The Kac problem for a gas of interacting Brownian loops

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Abstract

The following version of the inverse spectral problem of Kac is discussed for a system of interacting Brownian loops in a bounded admissible domain Λ of $\mathbb{R}^2$: Given a stable pair potential ϕ with nice decay properties, the logarithm of the associated partition function in the dilated region $R \cdot \Lambda$, $R \geq 1$, is expanded as $R \to +\infty$. The proof is based on the method of cluster expansion.

Introduction

For the Ginibre gas obeying Maxwell-Boltzmann (**MB**) statistics, defined below by a finite measure $\mathcal{W}_{z \cdot \rho_{\Lambda_R}}$ on the set of Brownian loops in Λ_R and an interaction potential ϕ, the associated log-partition function

$$ln\, Z(\Lambda_R, z) = ln \int exp(-\mathcal{U}^{\phi}(\mu)) W_{z \cdot \rho_{\Lambda_R}}(d\mu) \tag{1}$$

will be expanded in the infinite volume limit $R \to +\infty$. Here $\Lambda_R = R \cdot \Lambda$ and $\mathcal{U}^{\phi}$ is the energy, defined by ϕ, and $z > 0$ an activity or intensity parameter. We shall show that

$$ln\, Z(\Lambda_R, z) = R^2 \cdot p(\phi, z) \cdot |\Lambda| - R \cdot b(\phi, z) \cdot |\partial\, \Lambda| + o(R) \text{ as } R \to +\infty. \tag{2}$$

Here $|\Lambda|$ is the volume and $|\partial\, \Lambda|$ the surface measure of $|\Lambda|$. The coefficents $p(\phi, z)$, and $b(\phi, z)$ are explicitly expressed as functional integrals by means of ϕ and z. $p(\phi, z)$ can be interpreted as the pressure and $b(\phi, z)$ as the surface tension. We consider the case $\nu = 2$ only for simplicity. The same arguments give in the general case $\nu \geq 2$ a similar result:

$$\ln\, Z(\Lambda_R, z) = R^{\nu} \cdot p(\phi, z) \cdot |\Lambda| - R^{\nu-1} \cdot b(\nu, \phi, z) \cdot |\partial\, \Lambda| + o(R^{\nu-1}) \text{ as } R \to +\infty.$$

Since the left hand side is determined by the potential (if Λ and z are given), the expansion (2) implies that ϕ uniquely determines the volume, and the surface measure of the domain in which the Ginibre gas is living. In this sense this work is a contribution to the program of Mark Kac.

The model which we consider in this paper is actually a quantum gas with (**MB**) statistics in the "loop" representation [1]. Hence (2) gives the first two terms of the asymptotic expansion of the log-partition function of a quantum gas in thermodynamic limit. For classical gases a complete expansion of ln $Z(\Lambda_R, z)$ was obtained in [5].

We make some comments on the methods used to obtain this result: The starting point is the representation

$$ln\, Z(\Lambda_R, z) = W_{z\cdot\rho_{\Lambda_R}}(g), z \text{ small enough}, \tag{3}$$

by means of the Ursell function g. It is based on the **method of cluster expansion**. See [6] for details.

In the second step the right hand side of (3) is written by means of the Campbell measure $\mathcal{C}_\rho$ of W_ρ:

$$W_{z\cdot\rho_{\Lambda_R}}(g) = \mathcal{C}_\rho\left(1_{\mathcal{X}(\Lambda_R)\times\mathcal{M}(\Lambda_R)}\cdot\hat{g}_z\right),$$

where

$$\hat{g}_z(x,\mu) = \tilde{g}_z(\mu) = z^{|\mu|}\cdot\frac{g(\mu)}{|\mu|}, \text{ if } \mu(\{x\}) \geq 1.$$

The **method of the Campbell measure** is explained in appendix A. One of its main elements is the following **partial integration formula**

$$\mathcal{C}_\rho(h) = \int_{\mathcal{M}(\mathcal{X})}\int_{\mathcal{X}} h(x, \mu+\delta_x)\rho(dx)W_\rho(d\mu), \tag{4}$$

where h is non-negative and measurable.

Writing the integration with respect to ρ more explicitly we obtain the representation

$$ln\, Z(\Lambda_R, z) = \int_{\Lambda_R} du \int_{\mathcal{X}^0} P^0(dx^0)1_{\mathcal{X}(\Lambda_R)}(u+x^0)\int_{\mathcal{M}(\Lambda_R)} W_\rho(d\mu)\tilde{g}_z(\mu+\delta_{x^0+u}). \tag{5}$$

The main strategy then will be, to make successively finer decompositions with respect to the variables (u, x^0, μ) of the integral on the right side of (5), thereby isolating the relevant global and correction terms.

The analysis of the correction terms is based on the estimates for the decay of correlations in our system which had been obtained by the authors in [6] . It is summarized in the socalled **main lemma**, which is stated in appendix C.

We also mention that some technics developed in Macris et al. [3] are used throughout the paper.

The Ginibre gas. Formulation of the main result

In this paper we consider domains Λ which are open convex bounded subsets of $\mathbb{R}^2$) with n convex closed holes. We assume that the connected parts of the boundary of Λ are one dimensional closed C^2-manifolds. Such Λ we will call admissible domain. Thus for each admissible Λ and $u \in \Lambda$ with $d(u, \partial\Lambda) < \delta$, for $\delta > 0$ sufficiently small, there is a unique point $r \in \partial\Lambda$ such that $d(u,r) = d(u,\partial\Lambda)$. Here $\partial\Lambda$ denotes the boundary of Λ, and d the Euclidean distance in $\mathbb{R}^2$.

