Semiharmonic graphs with fixed cyclomatic number *

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Abstract

Let the trunk of a graph G be the graph obtained by removing all leaves of G. We prove that, for every integer $c \geq 2$, there are at most finitely many trunks of *semiharmonic* graphs with cyclomatic number c — in contrast to the fact established by the last two of the present authors in their paper *Semiharmonic Bicyclic Graphs* (this journal) that there are infinitely many connected semiharmonic graphs with given cyclomatic number. Further, we prove that there are at most finitely many semiharmonic but not *almost semiregular* graphs with cyclomatic number c.

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1 Introduction

We consider locally finite simple graphs $G = (V, E)$ without isolated vertices and with vertex set $V = V_G$, edge set $E = E_G \subseteq \binom{V}{2}$, adjacency matrix $A = A_G$ (indexed by the elements of V), and *trunk*

$$\mathrm{sk}(G) := (V_G - V_1(G), E_G - E_1(G))$$

where $V_1(G)$ denotes the set of leaves of G and $E_1(G)$ denotes the set of *pending* edges of G, i.e. the set of edges of G that are incident with a leaf. The degree of a vertex $v \in V$ and the set of neighbors of v are denoted by $\mathrm{d}(v) = \mathrm{d}_G(v)$ and $N(v) = N_G(v)$, respectively. The number of walks of length k of G starting at v is denoted by $\mathrm{d}_k(v)$. Clearly, one has

$$\mathrm{d}_0(v) = 1,\ \mathrm{d}_1(v) = \mathrm{d}(v),\ \text{and } \mathrm{d}_{k+1}(v) = \mathrm{d}_k(N(v))$$

for every $v \in V$ and $k \in \mathbf{N}_0$ where we use the convention to denote, for any set X, any finite subset Y of X, and any map $f : X \to \mathbf{R}$, by $f(Y)$ the sum $\sum_{y \in Y} f(y)$.

G is called *regular* if there exists a constant r such that $\mathrm{d}(v) = r$ holds for every $v \in V$ in which case G is also called r-regular. Further: G is called a, b*-semiregular* for some integers $a, b \in \mathbf{N}$ if $\{\mathrm{d}(v), \mathrm{d}(w)\} = \{a, b\}$ holds for all edges $\{v, w\} \in E$; G is called *almost semiregular* if there exist constants $a, b \in \mathbf{R}$ and a bipartition $V = V^+ \cup V^-$ of its vertex set V (with $E \subseteq \{\{v^+, v^-\} | v^+ \in V^+, v^- \in V^-\}$, as usual) such that $\mathrm{d}(v^+) = a$ and $\mathrm{d}_2(v^+) = ab$ holds for every vertex $v^+ \in V^+$ in which case G is also called *almost a,b-semiregular* and the set V^+ is called a *constant part* of G; G is called *harmonic* if there exists a constant μ such that $\mathrm{d}_2(v) = \mu\mathrm{d}(v)$ holds for every $v \in V$ in which case G is also called $\mu-$harmonic; and G is called *semiharmonic* if there exists a constant μ such that $\mathrm{d}_3(v) = \mu\mathrm{d}(v)$ holds for every $v \in V$ in which case G is also called μ-semiharmonic. Every μ-harmonic

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graph is also μ^2-semiharmonic in view of $\mathrm{d}_3(v) = \mathrm{d}_2(N(v)) = \mu\mathrm{d}_1(N(v)) = \mu\mathrm{d}_2(v) = \mu^2\mathrm{d}_1(v)$. Similarly, almost semiregular graphs are halfway between semiregular and semiharmonic graphs: Indeed, every a,b-semiregular graph is almost a,b-semiregular and every almost a,b-semiregular graph is ab-semiharmonic since $\mathrm{d}_2(v^-) = a\,\mathrm{d}(v^-)$ and $\mathrm{d}_3(v^-) = ab\,\mathrm{d}(v^-)$ holds for every $v^- \in V^-$, and $\mathrm{d}_3(v^+) = a\,\mathrm{d}_2(v^+) = a\,(ab) = ab\,\mathrm{d}(v^+)$ holds for every $v^+ \in V^+$. A connected semiharmonic graph that is not harmonic will be called *strictly semiharmonic.*

Classification of harmonic and semiharmonic graphs according to their cyclomatic number became of interest recently. All (finite or infinite) harmonic trees were constructed in [4]. All finite harmonic graphs with up to four independent cycles were characterized in [1] where it was also shown that, while the number of finite harmonic trees is infinite, the number of finite harmonic graphs with a fixed positive cyclomatic number is finite.

