi}$ of cardinality 3 – so, they are uniquely determined by the isolation indices of the splits in the unique octahedral split system containing $\mathcal{S}(d)$.

The converse, i.e. the assertion that the metrics $d_{c_1,\dots,c_t}$ and $\widehat{d}_{c_1,\dots,c_4}$ described in Theorem 1 are antipodal and totally split decomposable, now follows also easily from the above definitions and identifications, and the facts collected in Section 2. ∎

4 Octahedral Split Systems Revisited

Theorem 1 has an interesting consequence for totally split-decomposable metrics that are in addition *consistent*, that is, totally split-decomposable metrics d for which the associated split system $\mathcal{S}(d)$ does not contain an octahedral subsystem [4]. Namely, a proper consistent totally split-decomposable metric d is antipodal if and only if it is isometric to a metric of the form $d_{c_1,\dots,c_t}$ for some $t \geq 1$ and some positive constants $c_1, \dots, c_t$.

In view of this fact, it is of some interest to understand and characterize octahedral split systems in terms of metrics. In [6, Theorem 4.1] we characterized octahedral split systems in various ways using properties of splits, and we now extend these results, deriving several additional characterizations that refer to the split metrics associated to a split system (assertions (vii) to (ix'') below).

Theorem 3 *Let $\mathcal{S} \subseteq \mathcal{S}(X)$ be a weakly compatible, yet incompatible split system of cardinality at least 2. Then the following statements are equivalent:*

(i) *$\mathcal{S}$ is an octahedral split system;*

(ii) *$\mathcal{S}$ contains an octahedral split system;*

(iii) *for every $x \in X$, there exist $S_1, S_2, S_3, S_4 \in \mathcal{S}$ such that $S_1(x) \cap S_2(x) = S_3(x) \cap S_4(x)$ and $\{S_1, S_2\} \cap \{S_3, S_4\} = \emptyset$;*

(iii′) *for every $x \in X$, there exist $S_1, S_2, S_3, S_4 \in \mathcal{S}$ such that $\overline{S_1}(x) \cap \overline{S_2}(x) = \overline{S_3}(x) \cap \overline{S_4}(x)$ and $\{S_1, S_2\} \cap \{S_3, S_4\} = \emptyset$;*

(iv) *there exists some $x \in X$ and $S_1, S_2, S_3, S_4 \in \mathcal{S}$ with $S_1(x) \cap S_2(x) = S_3(x) \cap S_4(x)$ and $\{S_1, S_2\} \neq \{S_3, S_4\}$;*

(iv′) *there exists some $x \in X$ and $S_1, S_2, S_3, S_4 \in \mathcal{S}$ with $\overline{S_1}(x) \cap \overline{S_2}(x) = \overline{S_3}(x) \cap \overline{S_4}(x)$ and $\{S_1, S_2\} \neq \{S_3, S_4\}$;*

(v) *$\#\{\bigcap_{S \in \mathcal{S}} S(x) : x \in X\} \neq \#\bigcup \mathcal{S}$ where $\bigcup \mathcal{S} := \{A \subseteq X : A \in S \text{ for some } S \in \mathcal{S}\} = \{S(x) : S \in \mathcal{S}, x \in X\}$;*

(vi) *there exists a subset $Y \subseteq X$ with $\#Y = 6$ so that the induced split system*

$$\mathcal{S}|_Y := \{T \in \mathcal{S}(Y) \;:\; \text{there exists some } S = \{A, B\} \in \mathcal{S} \text{ with } T = S|_Y := \{A \cap Y, B \cap Y\}\}$$

is octahedral;

[4] In [5], we give a six-point condition that characterizes consistent totally split-decomposable metrics and in [7, 8], more can be found regarding these metrics and their tight span.