We consider classical stable pair interaction ϕ which is continuous even function on $\mathbb{R}^2 \setminus 0$. We assume also that $\int_{\mathbb{R}^2} |\phi(u)| \cdot (1+|u|)^l du < +\infty$ where $l \geq 0$ will be chosen later. The class of such potentials we denote by $\mathcal{P}_l$.

We recall briefly the notion of a Ginibre gas obeying MB-statistics. Fix $\beta > 0$ and consider the σ-finite measure space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \rho)$, where

$$\mathcal{X} = \{x \in \mathcal{C}([0,\beta], \mathbb{R}^2) \mid x(0) = x(\beta)\}$$

denotes the space of continuous loops provided with the topology of uniform convergence. $\mathcal{B}_{\mathcal{X}}$ denotes the Borel σ-field in $\mathcal{X}$, and

$$\rho = \int_{\mathbb{R}^2} P^u \, du \tag{6}$$

the locally finite measure on $\mathcal{X}$ defined by means of the Brownian bridge measure P^u, being concentrated on the set $\mathcal{X}^u$ of loops which start and end in u. (For a precise definition and construction of P^u see [1]). P^u is not normalized and satisfies $P^u(\mathcal{X}^u) = (\pi\beta)^{-\nu/2}$.

Now let Λ be an admissible domain in $\mathbb{R}^2$, and consider the restriction of ρ to the bounded Borel set

$$\mathcal{X}(\Lambda) = \{x \in \mathcal{X} \mid x(t) \in \Lambda \text{ for each } 0 \leq t \leq \beta\}.$$

This restriction is denoted by ρ_Λ. It is a finite measure with $\rho_\Lambda(\mathcal{X}_\Lambda) \leq |\Lambda| \cdot (\pi\beta)^{-\nu/2}$.

Given in addition a parameter $z > 0$, a measure $W_{z\cdot\rho}$ is defined on $\mathcal{M} = \mathcal{M}(\mathcal{X})$ by

$$W_{z\cdot\rho}(\varphi) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \, \rho(dx_1) \ldots \rho(dx_n),$$

where φ is any non-negative, measurable function on $\mathcal{M}$. (Here $\mathcal{M}$ is the set of all finite, simple point measures on $\mathbb{R}^2$. For details see [4].) We are interested in the finite measure $W_{z\cdot\rho_\Lambda}$ on $\mathcal{M}(\Lambda) = \mathcal{M}(\mathcal{X}(\Lambda))$, the set of finite configurations of loops in Λ.

On the other hand, by means of a given potential $\phi \in \mathcal{P}_l$ we introduce the following energy functional $\mathcal{U}$:

$$\mathcal{U}(\mu) = \frac{1}{2} \cdot \sum_{\substack{x,y\in\mu,\\ x\neq y}} \int_0^\beta \phi(x(s) - y(s))ds, \mu \in \mathcal{M}.$$

On account of the stability assumption on ϕ the functional $f := \exp(-\mathcal{U})$, the so called **Boltzmann factor**, is integrable with respect to $W_{z\cdot\rho_\Lambda}$.

The triple $(\mathcal{M}(\Lambda), W_{z\cdot\rho_\Lambda}, \phi)$ we call the **Ginibre gas in Λ with intensity z, potential ϕ and MB statistics**.

Of central importance is its **partition function** $Z(\Lambda, z) = W_{z\cdot\rho_\Lambda}(\exp(-\mathcal{U}))$ which is strictly positive and finite, because the stability of ϕ implies $f(\mu) \leq \exp(\mu(X) \cdot \beta \cdot B), \mu \in \mathcal{M}$.

Given ϕ, β, z and some admissible Λ, we are interested in the asymptotic expansion of $\ln Z(\Lambda_R, z)$ as $R \to +\infty$, where $\Lambda_R = R\Lambda = \{Ru \mid u \in \Lambda\}$, $R \geq 1$. As in [7] we shall use the representation of the log-partition function by means of the Ursell function g:

$$\ln Z(\Lambda_R, z) = W_{z\cdot\rho_{\Lambda_R}}(g) \text{ if } z \text{ is sufficiently small.}$$

The following theorem will be obtained in this paper. Let $W_\rho^\mu(\varphi) = \int_\mathcal{M} \varphi(\mu+\overline{\mu})W_\rho(d\overline{\mu})$, $\varphi \in L^1(W_\rho)$, $\mu \in \mathcal{M}$. Instead of $W_\rho^{\delta_x}$ we shall write W_ρ^x.

Theorem If $\phi \in \mathcal{P}_l$ with $l > 16$, and z is from the interval $0 < z < [(2^{l+1} + e) \cdot e^{2\beta B} \cdot \|q\|_l]^{-1}$, where $\|\cdot\|_l$ is given in appendix B, then for any admissible domain in $\mathbb{R}^2$ the following asymptotic expansion of the log-partition function holds true:

$$\begin{aligned}\ln Z(\Lambda_R, z) =& R^2 \cdot \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{x^0}(\tilde{g}_z) \cdot |\Lambda| \\ & -R \cdot \int_{\partial\Lambda} \sigma(dr) \int_0^{+\infty} dt \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{r+t\cdot\mathbf{n}(r)+x^0}(1_{\mathcal{M}^c(\Pi_r^+)} \cdot \tilde{g}_z) \\ & +o(R) \text{ as } R \to +\infty,\end{aligned}$$

where $\mathbf{n}(r)$ is the inward drawn unit normal to Λ at the point $r \in \partial\Lambda$ and $\Pi_r^+ = \{x \in \mathbb{R}^2 \bigm| (x, \mathbf{n}(r)) \geq 0\}$.

