Rank one groups and trees

Pierre-Emmanuel Caprace

Rank one groups and trees

Pierre-Emmanuel Caprace
Tom De Medts
Yves de Cornulier - Nicolas Monod - Romain Tessera

Bernd Fischer

Bernd Fischer

CFSG

Bernd Fischer

CFSG

Classification of the Fischer Simple Groups

Bernd Fischer

CFSG

Classification of the Fischer Simple Groups

CCSG

Bernd Fischer

CFSG

Classification of the Fischer Simple Groups

CCSG

Classification of the Compact Simple Groups

Bernd Fischer

CFSG

Classification of the Fischer Simple Groups

CCSG

Classification of the Compact Simple Groups

Theorem (Von Neumann 1935)

Every compact simple group is connected Lie or finite.

Theorem (Von Neumann 1935)

Every compact simple group is connected Lie or finite.

Theorem (Von Neumann 1935)

Every compact simple group is connected Lie or finite.

Theorem (Gleason; Montgomery-Zippin 1950)

Every almost connected simple locally compact group is connected Lie or finite.

Theorem (Von Neumann 1935)

Every compact simple group is connected Lie or finite.

Theorem (Gleason; Montgomery-Zippin 1950)

Every almost connected simple locally compact group is connected Lie or finite.

Beyond?

Compactly generated simple locally compact groups

Compactly generated simple locally compact groups

- Finite simple groups

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???
- Finitely generated simple groups

Theorem (Ph. Hall; P. Schupp 1974)

Any countable group embeds in a 2-generated simple group.

Theorem (Ph. Hall; P. Schupp 1974)

Any countable group embeds in a 2-generated simple group.

Wrong for non-discrete groups

Theorem (Ph. Hall; P. Schupp 1974)

Any countable group embeds in a 2-generated simple group.

Wrong for non-discrete groups

Non-discrete topology

$\Rightarrow$ algebraic restrictions

Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???
- Finitely generated simple groups

Non-discrete Compactly generated simple locally compact groups

- Finite simple groups
- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???
- Finitely generated simple groups

Non-discrete Compactly generated simple locally compact groups

- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???

Non-discrete Compactly generated simple locally compact groups

linear

- Connected simple Lie groups
- Simple algebraic groups over local fields
- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???

non-linear

Non-discrete Compactly generated simple locally compact groups

- Connected simple Lie groups
- Simple algebraic groups over local fields

linear

- Complete Kac-Moody groups over finite fields
- Automorphism groups of trees, npc complexes
- Avatars
- ???

non-linear

Non-Lie type simple groups are not sporadic!

Problem

Characterise linear groups among (simple) l.c. groups?

Problem

Characterise linear groups among (simple) l.c. groups?

- Solved Gleason; Montgomery-Zippin (1950) for **connected** groups

Problem

Characterise linear groups among (simple) l.c. groups?

- Solved Gleason; Montgomery-Zippin (1950) for **connected** groups
- Solved by Lazard (1960) and Lubotzky-Mann (1989) for ***p*-adic fields**

Problem

Characterise linear groups among (simple) l.c. groups?

- Solved Gleason; Montgomery-Zippin (1950) for **connected** groups
- Solved by Lazard (1960) and Lubotzky-Mann (1989) for ***p*-adic fields**
- Useful for discrete groups

Problem

Characterise linear groups among (simple) l.c. groups?

- Solved Gleason; Montgomery-Zippin (1950) for **connected** groups
- Solved by Lazard (1960) and Lubotzky-Mann (1989) for ***p*-adic fields**
- Useful for discrete groups
- **Open in characteristic >0**

Problem

Characterise linear groups among (simple) l.c. groups?

- Solved Gleason; Montgomery-Zippin (1950) for **connected** groups
- Solved by Lazard (1960) and Lubotzky-Mann (1989) for ***p*-adic fields**
- Useful for discrete groups
- **Open in characteristic >0**
- Unified solution in all characteristics?

Theorem (Tits 1974; Tits-Weiss 2002)

Let G be a simple group.

If G has an irreducible split spherical BN-pair of rank ≥ 2, then G is linear (possibly over a skew- field).

Theorem (Tits 1974; Tits-Weiss 2002)

Let G be a simple group.