(vi′) *there exists a subset* $Y \subseteq X$ *with* $\#Y = 6$ *and a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *so that the induced split system* $S'|_Y$ *is octahedral;*

(vii) *one has* $d_{\mathcal{S}}(x, y) \neq 1$ *for all* $x, y \in X$*;*

(vii′) *there exists a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality at least* 2 *with* $d_{\mathcal{S}'}(x, y) \neq 1$ *for all* $x, y \in X$*;*

(vii″) *there exists a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality* 4 *with* $d_{\mathcal{S}'}(x, y) \neq 1$ *for all* $x, y \in X$*;*

(viii) *one has* $d_{\mathcal{S}}(x, y) \in \{0, 2, 4\}$ *for all* $x, y \in X$*;*

(viii′) *there exists a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality at least* 2 *with* $d_{\mathcal{S}'}(x, y) \in \{0, 2, 4\}$ *for all* $x, y \in X$*;*

(viii″) *there exists a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality* 4 *with* $d_{\mathcal{S}'}(x, y) \in \{0, 2, 4\}$ *for all* $x, y \in X$*;*

(ix) *there exists a subset* $Y \subseteq X$ *with* $\#Y = 6$ *and* $d_{\mathcal{S}}(x, y) \in \{2, 4\}$ *for all* $x, y \in Y$ *with* $x \neq y$*;*

(ix′) *there exists a subset* $Y \subseteq X$ *with* $\#Y = 6$ *and a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality at least* 2 *with* $d_{\mathcal{S}'}(x, y) \in \{2, 4\}$ *for all* $x, y \in Y$ *with* $x \neq y$*;*

(ix″) *there exists a subset* $Y \subseteq X$ *with* $\#Y = 6$ *and a subset* $\mathcal{S}' \subseteq \mathcal{S}$ *of cardinality* 4 *with* $d_{\mathcal{S}'}(x, y) \in \{2, 4\}$ *for all* $x, y \in Y$ *with* $x \neq y$*.*

Proof: The equivalence of the assertions (i) to (vi') has been established in [6]. The implications $(i) \Rightarrow (viii) \Rightarrow (viii') \Rightarrow (vii')$, $(viii) \Rightarrow (vii) \Rightarrow (vii')$, $(i) \Rightarrow (viii'') \Rightarrow (vii'')$, $(i) \Rightarrow (ix) \Rightarrow (ix')$, and $(i) \Rightarrow (ix'')$ are obvious.

The implications $(vii') \Rightarrow (ii)$ and $(vii'') \Rightarrow (ii)$ follow from the fact that one has $1 \in \{d_{\mathcal{S}''}(x, y) : x, y \in X\}$ for every strictly circular split system $\mathcal{S}'' \subseteq \mathcal{S}(X)$.

And the remaining implications $(ix') \Rightarrow (ii)$ and $(ix'') \Rightarrow (ii)$ follow from the following observations:

- Given a strictly circular split system $\mathcal{S}''$ of cardinality t defined on a finite set X and suppose that there exists a subset $Y \subseteq X$ with $\#Y = 6$ and $d_{\mathcal{S}''}(x, y) \neq 0, 1$ for all $x, y \in Y$ with $x \neq y$. Then $t > 5$ must hold, and one must have $6 \in \{d_{\mathcal{S}''}(x, y) : x, y \in Y\}$ in case $t = 6$.

- Given a strictly circular split system $\mathcal{S}''$ of cardinality $t > 5$ defined on a finite set X and suppose that $d_{\mathcal{S}''}(x,y), d_{\mathcal{S}''}(y,z), d_{\mathcal{S}''}(z,x) \in \{2,4\}$ holds for some $x,y,z \in X$. Then either $t = 6$ and $d_{\mathcal{S}''}(x,y) = d_{\mathcal{S}''}(y,z) = d_{\mathcal{S}''}(z,x) = 4$ holds, or exactly two of the three values $d_{\mathcal{S}''}(x,y), d_{\mathcal{S}''}(y,z), d_{\mathcal{S}''}(z,x)$ are equal to 2.

