Classification Theorems for Fusion Systems

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Main theme of the talk

Many theorems in finite group theory classify groups with certain structural properties.

The most prominent example is the classification of finite simple groups, but there are also many other examples.

This experience is useful to classify fusion systems. On the other hand, there is some hope that fusion systems give a new tool to study groups.

For the remainder of this talk let G be a finite group.

p-local subgroups

Most of classification theorems for groups study groups via their p-local subgroups.

Definition

A subgroup M of G is called **p-local** if $M = N_G(P)$ for some non-trivial p-subgroup P of G.

Suppose $\mathcal{F}$ is the p-fusion system of G, i.e. $\mathcal{F} = \mathcal{F}_S(G)$ where $S \in Syl_p(G)$. Then

$$Aut_{\mathcal{F}_S(G)}(P) \cong N_G(P)/C_G(P) \text{ for every } P \leq S.$$

On the other hand, by Alperin's Fusion Theorem, fusion in G is controlled in certain p-local subgroups.

Classification of simple fusion systems

Aschbacher suggests the following strategy:

Step 1: Classify all simple saturated 2-fusion systems.

Step 2: Use this to give a new proof of the classification of finite simple groups.

Example

It is almost trivial to classify saturated 2-fusion systems on dihedral groups, but this doesn't tell us very much about finite groups with dihedral Sylow 2-subgroup.

Advantages

Fusion systems don't see normal p'-subgroups: If G is a finite group, $S \in Syl_p(G)$ and $\overline{G} := G/O_{p'}(G)$ then $\mathcal{F}_S(G) \cong \mathcal{F}_{\overline{S}}(\overline{G})$.

Normal p'-subgroups of p-local subgroups cause difficulties in the proof of the classification of finite simple groups.

Problems

- There is no established notion of an action of a fusion system. There is no "meaningful" representation theory of fusion systems.
- Some constructions which are easy in groups are difficult (or even impossible) in fusion systems.
- (The 2-fusion system of a simple group is not necessarily a simple fusion system.)

Further reasons to classify fusion systems

- Search for exotic fusion systems. Try to gain some understanding why exotic fusion systems arise.
- "... *to me, classification refers to an attempt to understand the* intrinsic structure *of a given mathematical system.*" (D. Gorenstein)

Assumption

For the remainder of this talk let $\mathcal{F}$ be a saturated fusion system on a finite p-group S.

Normalizers

There is a notion of normalizers of p-subgroups in fusion systems. If $S \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$ then

$$N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q)) \text{ for } Q \leq S.$$

Definition (Puig)

Define $N_{\mathcal{F}}(Q)$ (the **normalizer in $\mathcal{F}$ of Q**) to be the category whose objects are the subgroups of $N_S(Q)$ such that for $A, B \leq N_S(Q)$, $Mor_{N_{\mathcal{F}}(Q)}(A, B)$ is the set of all $\phi \in Mor_{\mathcal{F}}(A, B)$ which extend to an element of $Mor_{\mathcal{F}}(AQ, BQ)$ taking Q to Q.

$N_{\mathcal{F}}(Q)$ is a fusion system on $N_S(Q)$. In each $\mathcal{F}$-conjugacy class, there exists an element Q such that $N_{\mathcal{F}}(Q)$ is saturated. This is what we call a **p-local subsystem**.

Centralizers, normal and central subgroups, factor systems

Centralizers of subgroups of S are similarly defined as normalizers.

A subgroup $Q \leq S$ is called **normal** in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(Q)$, and Q is called **central** if $\mathcal{F} = C_{\mathcal{F}}(Q)$.

A subgroup $T \leq S$ is called **strongly closed** in $\mathcal{F}$ if $x\phi \in T$ for every $x \in T$ and $\phi \in Mor_{\mathcal{F}}(\langle x \rangle, S)$.

If $T \leq S$ is strongly closed in $\mathcal{F}$ then we can form a **factor system** $\mathcal{F}/T$ on S/T. This is a saturated fusion system.

Warning

There is no general definition of a normalizer or a centralizer of a subsystem of $\mathcal{F}$ **!!!**

However, there is a notion of normal and central subsystems.