Semiharmonic trees and unicyclic graphs were characterized in [2], and semiharmonic bicyclic graphs were determined in [5]. It turned out that, while there are infinitely many trunks (up to isomorphism) of semiharmonic unicyclic graphs, there are only finitely many trunks of semiharmonic bicyclic graphs. We are interested in this contrast and prove in Section 2 that, for every integer $c \geq 2$, there are at most finitely many trunks of semiharmonic graphs with cyclomatic number c.

In [5], the following theorem has been established:

Theorem 1 *Let G be an almost semiregular graph with constant part V^+ and consider, for any $k \in \mathbf{N}$, the graph $G^{+k} = G^{+k}(V^+)$ obtained by attaching k pendant vertices to each vertex of V^+. Then G^{+k} is almost semiregular (and thus semiharmonic).*

This theorem was used in [5] to show that the number of finite semiharmonic graphs with a fixed positive cyclomatic number is infinite. Remarkably, as will be shown here in Section 3, this construction gives almost all such graphs, because it will be shown there that, for every integer $c \geq 2$, there are at most finitely many semiharmonic graphs with cyclomatic number c that are not almost semiregular.

For our proofs, we'll need a corollary of the following result a proof of which can be found in [3]:

Lemma 1 *Given a locally finite semiharmonic graph G and two vertices v, v' that are both adjacent to a third vertex u, one has $\mathrm{d}_2(v)\mathrm{d}(v') = \mathrm{d}_2(v')\mathrm{d}(v)$.*

Corollary 1 *Any finite, connected, and strictly semiharmonic graph G is bipartite and the average degree*

$$\mathrm{ad}(v) := \frac{\mathrm{d}_2(v)}{\mathrm{d}(v)}$$

of the neighbours of a vertex $v \in V$ is constant on both bipartite classes of G.

In other words, a finite connected graph G is semiharmonic if and only if the set

$$\mathrm{ad}(G) := \{\mathrm{ad}(v) \mid v \in V\}$$

coincides with the set $\mathrm{ad}(e) := \{\mathrm{ad}(v) \mid v \in e\}$ for every edge $e \in E$, and G is strictly semiharmonic if and only if, in addition, $\mathrm{ad}(G)$ has cardinality exactly 2. We mention one more

Corollary 2 *Any strictly semiharmonic connected finite graph $G = (V, E)$ with bipartition $V = V^+ \cup V^-$ and average degrees $\mathrm{ad}(v^+) = \lambda^+$ for $v^+ \in V^+$ and $\mathrm{ad}(v^-) = \lambda^-$ for $v^- \in V^-$ for which every vertex $v^+ \in V^+$ is adjacent to a leaf is almost semiregular.*

Proof If $w \in V^-$ is a leaf that is adjacent to a vertex $v \in V^+$, one has $\mathrm{d}(v) = \mathrm{d}_2(w) = \lambda^-\mathrm{d}(w) = \lambda^-$. Thus, if every $v \in V^+$ is adjacent to a leaf, $\mathrm{d}(v)$ is constant on V^+ which in turn implies that G is indeed almost semiregular (with V^+ as a constant part). ■

2 Trunks of semiharmonic graphs

We define a vertex v in a graph G to be a *bud*, if v is not a leaf, yet all neighbors of v except at most one are leaves, and we define v to be a *knob* if exactly two of its neighbours are not leaves. Further, we define the graph $T_{a,b}$ for any $a, b \in \mathbf{N}$ to be the tree that contains a "central" vertex v of degree a all of whose neighbors are leaves in case $b = 1$ and buds of degree b in case $b > 1$. Clearly, any such tree is $(a + b - 1)$-semiharmonic. A proof of the following lemma can be found in [2]:

Lemma 2 *Let G be a connected semiharmonic graph.*

(i) If G contains a bud, it is a finite tree.

(ii) If G is a finite tree, there exist $a, b \in \mathbf{N}$ with $G \simeq T_{a,b}$.