If ϕ is also Euclidean invariant then

$$
\begin{aligned}
\ln Z(\Lambda_R, z) =& R^2 \cdot \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{x^0}(\tilde{g}_z) \cdot |\Lambda| \\
& - R \cdot \int\limits_0^{+\infty} dt \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{t\cdot\mathbf{n}^0 + x^0}(1_{\mathcal{M}^c(\Pi_0^+)} \cdot \tilde{g}_z) \cdot |\partial\Lambda| \\
& + o(R) \text{ as } R \to +\infty.
\end{aligned}
$$

Here $\mathbf{n}^0$ is any fixed unit vector and $\Pi_0^+ = \{x \in \mathbb{R}^2 \bigm| (x, \mathbf{n}^0) \geq 0\}$.

The proof of the theorem.

Let $\mathcal{I}(R) = \ln Z(\Lambda_R, z)$. By (A.4) from appendix A we see that $I(R) = W_{z\rho_{\Lambda_R}}(g) = C_\rho(1_{\mathcal{X}(\Lambda_R)\times\mathcal{M}(\Lambda_R)} \cdot \tilde{g}_z)$. Let

$$S(\Lambda_R) = \{x \in X \bigm| x(0) \in \Lambda_R\}.$$

Consider the set function $\mathcal{C}(A) := \mathcal{C}_\rho(1_A \cdot \hat{g}_z)$. Then $\mathcal{I}(R) = \mathcal{C}(\mathcal{X}(\Lambda_R) \times \mathcal{M}(\Lambda_R))$. $\mathcal{C}$ is well defined and real valued for all sets $S(\Lambda_R) \times \mathcal{M}$ because

$$
\begin{aligned}
|\mathcal{C}(S(\Lambda_R) \times \mathcal{M})| &\leq \mathcal{C}_\rho\left(1_{S(\Lambda_R)\times\mathcal{M}} \cdot |\tilde{g}_z|\right) \\
&= \int_{\Lambda_R} du \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{M}} W_\rho(d\mu)\ |\tilde{g}_z|(\mu + \delta_{u+x^0}).
\end{aligned}
$$

Here we used the partial integration formula (4). Now the integral $W_\rho^{u+x^0}(|\tilde{g}_z|)$ does not depend on u, because W_ρ as well as ϕ (and thereby $\tilde{g}_z$) are translation invariant. Since the functional $F(x^0) = W_\rho^{x^0}(|\tilde{g}_z|)$ is bounded [7] this implies the assertion.

We have that

$$\mathcal{I}(R) = \mathcal{C}(S(\Lambda_R) \times \mathcal{M}(\Lambda_R)) - \mathcal{C}([S(\Lambda_R) - \mathcal{X}(\Lambda_R)] \times \mathcal{M}(\Lambda_R))$$

It is clear that the second term on the right hand side vanishes. Then we decompose again

$$\mathcal{I}(R) = \mathcal{C}(S(\Lambda_R) \times \mathcal{M}(\mathcal{X})) - \mathcal{C}(S(\Lambda_R) \times \mathcal{M}^c(\Lambda_R)) \qquad (7)$$

This is our basic initial decomposition of $\mathcal{I}$.

We shall discuss the terms on the right hand side separately. The first term is a contribution to the volume, whereas the second will contribute to the boundary and the constant term.

By partial integration the first term of the decomposition (8) is given by

$$\mathcal{C}_\rho \left(1_{S(\Lambda_R)\times\mathcal{M}(\mathcal{X})} \cdot \tilde{g}_z \right) = \int_{\Lambda_R} du \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{u+x^0}(\tilde{g}_z)$$

As has been remarked already, the function $W_\rho^{u+x^0}(\tilde{g}_z)$ does not depend on u. Thus

$$\mathcal{C}(S(\Lambda_R)\times\mathcal{M}) = R^2 \cdot |\Lambda| \cdot \int_{\mathcal{X}^0} P^0(dx^0)\ W_\rho^{x^0}(\tilde{g}_z).$$

The integral on the right hand side is called the **pressure**.

It remains to expand the integral

$$\mathcal{T}(R) = \mathcal{C}(S(\Lambda_R)\times\mathcal{M}^c(\Lambda_R)) = \int_{S(\Lambda_R)} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}^c(\Lambda_R)} \cdot \tilde{g}_z \right).$$

We decompose it by decomposing Λ_R into Λ_R^{bd} and its complement Λ_R^{int}, where for some fixed constant $\varepsilon > 0$ and $\delta > 0$ (to be chosen later)

$$\Lambda_R^{bd} = \{ u \in \Lambda_R \bigm| d(u, \partial\Lambda_R) < \delta \cdot R^\varepsilon \}.$$

We obtain the following decomposition

$$\mathcal{T}(R) = \mathcal{C}(S(\Lambda_R^{bd})\times\mathcal{M}^c(\Lambda_R)) + \mathcal{C}(S(\Lambda_R^{int})\times\mathcal{M}^c(\Lambda_R))$$

By partial integration formula

$$\mathcal{T}^{int}(R) = \mathcal{C}(S(\Lambda_R^{int})\times\mathcal{M}^c(\Lambda_R)) = \int_{S(\Lambda_R^{int})} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}^c(\Lambda_R)} \cdot \tilde{g}_z \right)$$

The main lemma (see Appendix B) implies that this is a correction term:

$$|\mathcal{T}^{int}(R)| \leq const \cdot R^2 \cdot |\Lambda| \cdot \frac{1}{(1+\frac{2}{3}\delta R^\varepsilon)^l}.$$

Let us **choose from now on** $\varepsilon = \frac{1}{8}$ and $l > 16$. Then $\mathcal{T}^{int}(R) = o(1)$ as $R \to +\infty$. Therefore all relevant informations of $\mathcal{T}$ are contained in the integral

$$\mathcal{T}^{bd}(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x\left(1_{\mathcal{M}^c(\Lambda_R)} \cdot \tilde{g}_z\right)$$

To analyze $\mathcal{T}^{bd}$ we set up at each point $r \in \partial\Lambda$ local coordinates (ξ, η), where ξ is along the tangent vector $\mathbf{s} = \mathbf{s}(r)$ and η is along the inward drawn unit normal $\mathbf{n}$ to $\partial\Lambda$ at r. Then $\partial\Lambda$ is given locally by $\eta = f_r(\xi)$, $|\xi| < \delta R^\varepsilon$ for $\delta > 0$ small enough (for example we can take $\delta = \frac{1}{2}[\sup\limits_{r\in\partial\Lambda} |\kappa(r)|]^{-1}$, $\kappa(r) = f_r''(0)$), where f_r is a function of class $\mathcal{C}^2$.