- Given a strictly circular split system $\mathcal{S}''$ of cardinality $t > 5$ defined on a finite set X and suppose that $d_{\mathcal{S}''}(x,y) = d_{\mathcal{S}''}(y,z) = 2$ and $d_{\mathcal{S}''}(x,z) = 4$. Then $d_{\mathcal{S}''}(y,u) = 2$ for some u in X implies $d_{\mathcal{S}''}(x,u) = 0$ or $d_{\mathcal{S}''}(z,u) = 0$ while $d_{\mathcal{S}''}(y,u) = 4$ for some u in X implies $d_{\mathcal{S}''}(x,u) = 6$ or $d_{\mathcal{S}''}(z,u) = 6$.

- Consequently, if $\mathcal{S}''$ is a strictly circular split system of cardinality t defined on a finite set X and if $d_{\mathcal{S}'}(x,y)$ is an even positive number for all elements x,y with $x \neq y$ in a 6-subset $Y \subseteq X$, then $t > 5$ and $\max(d_{\mathcal{S}''}(x,y) : x,y \in Y) > 4$ must hold. ∎

5 Antipodal Hamming Metrics

In this section, we shall see that an antipodal Hamming metric d can have more than one representation as a (positively weighted) sum of split metrics, but that in case d is also totally split decomposable any such representation is necessarily unique.

It is a straight-forward matter to see why antipodal Hamming metrics might not necessarily have unique representations as (positively weighted) sums of split metrics. Indeed, given a set X of cardinality $2n$ with a fixed-point free involution $\sigma : X \to X : x \mapsto \overline{x}$, the split system

$$\mathcal{S}_\sigma := \{\{A,B\} \in \mathcal{S}(X) : \#(A \cap \{x,\overline{x}\}) = 1 \text{ for all } x \in X\}$$

of cardinality 2^{n-1} is obviously the unique largest antipodal split system $\mathcal{S} \subseteq \mathcal{S}(X)$ with $S(\overline{x}) \neq S(x)$ for all $S \in \mathcal{S}$ and all $x \in X$. Consequently, the split metrics derived from this split system must be linearly dependent for all n with $2^{n-1} > \binom{2n}{2}$, that is, for $n > 7$. Moreover, there must be positive as well as negative coefficients in any linear relation between linearly dependent split metrics. Thus, there must exist disjoint weighted split systems $(\mathcal{S},\alpha)$ and $(\mathcal{T},\beta)$ with $\mathcal{S},\mathcal{T} \subseteq \mathcal{S}_\sigma$ such that the associated antipodal Hamming metrics $d_{\mathcal{S},\alpha}$ and $d_{\mathcal{T},\beta}$ coincide, provided $n > 7$ holds.

More explicitly, such disjoint weighted split systems $(\mathcal{S}, \alpha)$ and $(\mathcal{T}, \beta)$ with $d_{\mathcal{S},\alpha} = d_{\mathcal{T},\beta}$ exist already for $n = 4$: The sum of the four split metrics associated with the four splits of the form $\{A, -A := \{-a : a \in A\}\}$ of the set $X := \{1,2,3,4\} \cup -\{1,2,3,4\}$ for which A contains an even number of positive elements coincides necessarily with the sum of the split metrics associated with the four remaining splits of that form, i. e. those for which A contains an odd number of positive elements.

Note also that an arbitrary totally split-decomposable metric may have representations as a weighted sum of split metrics that differ from its "canonical" representation in terms of its associated weakly compatible split system – for instance, it is well known and easy to see that, for every set X of cardinality 4, the sum of all split metrics of the form $\delta_{\{A,B\}}$ with $1 \in \{\#A, \#B\}$ coincides with the sum of all split metrics of the form $\delta_{\{A,B\}}$ with $\#A = \#B = 2$. However, in contrast to this, we have the following:

Theorem 4 *Every totally split-decomposable antipodal metric can be uniquely represented as a positively weighted sum of split metrics.*