If G has an irreducible split spherical BN-pair of rank ≥ 2, then G is linear (possibly over a skew- field).

- **Rank one case?**

Theorem (Tits 1974; Tits-Weiss 2002)
Let G be a simple group.
If G has an irreducible split spherical BN-pair of rank ≥ 2, then G is linear (possibly over a skew- field).

- **Rank one case?**
- More natural conditions in the l.c. context?

Contraction groups

Let G be a l.c. group, let $\alpha \in G$.

Contraction groups

Let G be a l.c. group, let $\alpha \in G$.

The **contraction group** associated with α is

$$U_\alpha = \{\, g \in G \mid \lim_{n\to\infty} \alpha^n g \alpha^{-n} = 1 \,\}.$$

Contraction groups

Let G be a l.c. group, let $\alpha \in G$.

The **contraction group** associated with α is

$$U_\alpha = \{ g \in G \mid \lim_{n\to\infty} \alpha^n g \alpha^{-n} = 1 \}.$$

The **parabolic group** associated with α is

$$P_\alpha = \{ g \in G \mid \alpha^n g \alpha^{-n} \text{ is bounded} \}.$$

Examples

- $G = \mathbf{R}^n \rtimes \mathbf{R}$ with $\mathbf{R}$-action by homotheties

  $0 \neq \alpha \in \mathbf{R}$ $\quad U_\alpha = \mathbf{R}^n$ $\quad P_\alpha = G$

Examples

- $G = \mathbf{R}^n \rtimes \mathbf{R}$ with $\mathbf{R}$-action by homotheties

  $0 \neq \alpha \in \mathbf{R}$ $\qquad U_\alpha = \mathbf{R}^n$ $\qquad P_\alpha = G$

- $G = SL_2(\mathbf{R})$

$$\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$U_\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbf{R} \right\}$$

$$P_\alpha = \left\{ \begin{pmatrix} s & 0 \\ x & s^{-1} \end{pmatrix} \mid s, x \in \mathbf{R} \right\}$$

F = non-trivial finite group

$G = \Pi_{\mathbf{Z}} F \rtimes_{\alpha} \mathbf{Z}$ α = positive shift

$U_{\alpha} = (\oplus_{n<0} F) \oplus (\Pi_{n \geq 0} F)$

$P_{\alpha} = G$

F = non-trivial finite group

$G = \prod_{\mathbf{Z}} F \rtimes_\alpha \mathbf{Z}$ α = positive shift

$$U_\alpha = (\oplus_{n<0} F) \oplus (\prod_{n\geq 0} F)$$

$$P_\alpha = G$$

Warning: The contraction group U_α need not be closed in general!

Warning: The contraction group U_α need not be closed in general!

However, all contraction subgroups $U_\alpha < G$ are closed when G is:

- a Lie group [Hazod-Siebert, 1986]
- a p-adic analytic group [Wang, 1988]

Warning: The contraction group U_α need not be closed in general!

However, all contraction subgroups $U_\alpha < G$ are closed when G is:

- a Lie group [Hazod-Siebert, 1986]
- a p-adic analytic group [Wang, 1988]

Relationship between closedness of contraction groups and smoothness/linearity?

Compaction groups

Let G be a l.c. group, let $\alpha \in G$ and $C \leq G$ be an α-stable compact subgroup.

Compaction groups

Let G be a l.c. group, let $\alpha \in G$ and $C \leq G$ be an α-stable compact subgroup.

Conjugation by α descends to a homeo of G/C.

Compaction groups

Let G be a l.c. group, let $\alpha \in G$ and $C \leq G$ be an α-stable compact subgroup.

Conjugation by α descends to a homeo of G/C.

The **compaction group** associated with α and C is

$$U_{C;\alpha} = \{ g \in G \mid \lim_{n\to\infty} \alpha^n g \alpha^{-n} = 1 \text{ modulo } C \}.$$

Compaction groups

Let G be a l.c. group, let $\alpha \in G$ and $C \leq G$ be an α-stable compact subgroup.

Conjugation by α descends to a homeo of G/C.