Normal subsystems

There is a notion of normal subsystems such that, for $N \trianglelefteq G$, $S \in Syl_p(G)$ and $T := S \cap N$,

$$\mathcal{F}_T(N) \trianglelefteq \mathcal{F}_S(G).$$

A subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leq S$ is called **normal** in $\mathcal{F}$ if $\mathcal{E}$ is saturated, T is strongly closed in $\mathcal{F}$, $\mathcal{E}$ is invariant under conjugation with morphisms of $\mathcal{F}$, and some additional properties hold.

Example

Let G be a simple group of Lie type in defining characteristic p and of Lie rank 1. Let $S \in Syl_p(G)$. Then $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, so S is normal in $\mathcal{F}_S(G)$. Thus, $\mathcal{F}_S(S) \trianglelefteq \mathcal{F}$.

Simple fusion systems

Definition

The fusion system $\mathcal{F}$ is called **simple** if 1 and $\mathcal{F}$ are the only normal subsystems of $\mathcal{F}$.

Example

Let G be a simple group of Lie type in defining characteristic p and of Lie rank 1. Let $S \in Syl_p(G)$. Then $\mathcal{F}_S(G)$ is not simple.

Composition factors

If $\mathcal{E}$ is normal then set $\mathcal{F}/\mathcal{E} := \mathcal{F}/T$.

There is a notion of composition factors and a Jordan–Hölder Theorem for fusion systems.

Components

A component is a subnormal quasisimple subgroup. There is a notion of components of fusion systems.

Example

Let G be a finite group, $S \in Syl_p(G)$, K a component of G and $T := K \cap S$. Then $\mathcal{F}_T(K)$ is a component of $\mathcal{F}_S(G)$ if and only if the p-fusion system of $K/Z(K)$ is simple.

The group case

Theorem (Dichotomy Theorem)

Let G be a finite simple group of 2-rank at least 3. Then one of the following holds:

(1) *G is of* **component type**, *i.e. there exists an involution $t \in G$ such that $C_G(t)/O_{2'}(C_G(t))$ has a component.*

(2) *G is of* **characteristic 2-type** *(or of* **local characteristic 2***), i.e. for every 2-local subgroup M of G,*

$$C_M(O_2(M)) \leq O_2(M).$$

The Dichotomy Theorem for fusion systems

Call $\mathcal{F}$ constrained, if $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$.

Theorem (Dichotomy Theorem for fusion systems, Aschbacher)

One of the following holds:

(1) *$\mathcal{F}$ is of component type, i.e. there is a p-element $t \in S$ such that $C_{\mathcal{F}}(t)$ is saturated and has a component.*

(2) *$\mathcal{F}$ is of characteristic p-type, i.e. every p-local subsystem is constrained.*

One could also partition the 2-fusion systems into those of Baumann characteristic 2 and those of Baumann component type.

Examples

The generic examples for groups of component type are the groups of Lie type in odd characteristic. The generic examples for groups of characteristic p-type are the groups of Lie type in defining characteristic p.

Example

- *If G is a finite group of characteristic p-type then the p-fusion system of G is a fusion system of characteristic p-type. The converse is false.*
- *Assume there is an involution $t \in G$ and a component L of $C_G(t)/O_{2'}(C_G(t))$ such that $\mathcal{F}_T(L) \neq \mathcal{F}_T(N_L(T))$ for $T \in Syl_p(L)$. Then the p-fusion system of G is of component type.*

Ingredients of the proof

The proof of the dichotomy theorem for groups uses Bender's theorem about groups with a strongly 2-embedded subgroup, signalizer functor theory and the Gorenstein–Walter theorem on L-balance.

The proof of the dichotomy theorem for fusion systems requires only L-balance.

Roadmap for fusion systems of component type

- Prove a version of Aschbacher's classical involution theorem. (This is Aschbacher's work in progress on "quaternion fusion packets".)
- Develop a concept of a standard component for fusion systems, prove the existence of a standard component. (There is unpublished work of Aschbacher on tightly embedded subsystems of fusion systems.)
- Treat standard form problems. (Work of Justin Lynd.)

Standard form problems

Justin Lynd treated a problem which would presumably be "standard form problem" in almost any definition.