Proof: Without loss of generality, we may assume that d is proper. In case $\#X \leq 6$, the split metrics associated to an antipodal split system of the form $\mathcal{S}_\sigma$ for some fixed-point free involution σ of X are always linearly independent. So, we may assume $\#X > 6$, we may choose a strictly circular split system $\mathcal{S}$ representing d, and we may label the elements in X as $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n}$ so that $d(x_k, x_{k+i}) + d(x_{k+i}, x_{k+j}) = d(x_k, x_{k+j})$ holds for every integer k and all i, j with $0 \leq i \leq j \leq n+1$ (with labels computed modulo $2n$). All we need to observe now is that a split S involved in some representation of d as a weighted sum of split metrics and separating, say, x_1 from x_{2n} is necessarily the split $\{\{x_1, x_2, \ldots, x_n\}, \{x_{n+1}, \ldots, x_{2n}\}\}$. However, this follows from the fact that $S(x_1) = \overline{S}(x_{2n}) = S(x_n)$ and $S(x_i)$ must coincide for all i with $1 \leq i \leq n$ which in turn follows from applying our observation above regarding the case $\#X \leq 6$ to the 6-point subset $\{x_1, x_i, x_n, x_{n+1}, x_{n+i}, x_{2n}\}$ and the metric induced on this set. ∎

6 A Surprising Characterization of Linear Metric Spaces

In this section, we shall address the very simple, yet surprising observation that, using the full antipodal extensions (X^*, d^*_C) of a metric space (X, d) described in the introduction, the results established above allow us to conclude that – assuming that d is a proper metric – the associated antipodal metric spaces (X^*, d^*_C) are totally split decomposable for one or, equivalently, for all sufficiently large constants C if and only if either $\#X \leq 3$ holds or (X, d) is a *linear* metric space i.e. (X, d) is isometrically embeddable into the real line. To us, this simple observation was actually quite a surprise because, when beginning our work on antipodal and totally split-decomposable metric spaces, we did not expect them to be that closely related to linear metric spaces.

Continuing with the notation introduced above, define

$$A^+ := \{(a, +1) : a \in A\} \text{ and } A^- := \{(a, -1) : a \in A\}$$

for every subset A of X. Furthermore, let S^* denote the X^*-split

$$S^* := \{A^+ \cup B^-, A^- \cup B^+\}$$

for every X-split $S = \{A, B\}$, and let S_0^* denote the split $\{X^+, X^-\}$ of X^* (associated to the *degenerate* split $S_0 := \{X, \emptyset\}$ of X).

Next, given any symmetric map d from $X \times X$ into the reals, let d^* denote the map from $X^* \times X^*$ into the reals defined by $d^*((x, \epsilon), (y, \eta)) := \epsilon\eta d(x, y)$ for all x, y in X and all ϵ, η in $\{+1, -1\}$.

It is obvious that

- the transformation $d \mapsto d^*$ is linear,

and it is very easy to see that, given a positive constant C,

- the map $d^*_C := d^* + C\delta_{S_0^*}$ is a metric whenever

  (i) d is a metric

  and one has

(ii) $d(x,y)+d(y,z)+d(z,x)\leq 2C$ for all x,y,z in X,

- d_C^* is an antipodal metric in this case, and
- $\delta_S^* = \delta_{S^*} - \delta_{S_0^*}$ holds for every split S of X.

Consequently, if $(\mathcal{S},\alpha)$ is an arbitrary weighted split system and if we put $d := d_{\mathcal{S},\alpha}$, we have

$$d_C^* := \sum_{S\in\mathcal{S}} \alpha(S)\,\delta_{S^*} + \alpha\delta_{S_0^*}$$

with $\alpha := C - \sum_{S\in\mathcal{S}}\alpha(S)$ (see [4, p. 95]).

So, the antipodal metric d_C^* is a Hamming metric whenever d is a Hamming metric and the constant C is at least as large as the sum of the coefficients occurring in some representation of d as a sum of split metrics (note that this sum may depend on the representation under consideration).

However, this metric will almost never be totally split decomposable even if d is, because – even for a weakly compatible split system $\mathcal{S}$ – we cannot expect the (obviously antipodal and, hence, incompatible) split system

$$\mathcal{S}^* := \{S^* : S\in\mathcal{S}\}$$

to be weakly compatible, too (and explicit counterexamples are easily constructed).