The **compaction group** associated with α and C is

$$U_{C;\,\alpha} = \{\, g \in G \mid \lim_{n\to\infty} \alpha^n g \alpha^{-n} = 1 \text{ modulo } C \,\}.$$

Lemma

α acts as a compaction on the closure of U_α.

Lemma

α acts as a compaction on the closure of U_α.

F = non-trivial finite group

$G = \Pi_{\mathbf{Z}} F \rtimes_\alpha \mathbf{Z}$

α = positive shift

$U_\alpha = (\oplus_{n<0} F) \oplus (\Pi_{n \geq 0} F)$

$\overline{U_\alpha} = \Pi_{\mathbf{Z}} F$

Theorem 1 (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Theorem 1 (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ such that

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

Theorem 1 (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ such that

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

- Characteristic-free

Theorem 1 (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

- Characteristic-free
- Crucial step in the proof: U_α is closed

Theorem II (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) U_α *is closed and torsion-free.*

Theorem II (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) U_α *is closed and torsion-free.*

Then there is a l.c. field k of char 0 *and a semisimple algebraic k-group* **G** *of k-rank one such that* **G**(*k*) ≤ *G with finite index.*

Digression

Question (Milnor 1976)

Which connected Lie groups admit an invariant Riemannian metric of negative sectional curvature?

Digression

Question (Milnor 1976)

Which connected Lie groups admit an invariant Riemannian metric of negative sectional curvature?

Theorem (Heintze 1974)

A Lie group G is negatively curved iff $G \cong N \rtimes \mathbf{R}$ with N s.c. nilpotent and contracting action of $\mathbf{R}$ on N.

Question (Milnor 1976)
Which connected Lie groups admit an invariant Riemannian metric of negative sectional curvature?

Theorem (Heintze 1974)
A Lie group G is negatively curved iff $G \cong N \rtimes \mathbf{R}$ with N s.c. nilpotent and contracting action of $\mathbf{R}$ on N.

- Lie group $\Rightarrow$ l.c. group

Question (Milnor 1976)
Which connected Lie groups admit an invariant Riemannian metric of negative sectional curvature?

Theorem (Heintze 1974)
A Lie group G is negatively curved iff $G \cong N \rtimes \mathbf{R}$ with N s.c. nilpotent and contracting action of $\mathbf{R}$ on N.

- Lie group ⇒ l.c. group
- Soluble ⇒ amenable

Question (Milnor 1976)
Which connected Lie groups admit an invariant Riemannian metric of negative sectional curvature?

Theorem (Heintze 1974)
A Lie group G is negatively curved iff $G \cong N \rtimes \mathbf{R}$ with N s.c. nilpotent and contracting action of $\mathbf{R}$ on N.

- Lie group ⇒ l.c. group
- Soluble ⇒ amenable
- Negatively curved ⇒ Gromov hyperbolic

Question

Which l.c. groups are both amenable and hyperbolic?

Question

Which l.c. groups are both amenable and hyperbolic?

Theorem (C.-Cornulier-Monod-Tessera 2011)

A l.c. G is amenable and non-elementary hyperbolic iff $G \cong N \rtimes \mathbf{R}$ or $G \cong N \rtimes \mathbf{Z}$ with N closed and compacting action of $\mathbf{R}$ or $\mathbf{Z}$ on N.

Theorem (C.-Cornulier-Monod-Tessera 2011)

A l.c. G is amenable and non-elementary hyperbolic iff $G \cong N \rtimes \mathbf{R}$ or $G \cong N \rtimes \mathbf{Z}$ with N closed and compacting action of $\mathbf{R}$ or $\mathbf{Z}$ on N.

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ such that $G/\overline{\langle \alpha U_\alpha \rangle}$ is compact.

Theorem (C.-Cornulier-Monod-Tessera 2011)

A l.c. G is amenable and non-elementary hyperbolic iff $G \cong N \rtimes \mathbf{R}$ or $G \cong N \rtimes \mathbf{Z}$ with N closed and compacting action of $\mathbf{R}$ *or* $\mathbf{Z}$ *on N.*

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ such that $G/\overline{\langle \alpha U_\alpha \rangle}$ is compact.

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ *such that* $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact.*

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

Proof.