Furthermore, we associate to r the cylinder $\Pi_{r,\delta R^\varepsilon} = \{(\xi, \eta) \bigm| |\xi| < \delta \cdot R^\varepsilon\}$. We now decompose $\mathcal{T}^{bd}$ by decomposing $\mathcal{M}^c(\Lambda_R)$ as follows:

$$\mathcal{M}^c(\Lambda_R) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) + \mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon}), r \in \partial\Lambda_R.$$

By the main lemma

$$\int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x\left(1_{\mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})} \cdot \tilde{g}_z\right) = o(1) \text{ as } R \to +\infty$$

Thus the relevant term is

$$\mathcal{T}_A^{bd}(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x\left(1_{\mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon})} \cdot \tilde{g}_z\right)$$

To study this term we consider $\mathcal{F}^+_{r,\delta R^\varepsilon} = \{(\xi, \eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta > f_{r,R}(\xi)\}$, where $f_{r,R}(\xi) = R \cdot f_r\left(\frac{\xi}{R}\right)$. We now decompose $\mathcal{T}_A^{bd}$ by decomposing $\mathcal{D} := \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon})$ as follows:

$$\mathcal{D} = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}).$$

Again by the main lemma

$$\int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x\left(1_{\mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon})} \cdot \tilde{g}_z\right) = o(1)$$

because $\int_{S(\Lambda_R^{bd})} dx\ W_\rho^x(d\mu)$ realizes configurations of loops $\mu + \delta_x$ with $x(0) \in \Lambda_R^{bd}$ and with at least one loop touching the exterior of Λ_R in $\mathcal{F}^+_{r,\delta R^\varepsilon}$ (which can be a hole).

Therefore the term to be studied is

$$\mathcal{I}_2(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} \cdot \tilde{g}_z \right)$$

We decompose $\mathcal{I}_2$ by decomposing $\mathcal{D}' = \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$ by means of $\Pi^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta \geq 0\}$ and its complement and get

$$\mathcal{D}' = \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}) + \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}). \tag{8}$$

Remark. For shortness we consider only the case where r is a point of the convex part of the boundary $\partial\,\Lambda_R$. The concave situation can be treated similarly.

We first consider the integral

$$\mathcal{I}_2^A(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})} \cdot \tilde{g}_z \right)$$

To get rid of the restriction to the cylinder we decompose this integral by representing $\mathcal{D}'_2 = \mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})$ by means of $\Pi^+_r = \{(\xi,\eta) \bigm| \eta \geq 0\}$ and obtain

$$\mathcal{D}'_2 = \mathcal{M}^c(\Pi^+_r) - \mathcal{M}^c(\Pi^+_r)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})$$

The standard argument, with the help of the main lemma, shows that

$$\mathcal{I}_2^A(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}^c(\Pi^+_r)} \cdot \tilde{g}_z \right) + o(1) \text{ as } R \to +\infty. \tag{9}$$

Let $\kappa(r \bigm| R) = \frac{1}{R} \cdot \kappa(r)$ where $\kappa(r)$ is the curvature of $\partial\,\Lambda$ in $r \in \partial\,\Lambda$. We shall use now the disintegration of the 2-dimensional Lebesgue measure in the shell Λ_R^{bd}:

$$\begin{aligned} \tilde{\mathcal{I}}_2^A(R) &= \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x \left(1_{\mathcal{M}^c(\Pi^+_r)} \cdot \tilde{g}_z \right) \\ &= \int_{\partial\,\Lambda_R} \sigma_R(dr) \int_0^{\delta R^\varepsilon} dt\ [1 - t \cdot \kappa(r \bigm| R)] \cdot \mathcal{H}^{r+t\cdot\mathbf{n}(r)} \left(1_{\mathcal{M}^c(\Pi^+_r)} \right) \end{aligned} \tag{10}$$

where $\mathcal{H}^u$ is defined in appendix B.

Developing **here** the bracket, this is a difference of two integrals, say $\tilde{\mathcal{I}}_3(R)-\tilde{\mathcal{I}}_4(R)$. Furthermore, writing the integration $\int_0^{\delta R^\varepsilon} dt$ as the difference $\int_0^\infty dt - \int_{\delta R^\varepsilon}^\infty dt$, we obtain for these terms by the standard argument

$$\begin{aligned}
\tilde{\mathcal{I}}_3(R) &= \int_{\partial\Lambda_R} \sigma_R(dr) \int_0^{+\infty} dt\ \mathcal{H}^{r+t\cdot\mathbf{n}(r)}\left(1_{\mathcal{M}^c(\Pi_r^+)}\right) + o(1) \qquad (11)\\
&=: \mathcal{I}_3(R) + o(1) \text{ as } R\to+\infty;\\
\tilde{\mathcal{I}}_4(R) &= \int_{\partial\Lambda_R} \sigma_R(dr)\cdot dt\ t\cdot\kappa(r\mid R) \int_0^{+\infty} dt\ t\cdot\mathcal{H}^{r+t\cdot\mathbf{n}(r)}\left(1_{\mathcal{M}^c(\Pi_r^+)}\right) + o(1) \qquad (12)\\
&=: \mathcal{I}_4(R) + o(1) \text{ as } R\to+\infty.
\end{aligned}$$

Here, by virtue of the main lemma, both integrals $\tilde{\mathcal{I}}_3(R)$ and $\tilde{\mathcal{I}}_4(R)$ are convergent.

