A New Scale-invariant Geometry on L_1 Spaces

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Abstract

J.J. Nieto, A. Torres, and M.M. Vázquez-Trasande recently considered, for any $n \in \mathbb{N}$, the bivariate map

$$d : [0,1]^n \times [0,1]^n \to \mathbb{R} :$$

$$\big((p_1, \dots, p_n), (q_1, \dots, q_n)\big) \mapsto \frac{\sum_{i=1}^n |p_i - q_i|}{\sum_{i=1}^n \max\big(|p_i|, |q_i|\big)}$$

(with $d\big((p_1, \dots, p_n), (q_1, \dots, q_n)\big) := 0$ if $(p_1, \dots, p_n) = (q_1, \dots, q_n) = (0, \dots, 0)$) and showed that this map satisfies the triangle inequality and, hence, constitutes a metric. Here, we suggest to consider, more generally, for any space X with a positive measure μ, the bivariate map

$$d : L_1(X, \mu) \times L_1(X, \mu) \to \mathbb{R} : (p, q) \mapsto \frac{\int_X |p - q| d\mu}{\int_X \max\big(|p|, |q|\big) d\mu}$$

(with $d(p, q) := 0$ if $\int_X \max\big(|p|, |q|\big) d\mu = 0$) and show that, for maps $p, q \in L_1(X, \mu)^+ := \{f \in L_1(X, \mu) \mid f(x) \geq 0 \text{ for all } x \in X\}$, the length of a geodesic (relative to d) coincides with the sum

$$\ln \frac{\int_X \max(p, q) d\mu}{\int_X p \, d\mu} + \ln \frac{\int_X \max(p, q) d\mu}{\int_X q \, d\mu}$$

In particular, a continuous path $S : [0, 1] \to L_1(X, \mu)^+ : t \mapsto S_t$ is a geodesic in $L_1(X, \mu)$ relative to this metric if and only if there exists some $t_0 \in [0, 1]$ with $S_{t_0} = \max(p, q)$, and one has $S_t(x) \leq S_{t'}(x)$, for all $t, t' \in [0, t_0]$ with $t \leq t'$ and for all $t, t' \in [t_0, 1]$ with $t \geq t'$, almost all $x \in X$.

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1 Introduction

A well-known question arising in many contexts is how to quantify the (dis)similarity between any two probability distributions defined on the same set. In [1], J.J. Nieto, A. Torres, and M.M. Vázquez-Trasande suggested to use the number

$$d(p,q) := \sum_{i=1}^{n} |p_i - q_i| / \sum_{i=1}^{n} \max(p_i, q_i)$$

as a dissimilarity defined on the pairs (p,q) of probability distributions

$$p := (p_1, \ldots, p_n), q := (q_1, \ldots, q_n)$$

defined on the set $X := \{1, \ldots, n\}$, and showed that the triangle inequality holds indeed for this specific dissimilarity that, henceforth, we will denote by $d_{NTV}(p,q)$.

They proposed this definition in the context of analysing sequence profiles of protein- or DNA-sequence families and indicated that they expect it to lead to biologically more plausible results than using the dissimilarity measure d_1 defined by

$$d_1(p,q) := \sum_{i=1}^{n} |p_i - q_i|$$

(which nobody seems to like) or the similarity measure

$$s_2(p,q) := \sum_{i=1}^{n} \sqrt{p_i q_i},$$

i.e., the cosine of the two vectors $(\sqrt{p_1}, \ldots, \sqrt{p_n})$ and $(\sqrt{q_1}, \ldots, \sqrt{q_n})$ of euclidean length 1 (which appears to be quite popular).