Theorem (Lynd)

Suppose $p = 2$, $O_2(\mathcal{F}) = 1$ and $\mathcal{F} = O^2(\mathcal{F})$. Let $t \in S$ be an involution such that $C_{\mathcal{F}}(t)$ is saturated and $Baum(S) \leq C_S(t)$. Let $\mathcal{L}$ be a component of $C_{\mathcal{F}}(t)$ such that $\mathcal{L}$ is the fusion system of $L_2(q)$ for some odd prime power q. Assume $C_{C_S(t)}(\mathcal{L})$ is cyclic. Then $\mathcal{F}$ is the 2-fusion system of $L_4(q')$ for some $q' \equiv 3 \pmod 4$.

Introduction

In the original proof of the classification of finite simple groups, groups of characteristic 2-type were classified by switching attention to suitable odd primes and using similar methods as in the component type case.

This is presumably impossible for fusion systems, so one would have to use similar methods as in the program of Meierfrankenfeld, Stellmacher and Stroth (MSS-program) for fusion systems of characteristic 2-type.

Experience from the MSS-program suggests that odd primes could (to some extend) be treated simultaneously with little extra effort. Moreover, it's probably unnecessary to assume that the fusion system $\mathcal{F}$ in question is simple, the assumption $O_p(\mathcal{F}) = 1$ should be sufficient.

Introduction

Generic examples for fusion systems $\mathcal{F}$ of characteristic p-type with $O_p(\mathcal{F}) = 1$ are the fusion systems of simple groups of Lie type in defining characteristic p and Lie rank at least 2 extended by certain automorphisms.

In the cases which have been treated so far, the generic examples are fusion systems of simple groups of Lie type in defining characteristic p and Lie rank 2 extended by automorphisms.

For odd p, there occur some cases in which a complete classification seems to be difficult. However, these are well-defined cases in which the fusion system can be shown to be exotic.

Constrained systems

Recall: The fusion system $\mathcal{F}$ is called **constrained** if

$$C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F}).$$

A fusion system is of **characteristic p-type** if every p-local subsystem is constrained.

Theorem (Broto, Castellana, Grodal, Levi, Oliver)

If $\mathcal{F}$ is constrained, then $\mathcal{F}$ is the p-fusion system of a finite group G with $C_G(O_p(G)) \leq O_p(G)$, which is uniquely determined up to isomorphism.

If $\mathcal{F}$ is constrained and G is as above, then G is called a **model** for $\mathcal{F}$.

Theorem (Aschbacher)

Suppose $p = 2$ and $\mathcal{F}$ is of characteristic 2-type. Assume $\mathcal{F}$ is not generated by its maximal parabolic subsystems and, for every $P \leq S$, $Aut_{\mathcal{F}}(P)$ is a $\mathcal{K}$-group. Then $\mathcal{F}$ is the p-fusion system of a finite group G such that one of the following holds:

(a) *For some power q of 2, G is the extension of $L_3(q)$ or $Sp_4(q)$ by a graph automorphism (and automorphisms of odd order).*

(b) *S is dihedral and $G \cong L_2(r)$ or $PGL_2(r)$ for some odd prime power r.*

(c) *S is semidihedral and G is an extension of $L_2(r^2)$ by an automorphism of 2 for some odd prime power r.*

(d) *$G \cong L_3(3)$, $Aut(L_3(3))$ or J_3.*

In fact, Aschbacher proves a more general theorem.

Reduction to Amalgams

Let $\mathcal{F}$ be of characteristic p-type.

For $p = 2$, Aschbacher's Theorem classifies $\mathcal{F}$ under the assumption that $\mathcal{F}$ has a unique maximal parabolic subsystem.

Assume now that $\mathcal{F}$ has two different maximal parabolic subsystems $\mathcal{P}_1$ and $\mathcal{P}_2$. Then there are models G_1 and G_2 of $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the following hold:

- $G_1 \cap G_2 = S$.
- No non-trivial subgroup of S is normal in G_1 and G_2.

This is a setup in which principally the amalgam method works.

Minimal parabolic groups

Definition (McBride)

A group G is called minimal parabolic (with respect to p) if a Sylow p-subgroup S of G is not normal in G and is contained in a unique maximal subgroup of G.

