On Line Arrangements in the Hyperbolic Plane

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Abstract

Given a finite collection $\mathcal{L}$ of lines in the hyperbolic plane $\mathbb{H}$, we denote by $k = k(\mathcal{L})$ its *Karzanov* number, i.e. the maximal number of pairwise intersecting lines in $\mathcal{L}$, and by $C(\mathcal{L})$ and $n = n(\mathcal{L})$ the set and the number, respectively, of those *points at infinity* that coincide with at least one line from $\mathcal{L}$. It is shown that $\#\mathcal{L} \leq 2nk - \binom{2k+1}{2}$ always holds and that $\#\mathcal{L}$ equals $2nk - \binom{2k+1}{2}$ if and only if there is no collection $\mathcal{L}'$ of lines in $\mathbb{H}$ with $\mathcal{L} \subsetneq \mathcal{L}'$, $k(\mathcal{L}') = k(\mathcal{L})$ and $C(\mathcal{L}') = C(\mathcal{L})$.

1 Introduction

Given a non-empty finite collection $\mathcal{L}$ of lines in a plane, there are three ways to measure what might fashionably be called its "complexity": It can be measured (a) simply by the cardinality $\#\mathcal{L}$ of $\mathcal{L}$, (b) by the maximal number $k = k(\mathcal{L})$ of pairwise intersecting lines in $\mathcal{L}$, which might also be called its *Karzanov* number (cf.[6]), and (c) by the number $n = n(\mathcal{L})$ of those *points at infinity* that coincide with at least one line in $\mathcal{L}$. Obviously, in the Euclidean plane, we always have $0 \leq k = n \leq \#\mathcal{L}$, and there are no further restrictions regarding these three numbers. In particular, $\#\mathcal{L}$ can be arbitrarily large even if $k = n = 1$ holds. In contrast, in the hyperbolic plane, we always have $\#\mathcal{L} \leq \binom{n}{2}$ as well as $0 < 2k \leq n \leq 2\#\mathcal{L}$, and it is well known – and a simple exercise to show – that $\#\mathcal{L} \leq 2n - 3$ holds for every hyperbolic line arrangement $\mathcal{L}$ with $k(\mathcal{L}) = 1$. More generally, it follows from results in [2] that $\#\mathcal{L} \leq 2kn - \binom{2k+1}{2}$ holds in the extreme cases $k = 1, 2, 3$ and $n = 2k+1, 2k+2, 2k+3, 2k+4$ and that, for every $k \in \{1, \ldots, \lfloor n/2 \rfloor\}$, there exist line arrangements $\mathcal{L}$ with $n(\mathcal{L}) = n$ and $k(\mathcal{L}) = k$ of cardinality exactly $2kn - \binom{2k+1}{2}$ that are (n,k)-*maximal* (that is, for every larger arrangement $\mathcal{L}'$, one has either $n(\mathcal{L}') > n$ or $k(\mathcal{L}') > k$ i.e., one has $n(\mathcal{L}') + k(\mathcal{L}') > n(\mathcal{L}) + k(\mathcal{L})$).

In [2], it was conjectured (though in a more combinatorial and less geometric language) that every (n,k)-maximal line arrangement must be of this cardinality $2kn - \binom{2k+1}{2}$. In this paper, we show that this conjecture is, in fact, true. More explicitly, choose an orientation for the hyperbolic plane $\mathbb{H}$, and consider a subset C of cardinality $n \geq 2k+1$ of the set $\mathbb{S}$ of points at infinity of $\mathbb{H}$ considered as an oriented circle relative to the orientation induced by that of $\mathbb{H}$. For distinct $a, b \in \mathbb{S}$, let ab denote the line whose two points at infinity are a and b. Let $\mathcal{L}_k = \mathcal{L}_k(C)$ denote the arrangement of lines xy joining all those pairs of distinct points $x, y \in C$ for which the intersection of one of the two connected components of $\mathbb{S} - \{a, b\}$ with C has cardinality less than k (while the other component then necessarily contains at least $n - 2 - (k - 1) = n - k - 1 \geq k$ points from C). Then it is easy to see that $\#\mathcal{L}_k = nk$, and that $k(\mathcal{L}_k) = k$ both hold. Now select k consecutive points K from C, and add to $\mathcal{L}_k$ all lines of the form ab, with $a \in K$, $b \in C - K$, and $ab \notin \mathcal{L}_k$. This way, we add exactly $n - 2k - 1$ new lines for each point a in K. The resulting arrangement $\mathcal{L}^* = \mathcal{L}^*(C, K)$ therefore has cardinality