For $a, k \in \mathbf{N}$ with $k \geq 2$, let $M_a^{2k} = (V_a^{2k}, E_a^{2k})$ denote the connected graph containing a cycle $(v_1, \ldots, v_{2k} = v_0, v_1)$ of length $2k$ of knobs, alternatingly of degree $2 + a$ and 2, i.e. with $\mathrm{d}(v_{2i-1}) = 2 + a$, $\mathrm{d}(v_{2i}) = 2$ for $i = 1, \ldots, k$, and every neighbor of v_{2i-1} that is distinct from v_{2i-2} and v_{2i} being a leaf. Then M_a^{2k} is a unicyclic $(4 + a)$-semiharmonic graph, and every finite and strictly semiharmonic unicyclic graph is of this form (cf. [2]).

A proof of the following lemma can be found in [5]:

Lemma 3 *Let $G = (V, E)$ be a finite and strictly semiharmonic graph, let v_0, v_1, v_2, v_3, v_4 be a path or a cycle (if $v_0 = v_4$) of G such that v_1, v_2, and v_3 are knobs, and let $a_i := \mathrm{d}(v_i) - 2$ $(i = 1, 2, 3)$ denote the number of leaves adjacent to v_i. Then either one of the following three assertions holds:*

(i) $a_2 > 0$ and $G \cong M_{a_2}^{2k}$ for some $k \in \mathbf{N}$,

(ii) $a_2 = 0$, $a_1 = a_3 > 0$, and v_0, v_4 are not adjacent to a leaf,

(iii) $a_1 = a_2 = a_3 = 0$, and G is obtained from a $\mathrm{d}(v_0)$-regular multigraph by subdividing every edge by three new vertices.

Theorem 2 *For every integer $c \geq 2$, there are only finitely many (isomorphism classes of) trunks of connected semiharmonic graphs with cyclomatic number c.*

Proof Let G be a connected semiharmonic graph with cyclomatic number $c = c(G)$. If m is the number of edges and n the number of vertices of G, we have $c = m - n + 1$.

Since there are at most finitely many harmonic graphs with cyclomatic number c (cf. [1]) and, hence, only finitely many trunks of such graphs, we will further suppose that G is strictly semiharmonic.

Let $G' := \mathrm{sk}(G)$ denote the trunk of G. Clearly, G does not contain a bud (otherwise, G would be a tree by Lemma 2) and thus, all vertices of G' have degree at least 2. Further, G' also has cyclomatic number c since the cyclomatic number does not change by removing leaves. Therefore, $c = m' - n' + 1$, where m' is the number of edges, and n' is the number of vertices of G'.

We show that G does not contain a path $v_0, v_1, \ldots, v_5$ (or cycle if $v_0 = v_5$) for which the vertices $v_1, \ldots, v_4$ all are knobs. Otherwise, put $a_i := \mathrm{d}(v_i) - 2$ for $i = 1, \ldots, 4$ and note that $a_2 = a_3 = 0$ must hold in view of Lemma 3 and $c(G) > 1$. However, using Lemma 3 again, this would imply that also $a_1 = a_4 = 0$ must hold which in turn would imply that G is obtained from a $\mathrm{d}(v_1) = 2$-regular multigraph by subdividing every edge by three new vertices which is impossible, again in view of $c(G) > 1$.

Let p denote the number of vertices of G' having degree 2, and let Q denote the set of vertices of G' having degree larger than 2. We have

$$2m' = \sum_{u \in V(G')} \mathrm{d}_{G'}(u) = 2p + \sum_{u \in Q} \mathrm{d}_{G'}(u),$$

while, on the other hand, $m' = n' - 1 + c$ holds. Since $n' = p + \#Q$, we get

$$\sum_{u \in Q} \mathrm{d}_{G'}(u) = 2\#Q - 2 + 2c. \tag{1}$$

Moreover, since $\mathrm{d}_{G'}(u) \geq 3$ for all $u \in Q$, we also get

$$\#Q \leq -2 + 2c. \tag{2}$$

Further, contracting all vertices of degree 2 in G', we get a (multi)graph $G'' = (V'', E'')$ with vertex set $V'' = Q$ and cyclomatic number $c = \#E'' - \#Q + 1$ and, thus, $\#E'' = c + \#Q - 1 \leq 3c - 3$. So, there are only finitely many possibilities for G''.