Theorem (Chermak)

Assume $\mathcal{F}$ is of Baumann characteristic p and there is a centric linking system associated to $\mathcal{F}$. Let G_1 and G_2 be models of parabolic subsystems of $\mathcal{F}$. Suppose there are subgroups $X_1 \leq G_1$ and $X_2 \leq G_2$ such that the following hold:

- $X_1 \cap X_2 = S$.
- *No non-trivial subgroup of S is normal in X_1 and X_2.*
- *X_1 and X_2 are minimal parabolic.*
- *Neither X_1 nor X_2 centralizes $\Omega_1(Z(S))$.*

Then one of the following holds:

(1) $\mathcal{F}$ is the p-fusion system of a finite group G such that, for some power q of p, $F^*(G)$ is $PSL_2(q)$, or $Sp_4(q)$ (with $p = 2$), or $G_2(q)$ (with $p = 3$).

(2) $p = 2$ and $\mathcal{F}$ is the 2-fusion system of M_{23}.

(3) $p \in \{3, 5, 7, 13\}$ and S is non-abelian of order p^3 and exponent p.

Saturated fusion systems as in (3) have been classified by Ruiz and Viruel and include three exotic fusion systems for $p = 7$.

Chermak's proof is "classification-free" and relies only on a result of Bundy, Hebbinghaus and Stellmacher.

Under a more general hypothesis, Chermak shows that one of the following holds:

- The conclusion of the last theorem.
- $p = 2$ and $\mathcal{F}$ is the 2-fusion system of $Aut(SL_3(3))$, $PSU_4(3)$, M_{22} or J_3.
- $\mathcal{F}$ is of "3-exceptional type". This means $p = 3$, $\mathcal{F}$ is exotic and some further properties hold.

N-groups

Definition

An N-group is a finite group in which every p-local subgroup is solvable, for every prime p. An N2-group is a finite group in which every 2-local subgroup is solvable.

Properties:

- Every finite group which is minimal among the non-solvable groups is a non-abelian finite simple N-group.
- Every non-solvable group has a section which is a non-abelian finite simple N-group.

History of N-groups

Non-abelian finite simple N-groups were classified by Thompson. This work set a pattern for the classification of finite simple groups.

Thompson's result was generalized by Gorenstein and Lyons, Janko and Smith to N2-groups.

Using the amalgam method, Stellmacher gave a new proof for the classification of the 2-local structure of N2-groups of characteristic 2-type.

Solvable fusion systems

Recall: A finite group G is called solvable if one of the following two equivalent conditions holds:

(1) The commutator series reaches 1.

(2) Every composition factor of G has prime order.

There are two different notions of solvability in saturated fusion systems, one is due to Puig, one is due to Aschbacher.

Puig's definition is a translation of property (1) to fusion systems, Aschbacher's is a translation of property (2).

Characterizations of solvable fusion systems

Theorem (Puig/Aschbacher)

Every solvable fusion system is constrained.

In particular, fusion system analogues of N-groups will be fusion systems of characteristic p-type.

Theorem

A constrained fusion system $\mathcal{F}$ is Puig-solvable if and only if a model for $\mathcal{F}$ is p-solvable.

Characterizations of solvable fusion systems

Theorem (Aschbacher)

A constrained fusion system $\mathcal{F}$ is Aschbacher-solvable if and only if for a model G of $\mathcal{F}$ and every composition factor L of G, $T \trianglelefteq \mathcal{F}_T(L)$ for $T \in Syl_p(L)$.

A model for an Aschbacher-solvable fusion system can have non-abelian composition factors, for example:

- Simple groups of Lie type in defining characteristic p and of Lie rank 1.
- Simple groups with abelian Sylow p-subgroups.

The proof uses the classification of finite simple groups if p is odd. For $p = 2$, it uses the theorem of Goldschmidt about groups with a strongly closed abelian subgroup.

N-systems

Definition (Aschbacher)

The fusion system $\mathcal{F}$ is called an N-system if $p = 2$ and every 2-local subsystem is Puig-solvable.

Note: By the Theorem of Feit–Thompson, every solvable 2-fusion system has a solvable model.

Aschbacher classifies N-systems $\mathcal{F}$ with $O_p(\mathcal{F}) = 1$. The examples are mostly the 2-fusion systems of N-groups. Aschbacher used partly Stellmacher's approach.

Minimal Fusion Systems

Definition

The fusion system $\mathcal{F}$ is called **minimal** if $O_p(\mathcal{F}) = 1$ and every p-local subsystem is Aschbacher-solvable.

From now on "solvable" means always "Aschbacher-solvable".

- Every minimal non-solvable fusion system is minimal in the above sense.
- Every non-solvable fusion system has a section which is minimal.