Clearly, the definition of the metric d_{NTV} can be extended, wihout any additional effort, to work for any two L_1-integrable functions p, q defined on some measurable space X with a positive measure μ: Just put

$$d_{NTV}(p,q) := \frac{\int_X |p(x) - q(x)| d\mu}{\int_X \max(|p(x)|, |q(x)|) d\mu}$$

in case $\int_X \max(|p(x)|, |q(x)|) d\mu \neq 0$, and $d_{NTV}(p,q) := 0$ else so that $d_{NTV}(p,q) = 0$ holds for some L_1-integrable functions p, q defined on some X if and only if they represent the same element in the Banach space $L_1(X, \mu)$ of L_1-integrable functions defined on $X = (X, \mu)$.

It is easy to see that the method presented in [2] for proving that the triangle inequality holds in the case studied in [1] (i.e., essentially the case where X is finite), yields that it also holds in this much more general situation, thus establishing that the map

$$d_{NTV} : L_1(X, \mu) \times L_1(X, \mu) \to \mathbb{R} : (p,q) \mapsto d_{NTV}(p,q)$$

is indeed a well-defined metric on the space $L_1(X,\mu)$.

In [2], it was also observed that this metric is far from being "geodesic", i.e. that it differs considerably from the metric $d^{(0)}_{NTV}$ where, for any metric d defined on a set L, we denote by $d^{(0)}$ the map from $L \times L$ into $\mathbb{R} \cup \{+\infty\}$ that is defined as follows:

For any $\varepsilon > 0$ and $p, q \in L$, put

$$d^{(\varepsilon)}(p,q) := \inf\big(\sum_{i=1}^{N} d(r_{i-1}, r_i)\big)$$

where the infimum is taken over all finite sequences $(r_0, \ldots, r_N)$ of points in L with $r_0 := p$, $r_N := q$, and $d(r_{i-1}, r_i) < \varepsilon$ holds for all $i = 1, \ldots, N$ (with $d^{(\varepsilon)}(p,q) := \infty$, of course, if no such sequence exists), and put

$$d^{(0)}(p,q) := \sup_{\varepsilon>0}\big(d^{(\varepsilon)}(p,q)\big) \ .$$

Note, by the way, that

$$d^{(\varepsilon)}(p,q) \le d^{(\varepsilon')}(p,q)$$

holds for all $p, q \in L$ and $\varepsilon, \varepsilon' > 0$ with $\varepsilon' < \varepsilon$, implying that $d^{(0)}(p,q)$ can also be defined as the limit $\lim_{\varepsilon \to 0} d^{(\varepsilon)}(p,q)$.

Here, we will compute the $d^{(0)}_{NTV}$-distance for any two elements in the convex cone

$$L_1(X,\mu)^+ := \{p \in L_1(X,\mu) \,|\, p(x) \ge 0 \text{ for (almost) all } x \in X\}$$

of non-negative elements in $L_1(X,\mu)$. More specifically, we will establish:

Theorem 1 *Given any space X with a positive measure μ and any two elements $p, q \in L_1(X,\mu)^+$, we have*[1]

$$d^{(0)}_{NTV}(p,q) = \ln \frac{\int_X \max(p,q) d\mu}{\int_X p \, d\mu} + \ln \frac{\int_X \max(p,q) d\mu}{\int_X q \, d\mu} \ .$$

In the next section, we'll introduce some useful definitions and collect some simple relevant facts, in Section 3 we establish Theorem 1, and in the last section, we will discuss the structure of geodesics in $L_1(X,\mu)^+$ relative to the NTV-metric.

[1] Here, given any two non-negative real numbers a and b, we put $a/b := \infty$ in case $a > 0$ and $b = 0$, and $a/b := 1$ in case $a = b = 0$.

2 Some Basic Facts and Definitions

Given any two real-valued maps $p, q : X \to \mathbb{R}$,

- we'll write $p \leq q$ if $p(x) \leq q(x)$ holds for almost all $x \in X$,
- we'll write $p < q$ if $p \leq q$ holds and the set $\{x \in X : p(x) < q(x)\}$ is not a set of measure 0, i.e., if $p \leq q$, but not $q \leq p$ holds,
- we define p and q to be *comparable* if either $p \leq q$ or $q \leq p$ holds,
- and we define p and q to be *incomparable* if they are not comparable.