We remark that in case that ϕ is Euclidean invariant

$$\mathcal{H}^{r+t\cdot\mathbf{n}(r)}\left(1_{\mathcal{M}^c(\Pi_r^+)}\right) = \mathcal{H}^{t\cdot\mathbf{n}_0}\left(1_{\mathcal{M}^c(\Pi_0^+)}\cdot\tilde{g}_z\right),$$

where $\mathbf{n}_0$ is any unit vector and $\Pi_0^+ = \{x\in\mathbb{R}^2 \mid (x,n_0)\geq 0\}$. In this case

$$\begin{aligned}
\mathcal{I}_3(R) &= R\cdot|\partial\Lambda|\cdot\int_0^{+\infty} dt\ \mathcal{H}^{t\cdot\mathbf{n}_0}\left(1_{\mathcal{M}^c(\Pi_0^+)}\right);\\
\mathcal{I}_4(R) &= \int_{\partial\Lambda}\sigma(dr)\kappa(r)\int_0^{+\infty} dt\ t\cdot\mathcal{H}^{t\cdot\mathbf{n}_0}\left(1_{\mathcal{M}^c(\Pi_0^+)}\right)
\end{aligned}$$

Evidently $\mathcal{I}_3$ is a boundary term and $\mathcal{I}_4$ is a constant term. Combining (9) and (10) – (13), we find the following asymptotic expansion for $\mathcal{I}_2^A$:

$$\mathcal{I}_2^A(R) = \mathcal{I}_3(R) + o(R) \text{ as } R\to+\infty.$$

It remains to study the integral

$$\mathcal{I}_2^B(R) = \int_{S(\Lambda_R^{bd})} \rho(dx)\ W_\rho^x\left(1_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}\cdot\tilde{g}_z\right)$$

which corresponds to the first term in (9).

To complete the proof of the theorem we will show that $\mathcal{I}_2^B(R) = o(R)$ as $R\to+\infty$.

We have

$$\mathcal{I}_2^B(R) = \int_{S(\Lambda_R^{bd})} \rho(dx) \int_{\mathcal{M}} d\mu \; 1_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(\mu + \delta_x) \cdot \tilde{g}_z(\mu + \delta_x)$$

The gas $\int_{S(\Lambda_R^{bd})} dx \, W_\rho^x(d\mu)$ is now considered in the region $\mathcal{D}'' := S(\Lambda_R^{bd}) \times \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$. Thus its realizations $\mu + \delta_x$ consist of loops with $x(0) \in \Lambda_R^{bd}$, all are living in the cylinder $\Pi^+_{r,\delta R^\varepsilon}$, but at least one leaves $\mathcal{F}^+_{r,\delta R^\varepsilon}$

Observe now that $\mu + \delta_x \in \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})$ with $x \in S(\Lambda_R^{bd})$ iff

$$\begin{cases} x(0) \in \Lambda_R^{bd}, x \in \mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}), \mu \in \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon}) \text{ or} \\ x(0) \in \Lambda_R^{bd}, x \in \mathcal{X}(\mathcal{F}^+_{r,\delta R^\varepsilon}), \mu \in \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}). \end{cases}$$

This implies

$$\begin{aligned} |\mathcal{I}_2^B(R)| \leq & \int_{\mathcal{X}} \rho(dx) \; 1_{S(\Lambda_R^{bd})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}(x) \int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})} W_\rho(d\mu) \; |\tilde{g}_z|(\mu + \delta_x) \\ & + \int_{\mathcal{X}} \rho(dx) \; 1_{S(\Lambda_R^{bd})\mathcal{X}(\mathcal{F}^+_{r,\delta R^\varepsilon})}(x) \int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} W_\rho(d\mu) \; |\tilde{g}_z|(\mu + \delta_x) \\ & =: \mathcal{J}_2(R) + \mathcal{J}_1(R) \end{aligned} \tag{13}$$

We consider the terms on the right hand side separately.

Consider the inner integral of $\mathcal{J}_1(R)$. Separating a loop touching the outside of $\mathcal{F}^+_{r,\delta R^\varepsilon}$ yields

$$\int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} W_\rho(d\mu) \; |\tilde{g}_z|(\mu + \delta_x) \leq \int_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} \rho(dy) \; W_\rho^{\delta_x + \delta_y}(|\tilde{g}_z|)$$

This is estimated once more using the strong cluster property of the Ursell function g. It is proven in [6], (see p. 577) that

$$\int_{\mathcal{M}} W_\rho(d\mu) z^{|\mu|+2} \cdot g(\mu + \delta_x + \delta_y) \leq \frac{e^{2\beta B}}{1 - z \cdot e^{2\beta B + 1} \cdot \|q\|_l} \cdot Q_z(x, y), x, y \in \mathcal{X},$$

where $\|Q_z\|_\infty = \sup_{x,y \in \mathcal{X}} |Q_z(x, y)| < +\infty$. Thus

$$W_\rho^{\delta_x + \delta_y}(|\tilde{g}_z|) \leq D := \frac{e^{2\beta B}}{1 - z \cdot e^{2\beta B + 1} \cdot \|q\|_l} \cdot \|Q_z\|_\infty$$

and consequently

$$|\mathcal{J}_1(R)| \leq D \cdot \int_{S(\Lambda_R^{bd})} \rho(dx)\ \rho(\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}))$$

