Cohomology of Groups: A Crossroads in Mathematics

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1. Homological Algebra

- Let G be a group, k a commutative ring of coefficients

(the phrase "of coefficients" here has the empty meaning as usual)

Definition

$H^*(G,k) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},k) \cong \mathrm{Ext}^*_{kG}(k,k)$

In other words, take a Projective Resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

and form the complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_0,k) \to \mathrm{Hom}_{\mathbb{Z}G}(P_1,k) \to \cdots$$

Now take homology: $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},k)$ is kernel mod image in nth place

- The answer is independent of choice of projective resolution (up to natural isomorphism)
- More generally, if M is a $\mathbb{Z}G$-module we can take $\mathrm{Hom}_{\mathbb{Z}G}(P_*,M)$ and define $H^*(G,M)$ the same way.

If we tensor with k:

$$\cdots \to k \otimes_{\mathbb{Z}} P_n \to \cdots \to k \otimes_{\mathbb{Z}} P_1 \to k \otimes_{\mathbb{Z}} P_0 \to k \to 0$$

this is a projective resolution of k. Hence if M is a kG-module

$$\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M) \cong \mathrm{Ext}^n_{kG}(k, M).$$

Some Facts:

- $H^*(G,k)$ is a graded commutative ring:
$$yx = (-1)^{|x||y|}xy.$$
- $H^*(G,M)$ is a graded $H^*(G,k)$-module.
- (Evens) If G is finite and M is a Noetherian kG-module then $H^*(G,M)$ is a Noetherian $H^*(G,k)$-module.

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- (Evens) If G is finite and M is a Noetherian kG-module then $H^*(G, M)$ is a Noetherian $H^*(G, k)$-module.
- In particular, if k is Noetherian so is $H^*(G, k)$.
- If k is a field, then $H^*(G, k)$ is a finitely generated graded commutative k-algebra.
- If $\mathrm{char}(k)$ is zero or does not divide $|G|$ then there's nothing interesting here: you just get k in degree zero.
- More generally, for any k, $|G|$ annihilates positive degree elements.

Example (The Mathieu Group M_{11})

$G = M_{11}$, $\mathrm{char}(k) = 2$: $H^*(G, k) = k[x, y, z]/(x^2y + z^2)$ where $|x| = 3$, $|y| = 4$, $|z| = 5$.

2. Commutative Algebra

- Commutative algebraists usually write their theorems assuming that commutative means $xy = yx$; this is bad for us.
- They also often require their generators to be in the same degree; this is almost never the case for group cohomology.
- Nonetheless, we can talk about the usual commutative algebra concepts such as
  - Depth
  - Cohen–Macaulay
  - Gorenstein
  - Complete Intersection
  - Local Cohomology
  - Castelnuovo–Mumford regularity
  - etc.

2. Commutative Algebra, Contd.

Theorem (Quillen 1971)

If $\mathrm{char}(k) = p$ then the Krull dimension of $H^*(G,k)$ is equal to the p-rank of G, namely the largest r for which $(\mathbb{Z}/p)^r \le G$.

More generally, he described the prime ideal spectrum of $H^*(G,k)$ in terms of the elementary abelian subgroups:

$$H^*(G,k) \to \varprojlim H^*(E,k)$$

is an F-isomorphism — it induces an isomorphism of varieties.

Theorem (Duflot 1981)

The depth of $H^*(G,k)$ is at least the p-rank of the centre of a Sylow p-subgroup of G.

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2. Commutative Algebra, Contd.

Theorem (B–Carlson, 1994)

(i) *If $H^*(G,k)$ is Cohen–Macaulay then it's Gorenstein.*
(ii) *If $H^*(G,k)$ is a polynomial ring then the generators are all in degree one; in this case $p = 2$ and G modulo an odd order normal subgroup is $(\mathbb{Z}/2)^r$.*

Theorem (Conjectured by me in 2004, proved by Symonds 2010)

The Castelnuovo–Mumford regularity of $H^*(G,k)$ is always equal to zero.

As a consequence, $\dim H^n(G,k)$ is polynomial on residue classes, not just eventually so. So if you know $\dim H^n(G,k)$ for $n > 1000000$ then you know it for all $n \geq 0$.

$2\frac{1}{2}$. Representation Theory

Definition

The stable module category $\mathrm{StMod}(kG)$ is the quotient of the module category $\mathrm{Mod}(kG)$ by the projective modules. It is a compactly generated tensor triangulated category.

Theorem (BIK, Annals 2011)

There is a natural bijection between (tensor ideal) minimal localising subcategories of $\mathrm{StMod}(kG)$ *and nonmaximal homogeneous prime ideals in* $H^*(G,k)$.

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3. Topology

- EG - a contractible space on which G acts freely
- BG - the quotient EG/G
- $H^*(G,k) = H^*(BG;k)$
- Up to homotopy, BG is independent of choice of EG.
- $\Omega BG \simeq G$.
- Relationship with algebraic definition:
- $C_*(EG)$ is an acyclic complex of free $\mathbb{Z}G$-modules; i.e., a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

$$\begin{aligned} H^*(BG;k) &= H^*(\mathrm{Hom}_{\mathbb{Z}}(C_*(BG),k)) \\ &= H^*(\mathrm{Hom}_{\mathbb{Z}G}(C_*(EG),k)) \\ &\cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},k). \end{aligned}$$