As all this holds for any two maps $p, q : X \to \mathbb{R}$ if and only if it holds for any two maps $p', q' : X \to \mathbb{R}$ that differ from p and q, respectively, on a set of measure 0 only, we will use the same terminology also if p and q are just elements in $L_1(X, \mu)$.

Further, we put

$$\max(p, q)(x) := \max\big(p(x), q(x)\big)$$

and

$$\min(p, q)(x) := \min\big(p(x), q(x)\big),$$

for any two maps $p, q : X \to \mathbb{R}$ and all $x \in X$. And we define $\max(p, q)$ (or $\min(p, q)$, respectively) for any two elements $p, q \in L_1(X, \mu)$ to be the (well-defined) element in $L_1(X, \mu)$ represented by the map $\max(p', q')$ (or $\min(p', q')$) for any two maps $p', q' : X \to \mathbb{R}$ representing the elements p, q in $L_1(X, \mu)$. Note that $p, q \leq r$ holds for some real-valued maps $p, q, r : X \to \mathbb{R}$, or elements of $L_1(X, \mu)$, if and only if $\max(p, q) \leq r$ holds while $r \leq p, q$ holds for p, q, r as above if and only if $r \leq \min(p, q)$ holds.

Next, we define the map $|p| : X \to \mathbb{R}$ for any map $p : X \to \mathbb{R}$, or element p of $L_1(X, \mu)$, by

$$|p|(x) := |p(x)| \;\; (x \in X),$$

and we put

$$||p|| := \int_X |p| \, d\mu$$

for any $p \in L_1(X, \mu)$ so that

$$d_{NTV}(p, q) = \frac{\int_X |p(x) - q(x)| d\mu}{\int_X \max(|p(x)|, |q(x)|) d\mu} = \frac{||p - q||}{|| \max(|p|, |q|)||}$$

holds for all $p, q \in L_1(X, \mu)$,

$$d_{NTV}(p, q) = \frac{||p - q||}{|| \max(p, q)||}$$

holds for all $p, q \in L_1(X, \mu)^+$, and

$$||p - q|| = ||p|| - ||q||$$

as well as

$$d_{NTV}(p, q) = \frac{||q|| - ||p||}{||q||}$$

holds for all $p, q \in L_1(X, \mu)^+$ with $p \leq q$. In turn, this implies that

$$d_{NTV}(p, r) = 1 - \frac{||p||}{||r||} \leq 1 - \frac{||p||}{||q||} = d_{NTV}(p, q)$$

as well as

$$d_{NTV}(r, q) = 1 - \frac{||r||}{||q||} \leq 1 - \frac{||p||}{||q||} = d_{NTV}(p, q)$$

holds for all $p, q, r \in L_1(X, \mu)^+$ with $p \leq r \leq q$. Note also that

$$\begin{aligned} |a - b| &= (\max(a, b) - a) + (\max(a, b) - b) \\ &= |\max(a, b) - a)| + |\max(a, b) - b| \end{aligned}$$

holds for all $a, b \in \mathbb{R}$. So,

$$||p - q|| = ||\max(p, q) - p|| + ||\max(p, q) - q||$$

and, hence,

$$\begin{aligned} (1) \qquad d_{NTV}(p, q) &= \frac{||p - q||}{||\max(p, q)||} \\ &= \frac{||\max(p, q) - p|| + ||\max(p, q) - q||}{||\max(p, q)||} \\ &= d_{NTV}(\max(p, q), p) + d_{NTV}(\max(p, q), q) \end{aligned}$$

must also hold for any two elements $p, q \in L_1(X, \mu)^+$.