More explicitly

$$|\mathcal{J}_1(R)| \leq \frac{2 \cdot D}{\pi\beta} \cdot \delta R^\varepsilon \cdot \int_{\partial\,\Lambda_R} \sigma_R(dr)\rho(\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}))$$

Here we used that $P_0(\mathcal{X}^0) = \frac{1}{\pi\beta}$ and $|1 - t \cdot \kappa(r \mid R)| \leq 2$ if $0 \leq t \leq \delta R^\varepsilon$. Thus it remains to study the term

$$\mathcal{J}_1(r, R) := \rho(\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})), r \in \partial\,\Lambda_R.$$

This can be written as follows

$$\mathcal{J}_1(r, R) = \int\limits_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int\limits_0^\infty d\eta \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}((\xi, \eta) + y^0).$$

For each admissible domain Λ there exists a constant $C = C(\Lambda, \delta)$ such that for all $|\xi| < \delta R^\varepsilon$ we have $|f_r(\xi)| \leq C \cdot |\xi|^2$. Thus

$$|f_{r,R}(\xi)| \leq C \cdot \delta^2 \cdot R^{2\varepsilon - 1},\ \text{if } |\xi| < \delta R^\varepsilon.$$

Therefore any loop $y^0 \in \mathcal{X}^0$ such that $(\xi, \eta) + y^0 \in \mathcal{X}^c\left(\Pi^+_{r,\delta R^\varepsilon}\right)\mathcal{X}^c\left(\mathcal{F}^+_{r,\delta R^\varepsilon}\right)$ satisfies the bound $\sup \|y^0\| > \delta R^\varepsilon$ for all R large enough ($R > (\delta C)^{\frac{1}{1-\varepsilon}}$ e.g.) if only $\eta > 2\delta R^{1/8}$. Combining this with the well known fact that $P^0\{x^0 \in \mathcal{X}^0 \mid \sup |x^0| > \alpha\} \leq C \cdot e^{-C\alpha^2}$, we obtain for

$$\begin{aligned} \mathcal{J}_1(r, R) = & \int\limits_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int\limits_0^{2\delta R^\varepsilon} d\eta \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})}((\xi, \eta) + y^0) \\ & + o(1) = \tilde{\mathcal{J}}_1(r, R) + o(1) \end{aligned} \tag{14}$$

For $x^0 \in \mathcal{X}^0$ let $\tilde{x}^0 = x^0 - \langle x^0, \mathbf{n}\rangle \cdot \mathbf{n}$ be the projection of x^0 onto the tangent plane at r. Let

$$\begin{cases} A_{r,\delta R^\varepsilon}(\xi) = \{(\eta, y^0) \in [0, 2\delta R^\varepsilon] \times \mathcal{X}^0 \mid \sup -\langle y^0, \mathbf{n}\rangle < \eta < \\ \qquad\qquad\qquad\qquad \sup[f_{r,R}(\xi + \tilde{y}^0) - \langle y^0, \mathbf{n}\rangle]\} \\ \tilde{A}_{r,\delta R^\varepsilon} = \{(\xi, y^0) \in [-\delta R^\varepsilon, \delta R^\varepsilon] \times \mathcal{X}^0 \mid |\xi + \tilde{y}^0| < \delta R^\varepsilon\}. \end{cases}$$

By Fubini's theorem

$$
\begin{aligned}
\tilde{\mathcal{J}}_1(r,R) &= \int\limits_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\tilde{A}_{r,\delta R^\varepsilon}}(\xi, y^0) \int\limits_0^{2\delta R^\varepsilon} d\eta 1_{A_{r,\delta R^\varepsilon}(\xi)}(\eta, y^0) \\
&= \int\limits_{-\delta R^\varepsilon}^{\delta R^\varepsilon} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\tilde{A}_{r,\delta R^\varepsilon}}(\xi, y^0) \\
&\quad \cdot \min\{2\delta R^\varepsilon, \sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle\}
\end{aligned}
$$

Let $\tau(\mathbf{n})$ and $\tau_R(\xi)$ be such that

$$
\begin{cases}
\inf\langle y^0, \mathbf{n}\rangle = \langle y^0(\tau(\mathbf{n})), \mathbf{n}\rangle, \\
\sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] = f_{r,R}(\xi+\tilde{y}^0(\tau_R(\xi))) - \langle y^0(\tau_R(\xi)), \mathbf{n}\rangle
\end{cases}
$$

Then $f_{r,R}(\xi+\tilde{y}^0(\tau(\mathbf{n}))) \le \sup[f_{r,R}(\xi+\tilde{y}^0) - (y^0, \mathbf{n})] + \inf(y^0, \mathbf{n}) \le f_{r,R}(\xi+\tilde{y}^0(\tau_R(\xi)))$. Therefore, taking into account that $|f_{r,R}(\xi)| \le C \cdot R^{-1}|\xi|^2$ for all $|\xi| < \delta R^\varepsilon$ we obtain

$$
|\sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle| \le C \cdot \frac{1}{R} \cdot |\xi| + \sup \|y^0\|^2.
$$

Consider now the subset $D_R = \{x^0 \in X^0 \bigm| \sup|x^0| < R^{1/9}\}$ and recall that $\varepsilon = \frac{1}{8}$. Then for $y^0 \in D_R$

$$
|\sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle| \le C \cdot R^{-3/4}. \tag{15}
$$