3. Topology: Examples

1. $G = \mathbb{Z}$; $EG = \mathbb{R}$; $BG = \mathbb{R}/\mathbb{Z} = S^1$.
   $H^0(\mathbb{Z}, k) \cong H^1(\mathbb{Z}, k) \cong k$, $H^i(\mathbb{Z}, k) = 0$ for $i \geq 2$.
2. $G = \mathbb{Z}/2$; $EG = S^\infty$; $BG = \mathbb{R}P^\infty$.
   If $\mathrm{char}(k) = 2$ then $H^*(G, k) = k[x]$ with $|x| = 1$.
3. $G = \mathbb{Z}/2 \times \mathbb{Z}/2$; $EG = S^\infty \times S^\infty$; $BG = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$.
   If $\mathrm{char}(k) = 2$ then $H^*(G, k) = k[x, y]$ with $|x| = |y| = 1$.
4. $G = Q_8 \subseteq SU(2) \cong S^3 =$ unit quaternions
   G acts freely on S^3 by left multiplication
   Cellular chains $C_*(S^3)$:

   $$0 \to \mathbb{Z} \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

   Form an infinite splice:

   $$\begin{array}{ccccccccccccc}
\cdots \to C_1 \to C_0 & & \longrightarrow & & C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0 \\
& \searrow & & \nearrow & \\
& & \mathbb{Z} & & \\
& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}$$

Conclusion: Let k be a field of characteristic 2.
$H^*(Q_8, k)$ is periodic with periodicity 4.
In fact the periodicity is given by multiplication by $z \in H^4(Q_8, k)$ and $H^*(Q_8, k)/(z) \cong H^*(S^3/Q_8; k)$.

Theorem

If G acts freely on S^{n-1} then $H^*(G, k)$ is periodic with period dividing n.

Example

If $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ then G cannot act freely on any sphere of any dimension.

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Notice that S^3/Q_8 is a manifold so $H^*(Q_8, k)/(z)$ satisfies Poincaré duality.

Definition

We say that a finitely generated positively graded commutative k-algebra is Cohen–Macaulay if it is a finitely generated free module over a polynomial subring $k[z_1, \dots, z_r]$.

Noether Normalization: $\exists k[z_1, \dots, z_r]$ over which it's f.g., and whether it's free is independent of choice of normalization.

Theorem (B–Carlson)

If $H^*(G, k)$ is Cohen–Macaulay then $H^*(G, k)/(z_1, \dots, z_r)$ is a finite Poincaré duality ring with top degree $\sum_{i=1}^{r}(|z_i| - 1)$ i.e., $H^*(G, k)$ is Gorenstein.

More generally, even if $H^*(G,k)$ is not Cohen–Macaulay, there is a spectral sequence converging to a finite Poincaré duality ring.

Greenlees reformulated this more cleanly as a local cohomology spectral sequence

$$H^s_{\mathfrak{m}}H^t(G,k) \Rightarrow H_{-s-t}(G,k).$$

Symonds' theorem states that the E_2 page is zero for $s+t>0$. There is a sense in which the cochains on BG are always derived Gorenstein as a DGA (Dwyer-Greenlees-Iyengar)

5. A glimpse of p-completion

Let p be a prime. Bousfield–Kan p-completion is a functor from spaces to spaces together with a natural transformation $X \to X^\wedge_p$

- $X \to Y$ induces $H_*(X, \mathbb{F}_p) \xrightarrow{\cong} H_*(Y, \mathbb{F}_p)$ iff $X^\wedge_p \xrightarrow{\simeq} Y^\wedge_p$
- X is p-complete if $X \xrightarrow{\simeq} X^\wedge_p$
- X is p-good if $X^\wedge_p$ is p-complete
- Otherwise X is p-bad and $X^{\wedge\,\wedge}_{p\,p} \dots$ is still p-bad!
- X connected, $\pi_1 X$ finite implies X p-good
- In particular if G is finite BG is p-good
- BG is p-complete $\Leftrightarrow$ G is p-nilpotent $(G/O_{p'}G$ is a p-group$)$
- The Eilenberg–Moore spectral sequence whose E^2 page is $\mathrm{Tor}_{**}^{H^*(BG,\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$ doesn't converge to $\mathbb{F}_p G$ but rather to $H_*(\Omega BG^\wedge_p, \mathbb{F}_p)$.

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5. A glimpse of p-completion, Contd.

- $BG^\wedge_p$ only depends on the p-local structure of G.
- More precisely, there's a category $\mathcal{L}^c_S(G)$ defined as follows: Let $S \in \mathrm{Syl}_p(G)$.
- Objects: subgroups $H \le S$ satisfying
  $C_G(H) = Z(H) \times O_{p'}C_G(H)$
- Arrows:
  $\mathrm{Hom}_{\mathcal{L}^c_S(G)}(H, K) = \{g \in G \mid gHg^{-1} \subseteq K\}/O_{p'}C_G(H).$

Theorem (Broto–Levi–Oliver)

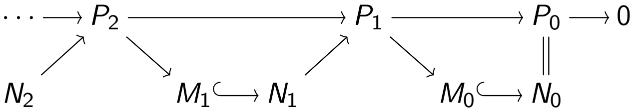
$|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$, and one can recover $\mathcal{L}^c_S(G)$ from $BG^\wedge_p$.

If M is an $\mathbb{F}_p G$-module, define $[O^p G, M]$ to be the linear span of the elements $g(m) - m$ with $g \in O^p G$ and $m \in M$
This is the smallest submodule of M such that the quotient has a filtration where G acts trivially on the filtered quotients

- $P_0 = N_0$ = projective cover of $\mathbb{F}_p$ as $\mathbb{F}_p G$-module

For $i \geq 1$,

- $M_{i-1} = [O^p G, N_{i-1}]$
- P_i = projective cover of M_{i-1}
- $N_i = \Omega M_{i-1}$



Theorem (B, 2009)

$$H_i(P_*) = N_i / M_i \cong H_i(\Omega BG^\wedge_p; \mathbb{F}_p).$$