Finally, we define a map

$$S : [a, b] \to L_1(X, \mu) : t \mapsto S_t$$

from a real interval $[a, b]$ into $L_1(X, \mu)$ to be monotonically increasing (or decreasing) if $S_t \leq S_{t'}$ (or $S_t \geq S_{t'}$, respectively) holds for all $t, t' \in [a, b]$ with $t < t'$. And we define S to be monotone if it is either monotonically increasing or monotonically decreasing. In the last section, we will establish the following result:

Theorem 2 *Given any space X with a positive measure μ and any two elements $p, q \in L_1(X,\mu)^+$, a continuous map*

$$S : [0,1] \to L_1(X,\mu) : t \mapsto S_t$$

with $S_0 = p$ and $S_1 = q$ is a geodesic path from p to q in $L_1(X,\mu)^+$ relative to the NTV-metric if and only if there exists some $t_0 \in [0,1]$ with $S_{t_0} = \max(p,q)$, S restricted to $[0,t_0]$ is monotonically increasing, and S restricted to $[t_0,1]$ is monotonically decreasing.

3 Proof of Theorem 1

Asssume that p and q are two arbitrary elements $L_1(X,\mu)^+$, put $m := \max(p,q)$, and choose some positive $\varepsilon < 1$. As $d_{NTV}(0,p) = 1$ holds for all non-zero elements $p \in L_1(X,\mu)$, we have

$$d_{NTV}^{(\varepsilon)}(0,p) = d_{NTV}^{(0)}(0,p) = \infty$$

for all such p.

So, we may assume from now on that $p, q \neq 0$ holds. To show that

$$d_{NTV}^{(\varepsilon)}(p,q) \leq \ln \frac{||m||}{||p||} + \ln \frac{||m||}{||q||}$$

holds in this case, consider, for some large integer N, the sequence $r_0, r_1, \ldots, r_{2N}$ of non-zero elements in $L_1(X,\mu)^+$ defined by

$$r_i := p + \frac{i}{N}(m-p)$$

and

$$r_{2N-i} := q + \frac{i}{N}(m-q)$$

for $i = 0, 1, \ldots, N$. Clearly, we have

$$||r_i|| = ||p|| + \frac{i}{N}||m-p|| = ||p|| + \frac{i}{N}(||m|| - ||p||) \geq ||p||$$

and

$$||r_{2N-i}|| = ||q|| + \frac{i}{N}||m-q|| = ||q|| + \frac{i}{N}(||m|| - ||q||) \geq ||q||$$

for all $i = 0, 1, \ldots, N$. Hence, we have

$$d_{NTV}(r_{i-1}, r_i) = \frac{||r_i|| - ||r_{i-1}||}{||r_i||} = \frac{||m-p||}{N\,||r_i||} \leq \frac{||m||}{N\,||p||} \leq \varepsilon$$

and

$$d_{NTV}(r_{2N-i}, r_{2N-i+1}) = \frac{||r_{2N-i}|| - ||r_{2N-i+1}||}{||r_{2N-i}||} = \frac{||m-q||}{N\,||r_{2N-i}||} \leq \frac{||m||}{N\,||q||} \leq \varepsilon$$

for all $i = 0, 1, \ldots, N$ provided $N \geq \frac{||m||}{\varepsilon||p||}$ and $N \geq \frac{||m||}{\varepsilon||q||}$ holds. Moreover, the well-known fact that

$$\frac{a-b}{a} \leq \ln\frac{a}{b}$$

holds for any two positive numbers a, b with $a \geq b$, implies

$$\begin{aligned}\sum_{i=1}^{2N} d_{NTV}(r_{i-1}, r_i) &= \sum_{i=1}^{N} \frac{||r_i|| - ||r_{i-1}||}{||r_i||} + \sum_{i=1}^{N} \frac{||r_{2N-i}|| - ||r_{2N-i+1}||}{||r_{2N-i}||} \\ &\leq \sum_{i=1}^{N} \ln\frac{||r_i||}{||r_{i-1}||} + \sum_{i=1}^{N} \ln\frac{||r_{2N-i}||}{||r_{2N-i+1}||} \\ &= \ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||}\end{aligned}$$

and, therefore, also

$$d_{NTV}^{(\varepsilon')}(p, q) \leq \ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||}$$

for all $\varepsilon' \in [0, 1)$, as claimed.