$$\begin{aligned}\#\mathcal{L}^* &= \#\mathcal{L}_k + k(n - 2k - 1) \\ &= 2nk - \binom{2k+1}{2}.\end{aligned}$$

Moreover, $\mathcal{L}^*$ is clearly (n, k)-maximal, since any line l not in $\mathcal{L}_k$ of the form bb' with $b, b' \in C - K$ would be clearly contained in a set of $(k + 1)$ pairwise intersecting lines from $\mathcal{L}^* \cup \{l\}$ (see Figure 1): Just let $a_1, a_2, \ldots, a_k$ denote the k consecutive points in K, let $b_1, b_2, \ldots b_k$ denote k consecutive points from C between b and b' in that connected component of $S - \{b, b'\}$ not containing K, and consider the lines $bb', a_1b_1, a_2b_2, \ldots, a_kb_k$.

As is easily seen, $\mathcal{L}^*$ is just one of the many (n, k)-maximal arrangements that can be constructed using the methods introduced in [2]. However, as we realized only recently, it is some sort of a "primordial" (n, k)-maximal line arrangement. Indeed, we will show here that, given any (n, k)-maximal ar-

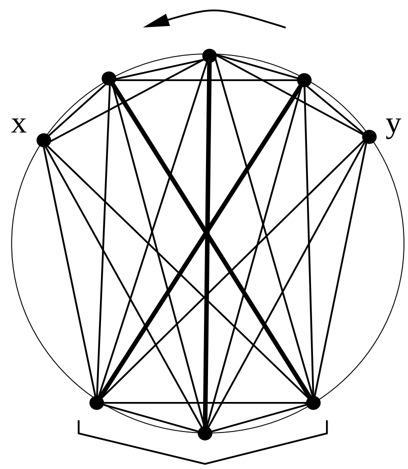

Figure 1: *The canonical line arrangement* $\mathcal{L}^*$ *in case* $n = 8$ *and* $k = 3$. *The three thick lines are those contained in* $\mathcal{L}^* - \mathcal{L}_3$.

rangement of lines $\mathcal{L}$, there exists a sequence of (n, k)-maximal arrangements

$$\mathcal{L}_0 := \mathcal{L}, \mathcal{L}_1, \dots, \mathcal{L}_p := \mathcal{L}^*$$

with $\mathcal{L}_i \Delta \mathcal{L}_{i+1} = 2$ for $0 \leq i \leq p - 1$, i.e. $\mathcal{L}_{i+1}$ can be obtained from $\mathcal{L}_i$ by removing one of the lines of $\mathcal{L}_i$ and replacing it by another. In particular, this will imply our main result:

Theorem 1.1 *Any* (n, k)*-maximal arrangement* $\mathcal{L}$ *of lines in the hyperbolic plane* $\mathbb{H}$ *has cardinality*

$$\#\mathcal{L} = \#\mathcal{L}^* = 2nk - \binom{2k+1}{2}.$$

Acknowledgment: We thank Fritz von Haeseler for making us aware of the close connection between cyclic split systems as considered in [2, 3], and the hyperbolic line arrangements considered here.