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ *such that* $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact.*

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

Proof.

$$\overline{\langle \alpha U_\alpha \rangle} \cong \langle \alpha \rangle \ltimes \overline{U_\alpha} \cong \mathbf{Z} \ltimes N$$

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ *such that* $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact.*

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

Proof.

$$\overline{\langle \alpha U_\alpha \rangle} \cong \langle \alpha \rangle \ltimes \overline{U_\alpha} \cong \mathbf{Z} \ltimes N$$

$$\Rightarrow \overline{\langle \alpha U_\alpha \rangle} \text{ hyperbolic}$$

Corollary

Let G be a l.c. group.

Assume there is $\alpha \in G$ *such that* $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact.*

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

Proof.

$\overline{\langle \alpha U_\alpha \rangle} \cong \langle \alpha \rangle \ltimes \overline{U_\alpha} \cong \mathbf{Z} \ltimes N$

$\Rightarrow \overline{\langle \alpha U_\alpha \rangle}$ hyperbolic

$\Rightarrow G$ hyperbolic.

Theorem (C.-Cornulier-Monod-Tessera 2011)

Let G be a l.c. group.

Assume there is $\alpha \in G$ *such that* $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact.*

Then G has a continuous proper cocompact action on a Gromov hyperbolic metric space.

If in addition G is unimodular, then modulo a compact normal subgroup, either:

- *G is a rank one simple Lie group, or*
- $G \leq \mathrm{Aut}(T)$ *and is 2-transitive on* ∂T *for some tree T.*

Back to Theorem I

Theorem I (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

Theorem I (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

(i) $\Rightarrow$ either G is a simple Lie group,

or $G \leq \mathrm{Aut}(T)$ is 2-transitive on ∂T.

Theorem I (C.-De Medts 2011)

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) P_α *is metabelian.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *for some l.c. field k.*

(i) $\Rightarrow$ either G is a simple Lie group,

or $G \leq \mathrm{Aut}(T)$ is 2-transitive on ∂T.

Let $G \leq \mathrm{Aut}(T)$ *closed non-compact act*

(i) 2-*transitively on* ∂T,

(ii) *with metabelian stabilisers.*

Then $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$ *and* $\partial T \cong \mathbf{P}^1(k)$ *with k locally compact field.*

Projective permutation groups

Theorem (Tits, 1949)

Let $G < \mathrm{Sym}(\Omega)$ be 3-transitive group.

If $G_{x,y}$ is abelian for $x \neq y \in \Omega$, then $G = \mathrm{PGL}_2(k)$ for some field k.

Projective permutation groups

Theorem (Tits, 1949)

Let $G < \mathrm{Sym}(\Omega)$ *be 3-transitive group.*

If $G_{x,y}$ *is abelian for* $x \neq y \in \Omega$*, then* $G = \mathrm{PGL}_2(k)$ *for some field* k*.*

Abelian transitive $\Rightarrow$ sharply transitive

Projective permutation groups

Theorem (Tits, 1949)

Let $G < \mathrm{Sym}(\Omega)$ *be* 3*-transitive group.*

If $G_{x,y}$ *is abelian for* $x \neq y \in \Omega$, *then* $G = \mathrm{PGL}_2(k)$ *for some field* k.

Abelian transitive ⇒ sharply transitive

Let $G < \mathrm{Sym}(\Omega)$ *be* 3*-transitive group.*

If G_x *is metabelian for* $x \in \Omega$, *then* $G = \mathrm{PGL}_2(k)$ *for some field* k.

Theorem (Tits, 1949)

Let $G < \mathrm{Sym}(\Omega)$ *be 3-transitive group.*

If $G_{x,y}$ *is abelian for* $x \neq y \in \Omega$, *then* $G = \mathrm{PGL}_2(k)$ *for some field k.*

Let $G < \mathrm{Sym}(\Omega)$ *be 3-transitive group.*

If G_x *is metabelian for* $x \in \Omega$, *then* $G = \mathrm{PGL}_2(k)$ *for some field k.*

$\mathrm{PGL}_2(k)/\mathrm{PSL}_2(k) \cong k/k^2$

Moufang sets

Definition (Tits, 1992; Timmesfeld, 1999)

A set Ω with a 2-transitive group $G < \mathrm{Sym}(\Omega)$ is called a **Moufang set** if for some (hence all) $\xi \in \Omega$, the stabiliser G_ξ has a normal subgroup U_ξ , called a **root group**, acting regularly on $\Omega \setminus \{\xi\}$.