To show that also

$$d_{NTV}^{(0)}(p, q) \geq \ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||}$$

holds, it will suffice to show that

$$d_{NTV}(R) \geq \ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||} - \varepsilon(\frac{1}{||p||} + \frac{1}{||q||})$$

holds for the *distance sum*

$$d_{NTV}(R) := \sum_{i=1}^{N} d_{NTV}(r_{i-1}, r_i)$$

of any *ε-sequence* R for p and q, i.e. any finite sequence $R = (r_0, r_1, \ldots, r_N)$ of elements in $L_1(X, \mu)$ with $r_0 = p$, $r_N = q$ and $d_{NTV}(r_{i-1}, r_i) \leq \varepsilon$ for all $i = 1, \ldots, N$.

So, assume that we are given an ε-sequence $R = (r_0, r_1, \ldots, r_N)$ for p and q. We will construct further ε-sequences $R', R'', \ldots$ for p and q from R with $d_{NTV}(R) \geq d_{NTV}(R') \geq d_{NTV}(R'') \geq \ldots$ such that the distance sum of (at least) the last of these ε-sequences is larger than $\ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||} - \varepsilon(\frac{1}{||p||} + \frac{1}{||q||})$.

To this end, note first that — in view of the fact that $|\,|a| - |b|\,| \leq |a - b|$ holds for all $a, b \in \mathbb{R}$ — we have $||\,|r| - |s|\,|| \leq ||\,r - s||$ and, hence, also

$$d_{NTV}(r, s) = \frac{||r - s||}{||\max(|r|, |s|)||} \geq \frac{||\,|r| - |s|\,||}{||\max(|r|, |s|)||} = d_{NTV}(|r|, |s|)$$

for all $r, s \in L_1(X, \mu)$. So, replacing each r_i in the above sequence by $r'_i := |r_i|$ yields a new ε-sequence $r'_0, r'_1, \ldots, r'_N$ for p and q whose members are now contained in $L_1(X, \mu)^+$, and for which $d_{NTV}(r'_{i-1}, r'_i) \leq d_{NTV}(r_{i-1}, r_i)$ holds for all $i = 1, \ldots, N$. So, we may assume from now on that $r_i \in L_1(X, \mu)^+$ holds for all $i = 0, \ldots, N$.

Next, we note that, given any ε-sequence $r_0, r_1, \ldots, r_N$ for p and q with $r_0, r_1, \ldots, r_N \in L_1(X, \mu)^+$, we can find another such sequence $r'_0, r'_1, \ldots, r'_{2N}$ by inserting into the given sequence $r_0, r_1, \ldots, r_N$ the map $\max(r_{i-1}, r_i)$ between r_{i-1} and r_i for all $i = 1, \ldots, N$, i.e., by putting $r'_{2i} := r_i$ for each $i = 0, \ldots, N$ and $r'_{2i-1} := \max(r_{i-1}, r_i)$ for each $i = 1, \ldots, N$. Clearly, any two consecutive elements in the new sequence are comparable while (1) implies that

$$\sum_{i=1,\ldots,2N} d_{NTV}(r'_{i-1}, r'_i) = \sum_{i=1,\ldots,N} d_{NTV}(r_{i-1}, r_i)$$

holds. Consequently, we can restrict our attention to ε-sequences for which the maps r_{i-1} and r_i are comparable for all $i = 1, \ldots, N$.