2 Cyclic n-Sets

Let C denote a *cyclic n-set*, i.e. a transitive $\mathbb{Z}$-set of cardinality $n < \infty$. For $x \in C$ and $k \in \mathbb{Z}$, let $x^{(k)}$ denote the image of (k, x) in C with respect to the structure map

$$\mathbb{Z} \times C \to C$$

– implying that $x^{(0)} = x$, $(x^{(k_1)})^{(k_2)} = x^{(k_1+k_2)}$ and $C = \{x^{(k)} \mid k \in \mathbb{Z}\}$ holds for all $x \in C$ and $k_1, k_2 \in \mathbb{Z}$ – and put

$$x^+ := x^{(1)}$$

and

$$x^- := x^{(-1)}$$

so that

$$x^+ = y \iff y^- = x$$

holds for all $x, y \in C$. For $a, b \in C$, put

$$d^+(a, b) := \min(k \in \mathbb{N}_0 \mid a^{(k)} = b)$$

and

$$d^-(a,b) := \min(k \in \mathbb{N}_0 \mid a^{(-k)} = b).$$

Clearly, we have

$$d^-(a,b) = d^+(b,a)$$

for all $a, b \in C$, and

$$d^+(a,b) + d^-(a,b) = n$$

for all distinct $a, b \in C$. Next, given a subset Y of C, the following assertions are easily seen to be equivalent:

(i) $\#\{y \in Y \mid y^- \notin Y\} = 1$,

(ii) $\#\{y \in Y \mid y^+ \notin Y\} = 1$,

(iii) there exist $a, b \in C$ with $a \neq b^+$ and

$$Y = I^+(a,b) := \{a^{(k)} \mid 0 \leq k \leq d^+(a,b)\},$$

(iv) there exist $a, b \in C$ with $b \neq a^-$ and

$$Y = I^-(b,a) := \{b^{(-k)} \mid 0 \leq k \leq d^-(b,a)\}.$$

Moreover, if this holds, then the elements a and b in C referred to in (iii) are the unique elements $y_1, y_2 \in Y$ with $y_1^- \notin Y$ and $y_2^+ \notin Y$, respectively, and they will also be denoted by $a(Y)$ and $b(Y)$.

Any subset Y of C satisfying the four assertions above will be called a (cyclic) interval. Note that a subset Y of C is an interval if and only if its complement $C - Y$ is an interval, and that $a(Y) = b(C - Y)^+$ and $b(Y) = a(C - Y)^-$ holds in this case.

3 $\mathcal{L}$-pegs

Given a finite arrangement $\mathcal{L}$ of lines in $\mathbb{H}$, we denote by $C = C(\mathcal{L})$ the set of points in $\mathbb{S}$, the circle of points at infinity of $\mathbb{H}$, that are incident with at least one line in $\mathcal{L}$ so that, by definition, $n(\mathcal{L}) = \#C$ holds. Moreover, as explained above, upon choosing one of the two orientations of the hyperbolic plane as our orientation of reference, we view the set C as a cyclic n-set with $n := n(\mathcal{L}) = \#C$ whose 'orientation' we derive from the chosen orientation of $\mathbb{H}$, i.e. we define $a^{(k)}$ for any $a \in C$ and $k \in \mathbb{N}_0$ to be the k-th element in C encountered when going around $\mathbb{S}$ according to the given orientation, starting at a (and we define $a^{(-k)}$ for $k \in \mathbb{N}_0$ accordingly). Given a pair of distinct points $x, y \in C$, we define

$$\mathcal{L}^+(x, y) := \min\{i \in \mathbb{N}_{>0} \, : \, xy^{(i)} \in \mathcal{L} \text{ or } y^{(i)} = x\}$$

and

$$v^+(x,y) = v^+_{\mathcal{L}}(x,y) := y^{(\mathcal{L}^+(x,y))}$$

and, similarly,

$$\mathcal{L}^-(x,y) := \min\{i \in \mathbb{N}_{>0} \ : \ xy^{(-i)} \in \mathcal{L} \text{ or } y^{(-i)} = x\}$$

and

$$v^-(x,y) = v^-_{\mathcal{L}}(x,y) := y^{(\mathcal{L}^-(x,y))}.$$

E.g., for $\mathcal{L} = \mathcal{L}^*$ and x and y as in Figure 1, we have $\mathcal{L}^+(x,y) = 2$ and $\mathcal{L}^-(x,y) = 1$.