Finally, every edge $e'' \in E''$ in G'' "carries" at most 3 vertices of degree 2 from G'. So, there are only finitely many possibilities for G'. ∎

3 Semiharmonic graphs that are not almost semiregular

Let us now switch our perspective and study, for every given connected graph $G = (V, E)$, the collection of all (isomorphism classes of) semiharmonic graphs $G' = (V', E')$ with $\mathrm{sk}(G') = G$. Clearly, there is a one-to-one correspondence between the isomorphism classes of arbitrary graphs $G' = (V', E')$ with $\mathrm{sk}(G') = G$ and maps $r : V \to \mathbf{N}_{\geq 2}$ with $r(v) \geq \mathrm{d}(v)$ for every vertex $v \in V$ given by associating, to each graph $G' = (V', E')$ with $\mathrm{sk}(G') = G$, the map

$$r = r_{G'} : V \to \mathbf{N}_{\geq 2} : v \mapsto \mathrm{d}_{G'}(v),$$

or, conversely, to each map $r : V \to \mathbf{N}_{\geq 2}$ with $r(v) \geq \mathrm{d}(v)$ for every vertex $v \in V$, the graph $G^{(r)} := (V^{(r)}, E^{(r)})$ defined by

$$V^{(r)} := V \cup \{(v, i) \,|\, v \in V, i \in \{1, 2, \ldots, r(v) - \mathrm{d}(v)\}\}$$

and

$$E^{(r)} := E \cup \{\{v, (v, i)\} \,|\, v \in V, i \in \{1, 2, \ldots, r(v) - \mathrm{d}(v)\}\}.$$

Thus, assuming that a map $r : V \to \mathbf{N}_{\geq 2}$ with $r(v) \geq \mathrm{d}(v)$ for every vertex $v \in V$ is given and using the superscript ${}^{(r)}$ throughout to indicate reference to the graph $G^{(r)}$ when dealing with parameters like N, d, or d_2 (and no superscript to indicate reference to the graph G), we have

$$d^{(r)}(v) = r(v)$$

and

$$d_2^{(r)}(v) = d^{(r)}(N^{(r)}(v)) = r(v) - \mathrm{d}(v) + r(N(v))$$

for all $v \in V$, as well as

$$d^{(r)}((v, i)) = 1$$

and

$$d_2^{(r)}((v, i)) = d^{(r)}(v) = r(v)$$

for all $v \in V$ and $i \in \{1, 2, \ldots, r(v) - \mathrm{d}(v)\}\}$.

In particular, $G^{(r)}$ is harmonic if and only if there exists a positive real number k with $r(v) = k$ for all v in

$$L(r) := \{v \in V \,|\, r(v) > \mathrm{d}(v)\},$$

and

$$k\, r(v) = r(v) - \mathrm{d}(v) + r(N(v)) = r(v) - \mathrm{d}(v) + \mathrm{d}(N(v) \setminus L(r)) + k\, \#(N(v) \cap L(r))$$

for all $v \in V$.

And $G^{(r)}$ is connected and strictly semiharmonic if and only if G is connected and bipartite with bipartition, say, $V = V^+ \cup V^-$ and there exist positive real numbers λ^+ and λ^- with

$$r(v) = \lambda^- \text{ for all } v \in L^+(r) := \{v \in V^+ \,|\, r(v) > \mathrm{d}(v)\},$$

$$r(v) = \lambda^+ \text{ for all } v \in L^-(r) := \{v \in V^- \,|\, r(v) > \mathrm{d}(v)\},$$

$$\lambda^+ \lambda^- = \lambda^- - \mathrm{d}(v) + r(N(v)) = \lambda^- - \mathrm{d}(v) + \mathrm{d}(N(v) \setminus L(r)) + \lambda^+ \#(N(v) \cap L(r))$$

for all $v \in L^+(r)$,

$$\lambda^- \lambda^+ = \lambda^+ - \mathrm{d}(v) + r(N(v)) = \lambda^+ - \mathrm{d}(v) + \mathrm{d}(N(v) \setminus L(r)) + \lambda^- \#(N(v) \cap L(r))$$

for all $v \in L^-(r)$,

$$\lambda^+ \mathrm{d}(v) = \mathrm{d}(N(v) \setminus L(r)) + \lambda^+ \#(N(v) \cap L(r))$$

for all v in $V^+ \setminus L^+(r)$, and

$$\lambda^- \mathrm{d}(v) = \mathrm{d}(N(v) \setminus L(r)) + \lambda^- \#(N(v) \cap L(r))$$

for all v in $V^- \setminus L^-(r)$.