Definition (Tits, 1992; Timmesfeld, 1999)

A set Ω with a 2-transitive group $G < \mathrm{Sym}(\Omega)$ is called a **Moufang set** if for some (hence all) $\xi \in \Omega$, the stabiliser G_ξ has a normal subgroup U_ξ, called a **root group**, acting regularly on $\Omega \setminus \{\xi\}$.

Examples:

- G sharply 2-transitive on Ω
- $G = \mathrm{PSL}_2(k)$ acting on $\Omega = \mathbf{P}^1(k)$
- G = any simple algebraic group of rel. rank one

Examples:

- G sharply 2-transitive on Ω
- $G = \mathrm{PSL}_2(k)$ acting on $\Omega = \mathbf{P}^1(k)$
- G = any simple algebraic group of rel. rank one

Hazardous conjecture (Tits, 2000)

All Moufang sets are of « algebraic origin », ie. remotely related to the examples on that list.

Theorem (De Medts-Weiss; Grüninger; Segev)

Let Ω, $G \leq \mathrm{Sym}(\Omega)$ *be a proper Moufang set.*

Assume that U_x *and* $G_{x,y}$ *are both abelian for* $x, y \in \Omega$.

Then there is a field k and:

either $\Omega = \mathbf{P}^1(k)$ *and* $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$,

or char$(k) = 2$ *and* $\Omega \subset \mathbf{P}^1(k)$ *is exceptional*.

Theorem (De Medts-Weiss; Grüninger; Segev)

Let Ω, $G \leq \mathrm{Sym}(\Omega)$ *be a proper Moufang set.*

Assume that U_x *and* $G_{x,y}$ *are both abelian for* $x, y \in \Omega$.

Then there is a field k and:

either $\Omega = \mathbf{P}^1(k)$ *and* $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$,

or $char(k) = 2$ *and* $\Omega \subset \mathbf{P}^1(k)$ *is exceptional*.

- If U_x is abelian, then $G_{x,y}$ is abelian if and only if $G_x = U_x \rtimes G_{x,y}$ is metabelian.

Theorem (De Medts-Weiss; Grüninger; Segev)

Let Ω, $G \leq \mathrm{Sym}(\Omega)$ be a proper Moufang set.

Assume that U_x and $G_{x,y}$ are both abelian for $x, y \in \Omega$.

Then there is a field k and:

either $\Omega = \mathbf{P}^1(k)$ and $\mathrm{PSL}_2(k) \leq G \leq \mathrm{PGL}_2(k)$,

or $\mathrm{char}(k) = 2$ and $\Omega \subset \mathbf{P}^1(k)$ is exceptional.

- If U_x is abelian, then $G_{x,y}$ is abelian if and only if $G_x = U_x \rtimes G_{x,y}$ is metabelian.
- Conjecturally: abelian root groups $\Rightarrow$ quadratic Jordan division algebras

Back to l.c. groups

Proposition

Let $G \leq \mathrm{Aut}(T)$ *closed non-compact such that*

(i) G *is transitive on* ∂X

(ii) U_α *is closed for some* α *hyperbolic.*

Back to l.c. groups

Proposition

Let $G \leq \mathrm{Aut}(T)$ closed non-compact such that

- *(i)* *G is transitive on ∂X*
- *(ii)* *U_α is closed for some α hyperbolic.*

Then ∂T is a Moufang set with U_α as a root group.

Moreover the system of root groups is unique.

Back to l.c. groups

Proposition

Let $G \leq \mathrm{Aut}(T)$ *closed non-compact such that*

(i) G *is transitive on* ∂X

(ii) U_α *is closed for some* α *hyperbolic.*

Then ∂T *is a Moufang set with* U_α *as a root group.*

Moreover the system of root groups is unique.

If the contraction group U_α *is abelian, then it is closed.*

T locally finite tree.

$G < \mathrm{Aut}(T)$ closed, acts 2-transitively on ∂T with metabelian stabilisers.