We then call a quadruple (x,y,u,v) of points from S an $\mathcal{L}$-*peg* if the following statements hold:

(i) $x,y,u,v \in C = C(\mathcal{L})$;

(ii) $1 = d^+(x,y) < d^+(x,u) < d^+(x,v)$;

(iii) $xv, yu \in \mathcal{L}$; and

(iv) $\mathcal{L}^+(x,v), \mathcal{L}^-(y,u) \geq d^+(u,v)$.

Clearly, if (x,y,u,v) is an $\mathcal{L}$-peg, then (y,x,v,u) is an $\mathcal{L}$-peg relative to the reverse orientation of $\mathbb{H}$.

As we shall see, $\mathcal{L}$-pegs will be crucial for our treatment of hyperbolic line arrangements. We begin by presenting two key properties of $\mathcal{L}$-pegs, using the notation $B + a$ for $B \cup \{a\}$ and $B - a$ for $B - \{a\}$ for any subset B of a set A and any element a of A.

Lemma 3.1 *Suppose that $\mathcal{L}$ is a finite arrangement of lines in $\mathbb{H}$, and that (x, y, u, v) is an $\mathcal{L}$-peg with $\{xu, yv\} \cap \mathcal{L} = \emptyset$ (see Figure 2). Then we have*

$$k(\mathcal{L}) = k(\mathcal{L} + xu) = k(\mathcal{L} + yv).$$

In particular, if $\mathcal{L}_1$ is a finite arrangement of lines in $\mathbb{H}$ and (x, y, u, v) is an $\mathcal{L}_1$-peg with $\#(\{xu, yv\} \cap \mathcal{L}_1) = 1$, then $k(\mathcal{L}_1) = k(\mathcal{L}_1 \Delta \{xu, yv\})$ holds.

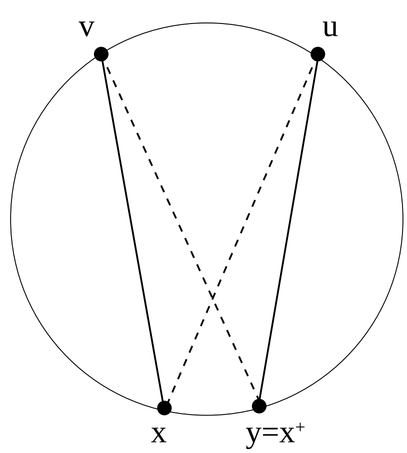

Figure 2: *The $\mathcal{L}_1$-peg in Lemma 3.1.*

Proof: We prove that $k(\mathcal{L}) = k(\mathcal{L} + xu)$ holds; the lemma then follows by symmetry upon reversing the orientation. Put $k := k(\mathcal{L})$, and suppose that $k(\mathcal{L} + xu) = k + 1$ holds. Then there must exist k pairwise intersecting lines $l_1, \ldots, l_k$ in $\mathcal{L}$ all of which intersect xu. Clearly, every one of these lines must either intersect uy, too, or it must be of the form yz for some $z \in I^+(u^+, x^-)$. Consequently, exactly all but one of these lines must intersect yu, and precisely one must be of the form yz as just described. However, since (x, y, u, v) is an $\mathcal{L}$-peg and we are assuming $yv \notin \mathcal{L}$, we see that z must, in fact, be contained in $I^+(v^+, x^-)$. But this would imply that the k lines $l_1, \ldots, l_k$ must all intersect the line xv, in contradiction to $k(\mathcal{L}) = k$. ∎