This suggests to define, for any subset L of V and any two integers $k^+, k^- \in \mathbf{N}_{\geq 2}$, the map $r_{(L|k^+,k^-)} : V \to \mathbf{N}_{\geq 2}$ that maps any $v \in V \setminus L$ onto $\mathrm{d}(v)$, any $v \in L^+ := V^+ \cap L$ onto k^+, and any $v \in L^- := V^- \cap L$ onto k^-, because any map r in the set

$$\mathrm{sh}(L) := \{r : V \to \mathbf{N}_{\geq 2} \,|\, L(r) = L \text{ and } G^{(r)} \text{ is semiharmonic}\},$$

is necessarily of this form. Note also that

$$r_{(L_1|k_1^+,k_1^-)} = r_{(L_2|k_2^+,k_2^-)}$$

holds for some L_1, k_1^+, k_1^- and L_2, k_2^+, k_2^- as above if and only if one has $L_1 = L_2$ as well as $k_1^+ = k_2^+$ in case $L_1^+ \neq \emptyset$ and $k_1^- = k_2^-$ in case $L_1^- \neq \emptyset$. Moreover, the above assertions imply that $r_{(L|k^+,k^-)} \in \mathrm{sh}(L)$ holds for some integers $k^+, k^- \in \mathbf{N}_{\geq 2}$ if and only if there exist positive real numbers $\lambda^+ = \lambda^+(L|k^+,k^-)$ and $\lambda^- = \lambda^-(L|k^+,k^-)$, uniquely determined by L and k^+, k^- such that the following holds:

(sh1) $L^+ \neq \emptyset$ implies $\lambda^- = k^+$, and $v \in L^+$ implies $k^+ > \mathrm{d}(v)$ and

$$\lambda^+ k^+ = k^+ - \mathrm{d}(v) + \mathrm{d}(N(v) \setminus L) + \lambda^+ \#(N(v) \cap L)$$

(sh2) $L^- \neq \emptyset$ implies $\lambda^+ = k^-$, and $v \in L^-$ implies $k^- > \mathrm{d}(v)$ and

$$\lambda^- k^- = k^- - \mathrm{d}(v) + \mathrm{d}(N(v) \setminus L) + \lambda^- \#(N(v) \cap L)$$

(sh3) $v \in V^+ \setminus L^+$ implies $\lambda^+ \mathrm{d}(v) = \mathrm{d}(N(v) \setminus L) + \lambda^+ \#(N(v) \cap L)$ or, equivalently,

$$\lambda^+ \#(N(v) \setminus L) = \mathrm{d}(N(v) \setminus L)),$$

(sh4) and $v \in V^- \setminus L^-$ implies $\lambda^- \mathrm{d}(v) = \mathrm{d}(N(v) \setminus L) + \lambda^- \#(N(v) \cap L)$ or, equivalently,

$$\lambda^- \#(N(v) \setminus L) = \mathrm{d}(N(v) \setminus L)).$$

We are interested in the cardinality of the sets $\mathrm{sh}(L)$. Clearly, we have $\#\mathrm{sh}(L) \leq 1$ whenever there exists at least one edge $e = \{v^+, v^-\} \in E$ with $v^+ \in V^+ \setminus L^+$ and $v^- \in V^- \setminus L^-$ because this implies $\#(N(v^+) \setminus L) \neq 0$ and

$$\lambda^+ = \frac{\mathrm{d}(N(v^+) \setminus L))}{\#(N(v^+) \setminus L)}$$

as well as $\#(N(v^-) \setminus L) \neq 0$ and

$$\lambda^- = \frac{\mathrm{d}(N(v^-) \setminus L))}{\#(N(v^-) \setminus L)}$$

and therefore $k^- = \lambda^+ = \frac{\mathrm{d}(N(v^+) \setminus L))}{\#(N(v^+) \setminus L)}$ in case $L^+ \neq \emptyset$ and $k^+ = \lambda^- = \frac{\mathrm{d}(N(v^-) \setminus L))}{\#(N(v^-) \setminus L)}$ in case $L^- \neq \emptyset$ while $r_{(L|k^+,k^-)}$ does not depend on the choice of k^+ or k^- in case $L^+ = \emptyset$ or $L^- = \emptyset$, respectively. Thus, $\#\mathrm{sh}(L) > 1$ implies $N(v) \subseteq L$ for all $v \in V \setminus L$.