T locally finite tree.

$G < \mathrm{Aut}(T)$ closed, acts 2-transitively on ∂T with metabelian stabilisers.

Need to show: G is a projective group.

T locally finite tree.

$G < \mathrm{Aut}(T)$ closed, acts 2-transitively on ∂T with metabelian stabilisers.

Need to show: G is a projective group.

G 2-transitive on ∂T

$\Rightarrow G$ transitive on $\mathrm{E}(T)$

$\Rightarrow$ there is $\alpha \in G$ hyperbolic.

T locally finite tree.

$G < \mathrm{Aut}(T)$ closed, acts 2-transitively on ∂T with metabelian stabilisers.

Need to show: G is a projective group.

G 2-transitive on ∂T

$\Rightarrow G$ transitive on $\mathrm{E}(T)$

$\Rightarrow$ there is $\alpha \in G$ hyperbolic.

Let $\xi \in \partial T$ be the repelling fixed point of α.

Let $\xi \in \partial T$ be the repelling fixed point of α.

Then $U_\alpha < P_\alpha = G_\xi$ and for $g \in U_\alpha$, we have

Let $\xi \in \partial T$ be the repelling fixed point of α.

Then $U_\alpha < P_\alpha = G_\xi$ and for $g \in U_\alpha$, we have

$$g = \lim_{n\to\infty} g\alpha^n g^{-1}\alpha^{-n}$$

Let $\xi \in \partial T$ be the repelling fixed point of α.

Then $U_\alpha < P_\alpha = G_\xi$ and for $g \in U_\alpha$, we have

$$g = \lim_{n\to\infty} g\alpha^n g^{-1}\alpha^{-n}$$

$$\Rightarrow \quad g \in \overline{[G_\xi, G_\xi]}$$

Let $\xi \in \partial T$ be the repelling fixed point of α.

Then $U_\alpha < P_\alpha = G_\xi$ and for $g \in U_\alpha$, we have

$$g = \lim_{n\to\infty} g\alpha^n g^{-1}\alpha^{-n}$$

$$\Rightarrow \quad g \in \overline{[G_\xi, G_\xi]}$$

$$\Rightarrow \mathrm{U}_\alpha \subset \overline{[G_\xi, G_\xi]}$$

Let $\xi \in \partial T$ be the repelling fixed point of α.

Then $U_\alpha < P_\alpha = G_\xi$ and for $g \in U_\alpha$, we have

$$g = \lim_{n\to\infty} g\alpha^n g^{-1}\alpha^{-n}$$

$$\Rightarrow \quad g \in \overline{[G_\xi, G_\xi]}$$

$$\Rightarrow U_\alpha \subset \overline{[G_\xi, G_\xi]}$$

$$\Rightarrow \quad U_\alpha \text{ abelian.}$$

$\Rightarrow \quad U_\alpha$ abelian.

- By Proposition, U_α is closed and ∂T is a Moufang set with abelian root groups and metabelian stabilisers.

$\Rightarrow$ U_α abelian.

- By Proposition, U_α is closed and ∂T is a Moufang set with abelian root groups and metabelian stabilisers.
- Can invoke Theorem on Moufang sets.

$\Rightarrow$ U_α abelian.

- By Proposition, U_α is closed and ∂T is a Moufang set with abelian root groups and metabelian stabilisers.
- Can invoke Theorem on Moufang sets.
- Get a l.c. field.

$\Rightarrow$ U_α abelian.

- By Proposition, U_α is closed and ∂T is a Moufang set with abelian root groups and metabelian stabilisers.
- Can invoke Theorem on Moufang sets.
- Get a l.c. field.
- Exceptional case in char 2 does not occur over local fields.

Back to Theorem II

Let G be a non-compact unimodular l.c. group without non-trivial compact normal subgroup (eg. G simple).

Assume there is $\alpha \in G$ *such that*

(i) $G/\overline{\langle \alpha U_\alpha \rangle}$ *is compact*

(ii) U_α *is closed and torsion-free.*

Then there is a l.c. field k of char 0 *and a semisimple algebraic k-group* **G** *of k-rank one such that* **G***(k)* ≤ *G with finite index.*

Back to Theorem II

Enough to prove:

Let $G \leq \mathrm{Aut}(T)$ closed non-compact such that $(G, \partial T)$ is a Moufang set with closed torsion-free root groups.