Next, note that $\max(p, q) + \min(p, q) = p + q$ and, hence,

$$||\max(p, q)|| + ||\min(p, q)|| = ||p|| + ||q||$$

holds for all $p, q \in L_1(X, \mu)^+$ which in turn implies that also

$$d_{NTV}\big(p, \max(p, q)\big) = \frac{||\max(p, q)|| - ||p||}{||\max(p, q)||} = \frac{||q|| - ||\min(p, q)||}{||\max(p, q)||}$$

$$\leq \frac{||q|| - ||\min(p, q)||}{||q||} = d_{NTV}(\min(p, q), q)$$

as well as

$$d_{NTV}(\max(p, q), q) \leq d_{NTV}\big(p, \min(p, q)\big)$$

holds for all non-zero elements $p, q \in L_1(X, \mu)^+$.

Thus, replacing any r_i ($i = 1, \ldots, N-1$) in such an ε-sequence for which $r_i \leq r_{i-1}$ and $r_i \leq r_{i+1}$ holds, first by $\min(r_{i-1}, r_{i+1})$ and then by $\max(r_{i-1}, r_{i+1})$, one sees that we can also restrict our attention to ε-sequences $r_0, r_1, \ldots, r_N$ for p and q for which some $k \in \{0, 1, \ldots, N\}$ with

$$r_0 \leq r_1 \leq \cdots \leq r_k \geq r_{k+1} \geq \cdots \geq r_N$$

exists which implies in particular that $m \leq r_k$ must hold.

Moreover, using now the also very well-known fact that

$$\begin{aligned}\frac{a-b}{a} &= \frac{a-b}{b} - \frac{(a-b)^2}{ab} \\ &\geq \ln\frac{a}{b} - \frac{(a-b)^2}{ab} \\ &= \ln\frac{a}{b} - (a-b)(\frac{1}{b} - \frac{1}{a})\end{aligned}$$

holds for any two positive numbers a, b with $a \geq b$, we see that

$$\begin{aligned}d_{NTV}(R) &= \sum_{i=1}^{N} d_{NTV}(r_{i-1}, r_i) = \sum_{i=1}^{k} \frac{||r_i|| - ||r_{i-1}||}{||r_i||} + \sum_{i=k+1}^{N} \frac{||r_{i-1}|| - ||r_i||}{||r_{i-1}||} \\ &\geq \sum_{i=1}^{k} \ln\frac{||r_i||}{||r_{i-1}||} - \varepsilon \sum_{i=1}^{k} (\frac{1}{||r_{i-1}||} - \frac{1}{||r_i||}) \\ &\quad + \sum_{i=k+1}^{N} \ln\frac{||r_{i-1}||}{||r_i||} - \varepsilon \sum_{i=k+1}^{N} (\frac{1}{||r_i||} - \frac{1}{||r_{i-1}||}) \\ &= \ln\frac{||r_k||}{||p||} - \varepsilon(\frac{1}{||p||} - \frac{1}{||r_k||}) + \ln\frac{||r_k||}{||q||} - \varepsilon(\frac{1}{||q||} - \frac{1}{||r_k||}) \\ &> \ln\frac{||m||}{||p||} + \ln\frac{||m||}{||q||} - \varepsilon(\frac{1}{||p||} + \frac{1}{||q||})\end{aligned}$$

holds as claimed.

4 Proof of Theorem 2

To establish Theorem 2, note first that Theorem 1 implies that

$$d_{NTV}^{(0)}(p,q) = d_{NTV}^{(0)}(p,r) + d_{NTV}^{(0)}(r,q)$$

holds for any three elements $p, q, r \in L_1(X,\mu)^+$ with $p \leq r \leq q$. Further, assuming that p, q, r are three non-zero elements in $L_1(X,\mu)^+$ with $r \leq p, q$ and putting $m := \max(p,q)$, we have $m + r = \max(p+r, q+r) \leq p+q$ and, therefore, also $||m|| + ||r|| \leq ||p|| + ||q||$ as well as

$$||p||\,||q|| - ||m||\,||r|| = (||p|| - ||r||)\,(||q|| - ||r||) + (||p|| + ||q|| - ||m|| - ||r||)||r|| \geq 0$$

which implies that

$$d_{NTV}^{(0)}(p,m) = \ln\frac{||m||}{||p||} \leq \ln\frac{||q||}{||r||} = d_{NTV}^{(0)}(q,r)$$

and

$$d_{NTV}^{(0)}(q,m) = \ln\frac{||m||}{||q||} \leq \ln\frac{||p||}{||r||} = d_{NTV}^{(0)}(p,r)$$

must hold, and that equality holds for at least one of these two inequalities if and only if equality holds for both if and only if $||p|| + ||q|| - ||m|| - ||r|| = 0$ and $||p|| = ||r||$ or $||q|| = ||r||$ and, therefore, if and only if $p = r \leq q$ or $q = r \leq p$ holds.