The second property of $\mathcal{L}$-pegs concerns their behavior in (n, k)-maximal arrangements of lines. Clearly, if $\mathcal{L}_1$ is an (n, k)-maximal arrangement of lines in $\mathbb{H}$ and (x, y, u, v) is an $\mathcal{L}_1$-peg, then $\mathcal{L}_1$ contains, by the previous lemma, at least one of either xu or yv. More precisely, we have

Lemma 3.2 *If $\mathcal{L}_1$ is an (n, k)-maximal arrangement of lines in $\mathbb{H}$, (x, y, u, v) is an $\mathcal{L}_1$-peg, and $\{xu, yv\} \not\subseteq \mathcal{L}_1$, then*

$$\mathcal{L}_2 := \mathcal{L}_1 \Delta \{xu, yv\}$$

is an (n, k)-maximal arrangement, and (x, y, u, v) is an $\mathcal{L}_2$-peg, too.

Proof: Put $\mathcal{L} := \mathcal{L}_1 - \{xu, yv\}$ and assume, without loss of generality (see above), that $xu \in \mathcal{L}_1$ and, hence, $\mathcal{L}_1 = \mathcal{L} + xu$ and $\mathcal{L}_2 = \mathcal{L} + yv$ holds. It is straight forward to see that $k(\mathcal{L}_2) = k(\mathcal{L}_1) = k$ and $C(\mathcal{L}_2) = C(\mathcal{L}_1) = C$ holds, too, and that (x, y, u, v) is also an $\mathcal{L}_2$-peg. So, it only remains to show that $\mathcal{L}_2$ is (n, k)-maximal.

To this end, consider first an element $w \in C - x$ with $xw \notin \mathcal{L}_1$. Then, since $\mathcal{L}_1$ is (n, k)-maximal and xw and xu do not intersect, we have

$$k(\mathcal{L}_2 + xw) \geq k(\mathcal{L} + xw) = k(\mathcal{L}_1 + xw) = k(\mathcal{L}_1) + 1.$$

So, if there were a line ab with $a, b \in C$ not contained in $\mathcal{L}_2$ and $k(\mathcal{L}_2 + ab) = k(\mathcal{L}_2)$, then we would necessarily have $ab \notin \mathcal{L}_1$, i.e $ab \neq xu$, in view of $\mathcal{L}_2 + xu = \mathcal{L}_1 + yv$ and $k(\mathcal{L}_1 + yv) = k + 1$, as well as $x \neq a, b$. In addition, if (x, y, u, v) would be an $(\mathcal{L}_2 + ab)$-peg, then

$$\mathcal{L}_1 + ab = (\mathcal{L}_2 + ab)\Delta\{xu, yv\}$$

would imply

$$k(\mathcal{L}_1 + ab) = k((\mathcal{L}_2 + ab)\Delta\{xu, yv\}) = k(\mathcal{L}_2 + ab) = k(\mathcal{L}_2) = k(\mathcal{L}_1),$$

contradicting the maximality of $\mathcal{L}_1$. So, we must have $\{a, b\} = \{y, w\}$ with

$0 < d^+(u,w) < d^+(u,v)$. However, (x,y,w,v) would then be an $(\mathcal{L}_2+ab)$-peg, implying

$$\begin{aligned} k(\mathcal{L}_2) = k(\mathcal{L}_2+ab) &= k(\mathcal{L}_2+yw) \\ &= k((\mathcal{L}+yv+yw)\Delta\{xw,yv\}) \\ &= k(\mathcal{L}+yw+xw) \\ &\geq k(\mathcal{L}+xw) \\ &= k(\mathcal{L}_2)+1, \end{aligned}$$

the final contradiction. ∎

We now define two (n,k)-maximal arrangements $\mathcal{L}_1$ and $\mathcal{L}_2$ to be *peg neighbors* if they are related to each other the way described in the last lemma.