Moreover, $L^+, L^- \neq \emptyset$ implies $\lambda^- = k^+ \in \mathbf{N}_{\geq 2}$ and $\lambda^+ = k^- \in \mathbf{N}_{\geq 2}$ and, hence,

$$(k^- - 1)\,(k^+ - \#(N(v) \cap L)) = \mathrm{d}(N(v) \setminus L) - \#(N(v) \setminus L)$$

and, therefore, also $N(v) \not\subseteq L$ for every $v \in L^+$ in view of

$$(k^- - 1)\,(k^+ - \#(N(v) \cap L)) \geq (k^- - 1)(k^+ - \mathrm{d}(v)) > 0,$$

as well as

$$(k^+ - 1)\,(k^- - \#(N(v) \cap L)) = \mathrm{d}(N(v) \setminus L) - \#(N(v) \setminus L)$$

and, therefore, also $N(v) \not\subseteq L$ for every $v \in L^-$ in view of

$$(k^+ - 1)\,(k^- - \#(N(v) \cap L)) \geq (k^+ - 1)(k^- - \mathrm{d}(v)) > 0.$$

Thus, we have

$$0 < k^+ - \#(N(v) \cap L) \leq (k^- - 1)\,(k^+ - \#(N(v) \cap L)) = \mathrm{d}(N(v) \setminus L) - \#(N(v) \setminus L)$$

for every $v \in L^+$ and

$$0 < k^- - \#(N(v) \cap L) \leq (k^+ - 1)\,(k^- - \#(N(v) \cap L)) = \mathrm{d}(N(v) \setminus L) - \#(N(v) \setminus L)$$

for every $v \in L^-$ which implies in particular

$$\#\mathrm{sh}(L) \leq \min\{\mathrm{d}(N(v) \setminus L) - \#(N(v) \setminus L) + \#(N(v) \cap L) \,|\, v \in L\}.$$

It remains to consider the cases $L^- = \emptyset$ and $L^+ = \emptyset$. However, our analysis implies that, if $L^- = \emptyset$ holds, we have $\#\mathrm{sh}(L) > 1$ if and only if we have $L = L^+ = V^+$ and the map $\mathrm{d}_2 - \mathrm{d}$ is constant on V^+ in which case $G^{(r)}$ is necessarily an almost semiregular graph for every r of the form $r = r_{(V^+|k^+,k^-)}$ with $k^+ > \mathrm{d}(v)$ for all $v \in V^+$. By symmetry, we have $L^+ = \emptyset$ and $\#\mathrm{sh}(L) > 1$ if and only $L = L^- = V^-$ and the map $\mathrm{d}_2 - \mathrm{d}$ is constant on V^- in which case $G^{(r)}$ is an almost semiregular graph for every r of the form $r = r_{(V^-|k^+,k^-)}$ with $k^+ > \mathrm{d}(v)$ for all $v \in V^-$.

Let us note finally that our analysis implies in particular the following two theorems:

Theorem 3 *Let G be a connected, bipartite graph. Then there are at most finitely many semiharmonic, but not almost semiregular graphs with trunk G.*

Theorem 4 *Given a graph $G = (V, E)$ and a subset U of V with $\#(e \cap U) = 1$ for all $e \in E$, the following assertions are equivalent:*

(i) the graph $G^{(r)}$ is almost semiregular for every map $r : V \to \mathbf{N}_{\geq 2}$ with $r(v) = \mathrm{d}(v)$ for all $v \in V \setminus U$ and $r(u) = k$ for some $k \geq \max\{\mathrm{d}(u) \,|\, u \in U\}$ and all $u \in U$,

(ii) the graph $G^{(r)}$ is semiharmonic for every map $r : V \to \mathbf{N}_{\geq 2}$ with $r(v) = \mathrm{d}(v)$ for all $v \in V \setminus U$ and $r(u) = k$ for some $k \geq \max\{\mathrm{d}(u) \,|\, u \in U\}$ and all $u \in U$,

(iii) the graph $G^{(r)}$ is almost semiregular for the map $r = r_U : V \to \mathbf{N}_{\geq 2}$ defined by $r_U(v) := \mathrm{d}(v)$ for all $v \in V \setminus U$ and $r_U(u) = \max\{\mathrm{d}(u) \,|\, u \in U\}$ for all $u \in U$,

(iv) the graph $G^{(r_U)}$ is semiharmonic for the map r_U defined above,

(v) the map $\mathrm{d}_2 - \mathrm{d}$ is constant on U.

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