We now can decompose $\tilde{\mathcal{J}}_1(r,R)$ for R large enough as follows:

$$
\begin{aligned}
\tilde{\mathcal{J}}_1(r,R) = &\int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\tilde{A}_{r,\delta R^{1/8}}}(\xi, y^0) \cdot 1_{D_R}(y^0) \cdot \\
&\qquad \bigl(\sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle] + \inf\langle y^0, \mathbf{n}\rangle\bigr) \\
&+ \int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\tilde{A}_{r,\delta R^{1/8}}}(\xi, y^0) \cdot 1_{D_R^c}(y_0) \cdot \\
&\qquad [\min\{2\delta R^{1/8}, \sup[f_{r,R}(\xi+\tilde{y}^0) - \langle y^0, \mathbf{n}\rangle]\} + \inf\langle y^0, \mathbf{n}\rangle\}] \\
=:& \tilde{\mathcal{J}}_1^A(r,R) + \tilde{\mathcal{J}}_1^B(r,R).
\end{aligned}
$$

It is easy to estimate $\tilde{\mathcal{J}}_1^B(r,B)$:

$$|\tilde{\mathcal{J}}_1^B(r,R)| \leq \int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{D_R^c}(y^0) \cdot 2\delta R^{1/8} + \sup \|y^0\| = o(1).$$

Thus it remains to estimate $\tilde{\mathcal{J}}_1^A$:

$$\begin{aligned} |\tilde{\mathcal{J}}_1^A(r,R)| &\leq \frac{C}{R} \int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) 1_{\tilde{A}_{r,\delta R^{1/8}}}(\xi, y^0) \cdot 1_{D_R}(y^0) \cdot |\xi| + \sup \|y^0\|^2 \\ &\leq \frac{C}{R} \int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi \int\limits_{\mathcal{X}^0} P^0(dy^0) (|\xi| + \sup \|y^0\|)^2 \\ &\leq \frac{C}{R} \cdot \int\limits_{-\delta R^{1/8}}^{\delta R^{1/8}} d\xi (\xi^2 \cdot (\pi\beta)^{-\nu/2} + |\xi| \cdot a_1 + a_2) \end{aligned}$$

where $a_n = \int_{\mathcal{X}^0} dy^0 \sup \|y^0\|^n < \infty, n = 1, 2$. This implies

$$|\tilde{\mathcal{J}}_1^A(r,R)| \leq C \cdot R^{-5/8}$$

Combining this with (15) gives

$$|\mathcal{J}_1(r,R)| \leq C \cdot R^{-5/8}.$$

It remains to study $\mathcal{J}_2(R)$ to terminate the proof of the theorem. We have

$$\begin{aligned} |\mathcal{J}_2(R)| &\leq \int\limits_{\mathcal{X}} dx \; 1_{S(\Lambda_R^{bd})\mathcal{X}(\Pi^+_{2,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}_{2,\delta R^\varepsilon})}(x) \int\limits_{\mathcal{M}} W_\rho(\mu) |\tilde{g}_z| (\mu + \delta_x) = \\ &= \int\limits_{\Lambda_R^{bd}} du \int\limits_{\mathcal{X}^0} P^0(dx^0) 1_{S(\Lambda_R^{bd})\mathcal{X}(\Pi^+_{2,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}_{2,\delta R^\varepsilon})}(x^0 + u) W_\rho^{x^0}(|\tilde{g}_z|). \end{aligned}$$

Since $W_\rho^{x^0}(|\tilde{g}_z|)$ as a function of $x^0 \in \mathcal{X}^0$ is bounded (see [7], Lemma 1) we are in conditions of Proposition 2 from [3], hence

$$|\mathcal{J}_2(R)| = o(R)$$

Combining this with (16) and (14) we find that $\mathcal{J}^B(R) = o(R)$ which completes the proof of the Theorem.

Appendix A. The general mathematical structure of gazes

To make the mathematical structure and properties of the underlying gases more transparent we present them here in a general framework. For proofs we refer to Matthes et al. [4].

Let $\mathcal{X}$ be a polish space, $\mathcal{B}(\mathcal{X})$ its Borel σ-field, $\mathcal{B}_0(\mathcal{X})$ the collection of bounded elements of $\mathcal{B}(\mathcal{X})$. If ρ is a locally finite measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, i.e. ρ is finite on $\mathcal{B}_0(\mathcal{X})$, we define the measure W_ρ on the space $\mathcal{M}^{\cdot\cdot}(\mathcal{X})$ of finite sums of Dirac measures on $\mathcal{X}$ by

$$W_\rho(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}} \overset{n}{\cdots} \int_{\mathcal{X}} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \rho(dx_1) \ldots \rho(dx_n),$$

where φ is non-negative and measurable with respect to the σ-field $\mathcal{F}^{\cdot\cdot}(\mathcal{X})$, generated by the counting variables $\zeta_B : \mathcal{M}^{\cdot\cdot}(\mathcal{X}) \to \mathbb{N}_0, \zeta_B(\mu) = \mu(B), B \in \mathcal{B}_0(\mathcal{X})$.

If ρ is diffuse then W_ρ is supported by $\mathcal{M}^{\cdot}(\mathcal{X}) = \{\mu \in \mathcal{M}^{\cdot\cdot}(\mathcal{X}) \mid \mu(\{x\}) \in \{0,1\}$ for any $x \in \mathcal{X}\}$. Thus $\mathcal{M}^{\cdot}$ can be considered as the set of all finite subsets (configurations) of $\mathcal{X}$). One can show that $\mathcal{M}^{\cdot}(\mathcal{X}) \in \mathcal{F}^{\cdot\cdot}(\mathcal{X})$.