4 The Main Result

Given a line arrangement $\mathcal{L}$ in $\mathbb{H}$ and some fixed point $a \in C = C(\mathcal{L})$, let $r(a) = r_{\mathcal{L}}(a)$ denote the number of lines in $\mathcal{L}$ that coincide with a, and define, with $n = \#C$, the sequence

$$\underline{r}(\mathcal{L},a) := (r(a), r(a^{(1)}), r(a^{(2)}), \ldots, r(a^{(n-1)})).$$

Let $\mathcal{A}(C) = \mathcal{A}_k(C)$ be the set of line arrangements $\mathcal{L}'$ with $C(\mathcal{L}') \subseteq C$ and $k(\mathcal{L}') \leq k$. We define a linear order $\succ = \succ_a$ on $\mathcal{A}(C)$ by putting $\mathcal{L}_2 \succ_a \mathcal{L}_1$ for $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{A}(C)$ if $\underline{r}(\mathcal{L}_2, a)$ is lexicographically larger than $\underline{r}(\mathcal{L}_1, a)$.

Consider a set K of k consecutive points in C. By definition of the arrangement $\mathcal{L}^* = \mathcal{L}^*(C, K)$ given in the introduction, we see that if we fix a to be the first element of the set K with respect the given orientation on C, then $\mathcal{L}^*$ is clearly the maximal element of $\mathcal{A}(C)$ with respect to $\succ$, since $r_{\mathcal{L}^*}(x) = n - 1$ holds for all $x \in K$ and since $\mathcal{L}_k$ must be contained in any (n, k)-maximal arrangement; so any $\mathcal{L}' \subseteq \mathcal{A}(C)$ with $\mathcal{L}' \succ \mathcal{L}^*$ must contain, and, hence, coincide with $\mathcal{L}^*$. Consequently, the main result of this paper, Theorem 1.1, follows from the following more explicit result:

Theorem 4.1 *Suppose that K is a set of k consecutive points in a subset C of $\mathbb{S}$ of cardinality $n > 2k$, that $\mathcal{L}^* = \mathcal{L}^*(C, K)$ is the arrangement defined in the introduction, and that a is the first element of the set K with respect to a fixed orientation of $\mathbb{H}$. If $\mathcal{L}_1 \in \mathcal{A}(C)$ is an (n, k)-maximal arrangement that is distinct from $\mathcal{L}^*$, then $\mathcal{L}_1$ is the peg-neighbor of an (n, k)-maximal arrangement $\mathcal{L}_2 \in \mathcal{A}(C)$ with $\mathcal{L}_2 \succ_a \mathcal{L}_1$.*

Proof: Since $\mathcal{L}_1$ is distinct from $\mathcal{L}^*$, we can choose some $x \in C - K$ with $xz \notin \mathcal{L}_1$ for some $z \in K = \{a, a^{(1)}, \ldots, a^{(k-1)}\}$. Choose that element x among all such elements from $C - K$ for which $d^+(a, x)$ maximal. Put $y := x^+$ and note that yz then is necessarily in $\mathcal{L}_1$ and that $d^+(z, x) > k$ must hold (see Figure 3).

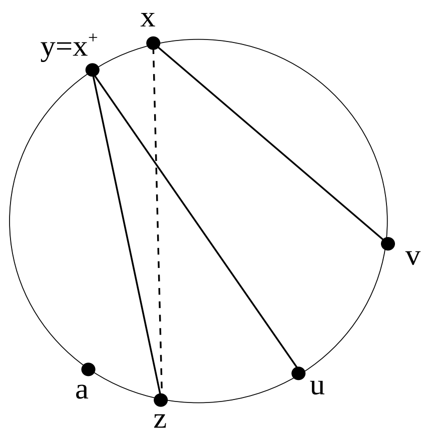

Figure 3: *Finding an $\mathcal{L}_1$-peg.*

Now, put $v := v^+(x, z)$ and $u := v^-(y, v)$. Clearly, we have $d^+(v, x) \geq k$, $d^-(v, u) \leq d^-(v, z)$, $xv \in \mathcal{L}$, $yu \in \mathcal{L}$, and