Goal

I hope to do the following:

- Classify all minimal fusion systems for $p = 2$.
- Classify fusion systems for odd primes except in some well-defined cases, in which the fusion system can be shown to be exotic.

This would imply a new result about groups of characteristic p-type. Moreover, one would get a characterization of **all** fusion systems of finite simple groups of Lie type in defining characteristic p and rank 2. Hence, one would get a complete overview what happens in the case of "small rank".

Rank 1

From now on assume $\mathcal{F}$ is a minimal fusion system.

If $\mathcal{F}$ has a unique maximal parabolic subsystem and $p = 2$ then the structure of $\mathcal{F}$ is given by Aschbacher's theorem. I gave an independent proof of that, using results of Bundy, Hebbinghaus, Stellmacher. For odd p, the arguments determine the structure of a certain p-local subsystem. This seems to be a difficult case in which many exotic fusion systems occur.

Rank 2

If $\mathcal{F}$ has at least two maximal parabolic subsystems, I try to adapt Stellmacher's approach and use the amalgam method.

Lemma

Assume $\mathcal{F}$ has at least two maximal parabolic subsystems and is not of 3-exceptional type. Then there exist subgroups G_1 and G_2 of models of parabolic subsystems such that the following hold:

- $G_1 \cap G_2 = S$.
- *No non-trivial subgroup of S is normal in both G_1 and G_2.*
- *G_1 and G_2 are "almost minimal parabolic".*
- *Another useful technical property.*

The amalgam method

We form the free amalgamated product $X := G_1 *_S G_2$.

Let Γ be the graph whose vertices are the right cosets $G_i x$, where $i = 1, 2$ and $x \in X$. Two vertices are joint by an edge if they intersect non-trivially.

The stabilizer of a vertex $\alpha = G_i x$ is $G_\alpha = G_i^x$. The amalgam method gives information about modules for G_α, in particular about the module structure of

$$Z_\alpha := \Omega_1(Z(O_p(G_\alpha))).$$

Work in progress

Theorem/ Conjecture: Assume $\mathcal{F}$ is not of 3-exceptional type. Let G_1, G_2 be as in the lemma before.

Case 1: $\Omega(Z(S)) \not\leq Z(G_i)$ for $i = 1, 2$.

One of the following holds:

(i) $\mathcal{F}$ is the p-fusion system of a finite group G where $F^*(G)$ is $PSL_2(p^n)$, or $Sp_4(2^n)$ (with $p = 2$), or $G_2(3^n)$ (with $p = 3$).

(ii) $\mathcal{F}$ is on a list of exceptions, including three exotic fusion systems for $p = 7$.

Work in progress

Case 2: $\Omega(Z(S))$ is centralized by G_1 or G_2. Let (α, α') be a "critical pair" and $b := dist(\alpha, \alpha')$.

Case 2.1: Assume $[Z_\alpha, Z_{\alpha'}] \neq 1$. Then $b = 2$, $G_\alpha / Q_\alpha \cong SL_2(q)$ and Z_α is a natural $SL_2(q)$-module.

Conjecture: $\mathcal{F}$ is the p-fusion system of a group G where $G \cong {}^3D_4(p^n)$, or $p \neq 3$ and $G \cong G_2(p^n)$, or $p = 2$ and $G \cong G_2(2)', J_2, Aut(J_2)$ or J_3.

Work in progress

Case 2.2: Assume $[Z_\alpha, Z_{\alpha'}] = 1$. Then b is odd. If $b \geq 5$ then G_α/Q_α has a normal subgroup which is the direct product of groups isomorphic to $SL_2(p^n)$, $U_3(p^n)$ or $Sz(2^n)$ (and $p = 2$).

Conjecture: $b \leq 5$. If $b = 5$ then $p = 3$, G_α/Q_α has a normal subgroup isomorphic to $SL_2(3)$, and $\mathcal{F}$ is the fusion system of F_3.

Examples with $b \in \{1, 3\}$:

- The 2-fusion systems of ${}^2F_4(2^n)$, ${}^2F_4(2)'$, M_{12} and $Aut(M_{12})$.
- The p-fusion system of $PSp_4(p^n)$, p odd.
- The p-fusion systems of $U_4(p^n)$ and $U_5(p^n)$.

Thank you!!!