The situation considered in this paper is $\mathcal{X} = \{x \in \mathcal{C}([0,\beta], \mathbb{R}^\nu) \mid x(0) = x(\beta)\}$ on which we consider the image ρ of the product measure $\lambda^\nu \otimes P^0$ under the mapping $(u, x^0) \to u + x^0$. Here λ^ν is the Lebesgue measure on $\mathbb{R}^\nu$ and P^0 the non-normalized Brownian bridge measure on $\mathcal{X}^0 = \{x \in \mathcal{X} \mid x(0) = x(\beta) = 0\}$. ρ is diffuse. Thus W_ρ is supported by $\mathcal{M}^{\cdot}(\mathcal{X})$. Moreover, W_ρ is translation invariant in the sense that W_ρ is invariant under shifts $\mu \mapsto \mu - a$ for any $a \in \mathbb{R}^\nu$.

The Campbell measure. Partial integration

W_ρ can be analysed by means of its **Campbell measure** (see [4] e.g.). This is defined by

$$\mathcal{C}_\rho(h) = \int_{\mathcal{M}^{\cdot\cdot}(\mathcal{X})} \int_{\mathcal{X}} h(x,\mu)\ \mu(dx) W_\rho(d\mu), h \geq 0 \text{ and measurable.}$$

It is well known that $\mathcal{C}_\rho$ is given by

$$\mathcal{C}_\rho(h) = \int_{\mathcal{M}^{\cdot\cdot}(\mathcal{X})} \int_{\mathcal{X}} h(x, \mu + \delta_x)\ \rho(dx) W_\rho(d\mu), h \geq 0 \text{ and measurable.} \quad \text{(A.2)}$$

This **partial integration formula** remains true for each $h \in \mathcal{L}^1(\mathcal{C}_\rho)$. $\mathcal{C}_\rho$ is concentrated on the measurable subset

$$\Gamma = \{(x,\mu) \in \mathcal{X} \times \mathcal{M}^{\cdot\cdot}(\mathcal{X}) \mid \mu(\{x\}) \geq 1\}.$$

The importance of the Campbell measure relies on the fact that it determines W_ρ uniquely outside the set $\{0\}$ which consists only on the void configuration. It is now evident that the following inversion formula is true

$$W_\rho(\varphi) = \mathcal{C}_\rho(\tilde{\varphi}), \varphi \geq 0, \text{ measurable with } \varphi(0) = 0.\text{(see [4])}$$

Here $\hat{\varphi}(x,\mu) = \frac{\varphi(\mu)}{|\mu|}, (x,\mu) \in \Gamma$. Hence for any $z > 0$ and $B \in \mathcal{B}_0(X)$

$$W_{z\cdot\rho_B}(\varphi) = \mathcal{C}_\rho\left(1_{B\times\mathcal{M}^{\cdot\cdot}(B)} \cdot \tilde{\varphi}_z\right), \varphi \in \mathcal{L}^1(W_{z\cdot\rho_B}) \text{ with } \varphi(0) = 0, \tag{A.3}$$

where $\hat{\varphi}_z(x,\mu) = z^{|\mu|} \cdot \hat{\varphi}(x,\mu)$. (Recall $|\mu| = \mu(\mathcal{X})$.)

Note that $\hat{\varphi}$ respectively $\hat{\varphi}_z$ do not depend on x.

Thus

$$W_{z\cdot\rho_B}(\varphi) = \int_B \int_{\mathcal{M}^{\cdot\cdot}(B)} \tilde{\varphi}_z(\mu + \delta_x)\ \rho(dx) W_\rho(d\mu), \varphi \in \mathcal{L}^1(W_{z\cdot\rho_B}) \text{ with } \varphi(0) = 0. \tag{A.4}$$

Appendix B. The main lemma

For the convenience of the reader we recall the main lemma the proof of which can be found in [6]: The underlying measure space $(\mathcal{X}, \rho)$ is the one of the Ginibre gas.

Consider the functional

$$\mathcal{H}^u(\varphi) = \int_{\mathcal{X}^u} P^u(dx^u) W^{x^u}(\varphi \cdot \tilde{g}_z), \varphi \geq 0 \text{ measurable.}$$

Main Lemma. (α) If $\Phi \in \mathcal{P}_0$ and $0 < z < \pi[e^{4\beta B+1} p_0(\Phi)]^{-1}$ then $\mathcal{H}^u$ defines a finite signed measure on $\mathcal{M}$ with $\mathcal{H}^u(1)$ being independent of u.

(β) If $\Phi \in \mathcal{P}_e$ and $0 < z < \pi[4^{e+3} e^{4\beta B} p_e(\phi)]^{-1}$ then there exists a constant c such that

$$|\mathcal{H}^u|(\mathcal{M}^c(K_R(u))) \leq \frac{c}{\left(1 + \frac{2}{3}R\right)^l} \text{ uniformly in } u. \tag{B.1}$$

Here $K_R(u)$ is the closed ν-ball centered in u with Radius R, and $|\mathcal{H}^u|$ the measure defined by (B.1) with $\tilde{g}_z$ replaced by $|\tilde{g}_z|$.

The proof follows from the estimate

$$|\mathcal{H}^u|(\mathcal{M}^c(K_R(u))) \leq const \cdot P^u(\mathcal{X}^c(K_R(u))) + \\ + \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{M}^c(K_R(u))} W_\rho(d\mu)|\tilde{g}_z|(\mu + \delta_{x^0+u}).$$

Here $P^u(\mathcal{X}^c(K_R(u))) \leq c \cdot e^{-c' \cdot R^2}$ and by corollary 5.1 of [6]

$$\int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{M}^c(K_R(u))} W_\rho(d\mu)|\tilde{g}_z|(\mu + \delta_{x^0+u}) \leq c'' \cdot \frac{1}{\left(1 + \frac{2}{3}R\right)^l}$$

uniformly in u. This implies the assertion.

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