$$\mathcal{L}^-(x, v) > d^+(z, v) \geq d^+(u, v)$$

– in particular $xu \notin \mathcal{L}$ and

$$\mathcal{L}^+(y, u) \geq \mathcal{L}^-(y, v) = d^+(u, v).$$

Consequently, (x, y, u, v) is an $\mathcal{L}_1$-peg for which then $yv \in \mathcal{L}_1$ must necessarily hold, and $\mathcal{L}_2 := \mathcal{L}_1 \Delta \{xu, yv\}$ is a peg-neighbour of $\mathcal{L}_1$ with $\mathcal{L}_2 \succ_a \mathcal{L}_1$, as claimed. ∎

Remark: For future reference, note that the argument above actually yields the following.

Lemma 4.2 *As above, let $\mathcal{L}$ be a finite collection of lines in the (oriented) hyperbolic plane $\mathbb{H}$ and let $C = C(\mathcal{L})$ denote the associated cyclic set of points at infinity. Furthermore, assume that $a, x, y \in C(\mathcal{L})$ are three distinct points with $y = x^+$, $ya, xx^- \in \mathcal{L}$, and $xa \notin \mathcal{L}$. Then the four points (x, y, u, v) with $v := v_{\mathcal{L}}^+(x, a)$ and $u := v_{\mathcal{L}}^-(y, v)$ always form an $\mathcal{L}$-peg (see Figure 4).*

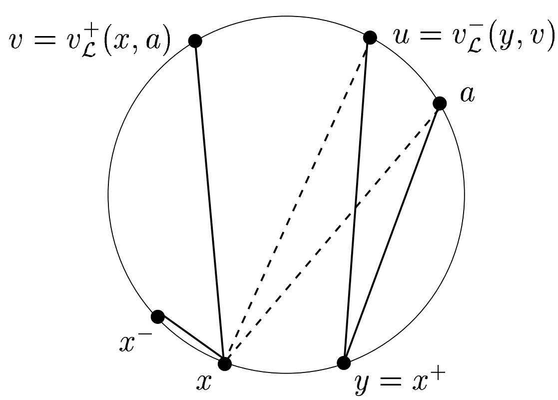

Figure 4: Forming $\mathcal{L}$-pegs.

More generally, if we have $aa^+ \in \mathcal{L}$ for all $a \in C$ and if $ab \notin \mathcal{L}$ holds for some $a, b \in C$, then the four points (x, y, u, v) with $y := v_{\mathcal{L}}^+(a, b)$, $x := y^-$, $v :=$

$v^+_{\mathcal{L}}(x,a)$ and $u := v^-_{\mathcal{L}}(y,v)$ always form an $\mathcal{L}$-peg, also denoted by $p^+_{\mathcal{L}}(a,b)$ (see Figure 5). In particular, if $\mathcal{L}$ is an (n,k)-maximal line arrangement, then $ab \notin \mathcal{L}$ always implies that the line $yv = v^+_{\mathcal{L}}(a,b)v^+_{\mathcal{L}}(v^+_{\mathcal{L}}(a,b)^-,a)$ belongs to $\mathcal{L}$.

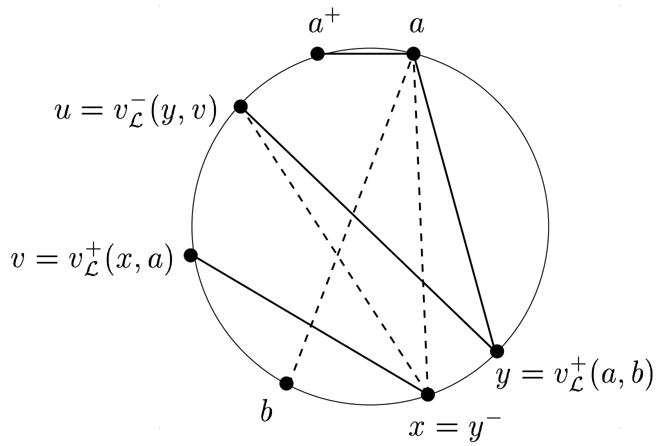

Figure 5: *The $\mathcal{L}$-peg $p^+_{\mathcal{L}}(a,b)$ in case $x \neq b$ and $u \neq a, a^+$.*