Actually, we can combine the above analysis and the facts established before to derive the following remarkable fact:

Theorem 5 *Given a finite set X with a proper metric*

$$d : X\times X \to \mathbb{R} : (x,y)\mapsto xy := d(x,y),$$

a metric of type d_C^* defined on X^* is totally split decomposable for some constant C with

$$2C \geq \max(xy+uv, xu+yv, xv+yu) + \min(xy+uv, xu+yv, xv+yu)$$

for all x,y,u,v in X if and only if it is totally split decomposable for all such C if and only if one has $\#X = 3$ or (X,d) is linear (i.e. isometric to a *subspace of the real line) in which case the maximum C_0 of all expressions of the form*

$$1/2(\max(xy + uv, xu + yv, xv + yu) + \min(xy + uv, xu + yv, xv + yu))$$

$(x, y, u, v \in X)$ *clearly coincides with*

$$\max(xy : x, y \in X) = 1/2 \max(xy + yz + zx : x, y, z \in X) = \sum_{S \in \mathcal{S}(d)} \alpha_d(S).$$

Proof: It is easily seen by direct inspection that every antipodal metric defined on a set of cardinality at most 6 is necessarily totally split decomposable. Thus, we may assume without loss of generality that $\#X > 3$ holds. It is easily also seen that d_C^* is a metric for which S_0^* is a split in $\mathcal{S}(d_C^*)$ if and only if $C > C_0$ holds.

Thus, if d_C^* is totally split decomposable and $C > C_0$ holds, the split system $\mathcal{S}(d_C^*)$ must be a strictly circular split system that contains S_0^*. Consequently, it must be possible to label the elements in X as $x_1, \ldots, x_n$ $(n = \#X)$ so that the splits in $\mathcal{S}(d_C^*)$ are exactly the splits of the form $\{A, B\}^*$ with $A = \{x_1, \ldots, x_j\}$ for some j with $1 \leq j \leq n$. Thus, the metric d on X must be totally split decomposable and

$$\mathcal{S}(d) = \{\{\{x_1, \ldots, x_j\}, \{x_{j+1}, \ldots, x_n\}\} : 1 \leq j < n\}$$

must hold. Clearly, this implies that $x_i x_k = x_i x_j + x_j x_k$ must hold for all integers i, j, k with $1 \leq i \leq j \leq k \leq n$ and, thus, it implies the linearity of (X, d) as claimed.

In addition, the same is easily seen to hold in case $C = C_0$ (rather than $C > C_0$) in view of the fact that $\mathcal{S}(d') \cup \{S'\} = \mathcal{S}(d' + \epsilon\delta_{S'})$ holds for every $\epsilon > 0$, every metric d' defined on a set X', and every split $S' = \{A', B'\}$ of X' satisfying the inequality

$$d'(a', a'') + d'(b', b'') \leq \max(d'(a', b') + d'(a'', b''), d'(a', b'') + d'(a'', b'))$$

for all a', a'' in A' and b', b'' in B'. This fact has been established in [1, Theorem 4], and it should be applied to $d' := d_{C_0}^*$ and $S' := S_0^*$, using the fact that $d_{C_0+\epsilon}^*$ coincides with $d_{C_0}^* + \epsilon\delta_{S_0^*}$ and the facts established just above in case $C := C_0 + \epsilon > C_0$.

The converse is obvious. ∎

Remark 2

In a similar vein, one can see that, given a split system $\mathcal{S} \subseteq \mathcal{S}(X)$, the corresponding split system $\{S_0^*\} \cup \mathcal{S}^*$ is weakly compatible if and only if the split system $\mathcal{S}$ is *nested* , i.e. one can label the splits in $\mathcal{S}$ as $S_1 = \{A_1, B_1\}, S_2 = \{A_2, B_2\}, \dots, S_k = \{A_k, B_k\}$ so that $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k$ holds (see [4] for more on nested split systems).

In relation to this it was noted in [4, Theorem 11.2.21] that a finite space (X, d) is linear if and only if d is totally split decomposable and the split system $\mathcal{S}(d)$ is nested. Consequently, using the notations and definitions introduced in the previous theorem, a metric of type d_C^* defined on X^* is totally split decomposable if and only if (X, d) is totally split decomposable and the split system $\mathcal{S}(d)$ is nested.

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