Then there is a l.c. field k of char 0 and a semisimple algebraic k-group $\mathbf{G}$ of k-rank one such that $\mathbf{G}(k) \leq G$ with finite index.

- Assume that the root group U_ξ is closed and torsion-free. Then by [Glöckner-Willis] we have

$$U_\xi \cong N_1 \times N_2 \times \ldots \times N_t,$$

with N_i nilpotent p_i-adic analytic for some prime p_i.

- Assume that the root group U_ξ is closed and torsion-free. Then by [Glöckner-Willis] we have

$$U_\xi \cong N_1 \times N_2 \times \ldots \times N_t,$$

with N_i nilpotent p_i-adic analytic for some prime p_i.

- Need to show t = 1.

- Assume that the root group U_ξ is closed and torsion-free. Then by [Glöckner-Willis] we have

$$U_\xi \cong N_1 \times N_2 \times \ldots \times N_t,$$

with N_i nilpotent p_i-adic analytic for some prime p_i.

- Need to show t = 1.
- **Strategy:** *show that* N_i *is a root subgroup*.

Sub-Moufang sets

- Given a compact subgroup $K \leq G_{\xi,\eta}$, the centraliser $Z_G(K)$ turns the fixed point set ∂T^K into a sub-Moufang set.

Sub-Moufang sets

- Given a compact subgroup $K \leq G_{\xi,\eta}$, the centraliser $Z_G(K)$ turns the fixed point set ∂T^K into a sub-Moufang set.
- Useful provided one can find K non-trivial.

Digression

Theorem (Kegel-Wielandt 1958)

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Digression

Theorem (Kegel-Wielandt 1958)

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Conjecture

The derived length of G is bounded in terms of the nilpotency classes of A and B.

Digression

Theorem (Kegel-Wielandt 1958)

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Conjecture

Let G be a profinite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Conjecture

Let G be a profinite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Given $G \leq \mathrm{Aut}(T)$ and $\xi, \eta \in \partial T$, if all compact subgroups $K \leq G_{\xi,\eta}$ are trivial, then have product decomposition of vertex stabilisers

$$G_{\tilde{v}} = (G_{\tilde{v}} \cap U_{\xi}) \,.\, (G_{\tilde{v}} \cap U_{\eta}).$$

Conjecture

Let G be a profinite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Given $G \leq \mathrm{Aut}(T)$ and $\xi, \eta \in \partial T$, if all compact subgroups $K \leq G_{\xi,\eta}$ are trivial, then have product decomposition of vertex stabilisers

$$G_v = (G_v \cap U_\xi) \,.\, (G_v \cap U_\eta).$$

However G_v cannot be soluble.

Conjecture

Let G be a profinite group.

If $G = A.B$ with $A, B \leq G$ nilpotent, then G is soluble.

Conjecture

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent of classes a and b,

then G is soluble of derived length $\leq f(a, b)$.

Conjecture

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent of classes a and b,

then G is soluble of derived length $\leq f(a, b)$.

- True if A, B are abelian [Ito 1955]

Conjecture

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent of classes a and b,

then G is soluble of derived length $\leq f(a, b)$.

- True if A, B are abelian [Ito 1955]
- True if A, B have coprime order [Hall-Higman 1956]

Conjecture

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent of classes a and b,

then G is soluble of derived length $\leq f(a, b)$.

- True if A, B are abelian [Ito 1955]
- True if A, B have coprime order [Hall-Higman 1956]
- Reduces to the case of p-groups [Pennington 1973]

Conjecture

Let G be a finite group.

If $G = A.B$ with $A, B \leq G$ nilpotent of classes a and b,

then G is soluble of derived length $\leq f(a, b)$.

- True if A, B are abelian [Ito 1955]
- True if A, B have coprime order [Hall-Higman 1956]
- Reduces to the case of p-groups [Pennington 1973]
- The derived length of G can be greater than the sum of the nilpotency classes of A and B [Cossey-Stonehewer 1998]