In the same vein, we can define $p^+_{\mathcal{L}}(b,a)$ as well as, using the reverse orientation, $p^-_{\mathcal{L}}(a,b)$ and $p^-_{\mathcal{L}}(b,a)$, thus producing four $\mathcal{L}$-pegs (not necessarily all distinct) from every line missing in $\mathcal{L}$ which, as above, implies that the three lines $v^+_{\mathcal{L}}(b,a)v^+_{\mathcal{L}}(v^+_{\mathcal{L}}(b,a)^-,b)$, $v^-_{\mathcal{L}}(a,b)v^-_{\mathcal{L}}(v^-_{\mathcal{L}}(a,b)^+,a)$ and $v^-_{\mathcal{L}}(b,a)v^-_{\mathcal{L}}(v^-_{\mathcal{L}}(b,a)^+,b)$ together with the line $v^+_{\mathcal{L}}(a,b)v^+_{\mathcal{L}}(v^+_{\mathcal{L}}(a,b)^-,a)$ mentioned above all must be contained in $\mathcal{L}$ whenever whenever $\mathcal{L}$ is an (n,k)-maximal line arrangment, for any n and k with $2k < n$.

5 Concluding Remarks

Finite line arrangements $\mathcal{L}$ as considered above are easily seen to correspond to *cyclic split systems* $\mathcal{S} = \mathcal{S}(\mathcal{L})$ defined on the (obviously also cyclic) set $X = X(\mathcal{L})$ of those open intervals in $\mathbb{S}$ that are connected components of $\mathbb{S} - C(\mathcal{L})$, by associating – to each line $l = ab$ in $\mathcal{L}$ – the split (or bipartition) $\{A, B\} := S(l) = S_{\mathcal{L}}(l)$ of X whose two parts A, B consist of the two complementary sets of intervals in X consisting of all connected components of $\mathbb{S} - C(\mathcal{L})$ contained in either one of the two connected components of $\mathbb{S} - \{a, b\}$. Using this simple observation, results on hyperbolic line arrangements can thus be rephrased easily in a purely combinatorial language, viz. as results regarding cyclic split systems (such as those discussed in [2, 3]), and vice versa.

Our present work arose as part of a continued study of *k-compatible split systems*, a study which was originally motivated by the fact that, due to early results by P.Buneman [1], 1-compatible split systems correspond to phylogenetic trees (see Figure 6) and that – correspondingly – more complicated split systems arising in the analysis of phylogenetic data due simply to noise that often blurs the true phylogenetic signal (or even to hybridization and horizontal gene transfer) might be classified according to the maximal number k of pairwise incompatible splits in the given system. This number had been studied by A.Karzanov in [6] who conjectured that the size of k-compatible split systems on an n-set can't be too large; see [2] – [7] for more details. In particular, in case the split systems in question are in addition *cyclic*, they are – as we have indicated above – intimately related to hyperbolic line arrangements and – translated into the language of split systems – Theorem 1.1 implies that Conjecture 1 in [2] is true, thus corroborating also the original expectations of A.Karzanov.

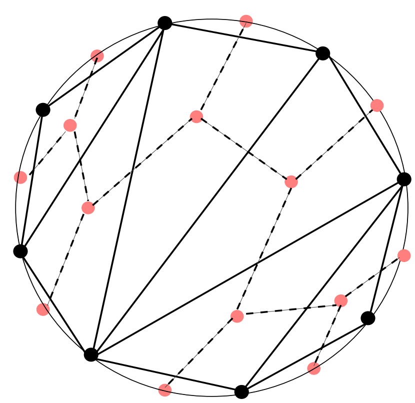

Figure 6: *A 1-compatible line arrangement and its associated "phylogenetic tree".*

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