Thus, the identity $d^{(0)}_{NTV}(p,q) = d^{(0)}_{NTV}(p,m) + d^{(0)}_{NTV}(q,m)$ implies that one has $d^{(0)}_{NTV}(p,q) = d^{(0)}_{NTV}(p,r) + d^{(0)}_{NTV}(r,q)$ if and only if $d^{(0)}_{NTV}(p,r)$ coincides with $d^{(0)}_{NTV}(q,m)$ and $d^{(0)}_{NTV}(q,r)$ coincides with $d^{(0)}_{NTV}(p,m)$ if and only if $p = r \leq q$ or $q = r \leq p$ holds.

Consequently, putting $m_p := \max(p,r)$, $m_q := \max(q,r)$, and $\overline{m} = \max(m_p, m_q)$ for some arbitrary elements $p, q, r \in L_1(X,\mu)^+$, we have

$$d^{(0)}_{NTV}(p,q) = d^{(0)}_{NTV}(p,r) + d^{(0)}_{NTV}(r,q)$$

for these three elements $p, q, r \in L_1(X,\mu)^+$ if and only if

$$\begin{aligned}
d^{(0)}_{NTV}(p,q) &= d^{(0)}_{NTV}(p,m) + d^{(0)}_{NTV}(m,q) \\
&= d^{(0)}_{NTV}(p,m_p) + d^{(0)}_{NTV}(m_p,r) + d^{(0)}_{NTV}(r,m_q) + d^{(0)}_{NTV}(m_q,q) \\
&\geq d^{(0)}_{NTV}(p,m_p) + d^{(0)}_{NTV}(m_p,m_q) + d^{(0)}_{NTV}(m_q,q) \\
&= d^{(0)}_{NTV}(p,m_p) + d^{(0)}_{NTV}(m_p,\overline{m}) + d^{(0)}_{NTV}(\overline{m},m_q) + d^{(0)}_{NTV}(m_q,q) \\
&= d^{(0)}_{NTV}(p,\overline{m}) + d^{(0)}_{NTV}(\overline{m},q) \\
&= d^{(0)}_{NTV}(p,m) + 2d^{(0)}_{NTV}(m,\overline{m}) + d^{(0)}_{NTV}(m,q)
\end{aligned}$$

or, equivalently, if and only if

$$d^{(0)}_{NTV}(m,\overline{m}) = 0 \text{ and } d^{(0)}_{NTV}(m_p,r) + d^{(0)}_{NTV}(r,m_q) = d^{(0)}_{NTV}(m_p,m_q)$$

holds. However, we have $d^{(0)}_{NTV}(m,\overline{m}) = 0$ if and only if $r \leq m$ holds. And we have just seen that $d^{(0)}_{NTV}(m_p,r) + d^{(0)}_{NTV}(r,m_q) = d^{(0)}_{NTV}(m_p,m_q)$ holds if and only if $m_p = r \leq m_q$ or $m_q = r \leq m_p$ holds. Thus, we have $d^{(0)}_{NTV}(p,q) = d^{(0)}_{NTV}(p,r) + d^{(0)}_{NTV}(r,q)$ for some non-zero elements p, q, r in $L_1(X,\mu)^+$ if and only $p \leq r \leq m$ or $q \leq r \leq m$ holds.

It is easy to see that Theorem 2 is directly equivalent to this assertion.

References

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