Definable subsets in free and hyperbolic groups

Olga Kharlampovich (CUNY, USA and McGill, Canada)
(joint results with A. Miasnikov)

March 14, 2012, Bielefeld University

Happy Birthday!

Abstract:
We give a description of definable subsets in a free non-abelian group and in a torsion-free non-elementary hyperbolic group G that follows from our work on the Tarski problems. As a corollary we show that proper non-abelian subgroups of F and G are not definable (solution of Malcev's problem for F) and prove Bestvina and Feighn's statement that definable subsets in a free group are either negligible or co-negligible.

Definable set

A set S is definable in a group G if there exists a first-order formula

$$\psi(p) = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \phi(p, x_1, y_1, \ldots, x_n, y_n),$$

where $\phi(p, x_1, y_1, \ldots, x_n, y_n)$ has no quantifiers, such that $p_0 \in S$ iff $\psi(p_0)$ is a true sentence in G.

The first order theory of a group G ($Th(G)$) is the set of all first-order sentences that are true in G.

The first order theory of a class of groups $\mathcal{C}$ ($Th(\mathcal{C})$) is the set of all first-order sentences that are true in every $G \in \mathcal{C}$.
One can also talk about $Th_{\forall}(\mathcal{C})$, $Th_{\exists}(\mathcal{C})$.

Malcev started in the 60s an extremely fruitful school in model theory and decidability of elementary theories.

The elementary theory of the class of all finite groups is undecidable

The elementary theory of a free nilpotent and of a free soluble non-abelian group is undecidable.

Ershov: Undecidability of the theories of symmetric and finite simple groups.

Two finite models are elementary equivalent iff they are isomorphic.

Malcev proved that the groups $G_n(K)$ and $G_m(L)$ (where $G = GL, SL, PGL, PSL, n, m \geq 3, K, L$ are fields of characteristics 0) are elementary equivalent if and only if $m = n$ and the fields K and L are elementary equivalent.

Malcev asked about $Th_\forall$ of finite groups, of finite nilpotent groups.
$Th_\forall$ of finite groups is undecidable (Slobodskoy, 1980)
$Th_\forall$ of finite nilpotent groups is undecidable (Kharlampovich, 1982), the theory of quasi-identities is undecidable.

$$\forall x_1, \dots, x_n (\wedge_{i=1}^k w_i(x_1, \dots, x_n) = 1 \to w(x_1, \dots, x_n) = 1)$$

.

Very interesting class: pseudofinite groups

We say that a group G is **pseudofinite** if it is an infinite model of the theory of finite groups; that is, if it is elementarily equivalent to an infinite ultraproduct of finite groups.

(J.S. Wilson) Any infinite pseudofinite simple group is elementarily equivalent to a Chevalley group (possibly of twisted type) over a pseudofinite field.

(D. Macpherson, K. Tent, 2007) Let G be a pseudofinite group with stable theory. Then G has a definable soluble normal subgroup of finite index.

I am not talking about this interesting class

Groups of finite Morley rank lie on the border between model theory and algebraic groups. Morley rank is a notion of dimension arising naturally in model theory. Morley rank behaves much like a dimension function for constructible sets over the complex numbers. Indeed, Hilberts Nullstellensatz shows that the Morley rank of an algebraic variety is equal to its Krull dimension.
The algebraicity conjecture in groups of finite Morley rank, due to Cherlin and Zilber, is that all simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields, i.e. arise from matrix groups.

Tarski

Alfred Tarski is widely considered as one of the greatest logicians of the twentieth century (often regarded as second only to Gödel), and thus as one of the greatest logicians of all time. Among philosophers he is especially known for his mathematical characterizations of the concepts of truth and logical consequence for sentences of classical formalized languages. Among logicians and mathematicians he is in addition famous for his work on set theory, model theory and algebra

Tarski's Type Problems: Complex numbers, 30s

Complex numbers $\mathbb{C}$

- $Th(\mathbb{C}) = Th(F)$ iff F is an algebraically closed field.
- $Th(\mathbb{C})$ is decidable.
- Definable sets are either finite or co-finite.

This led to development of the theory of algebraically closed fields.

Elimination of quantifiers: every formula is logically equivalent (in the theory ACF) to a boolean combination of quantifier-free formulas (something about systems of equations).

Tarski's Type Problems: Complex numbers, 30s

Complex numbers $\mathbb{C}$

- $Th(\mathbb{C}) = Th(F)$ iff F is an algebraically closed field.
- $Th(\mathbb{C})$ is decidable.
- Definable sets are either finite or co-finite.

This led to development of the theory of algebraically closed fields.

Elimination of quantifiers: every formula is logically equivalent (in the theory ACF) to a boolean combination of quantifier-free formulas (something about systems of equations).

Tarski's Type Problems: Reals, 30s and later

Reals $\mathbb{R}$

- $Th(\mathbb{R}) = Th(F)$ iff F is a real closed field.
- $Th(\mathbb{R})$ is decidable.

A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square.

Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin) (given a psd polynomial $f \in \mathbb{R}[x_l, \ldots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?)

Elimination of quantifiers (to equations): every formula is logically equivalent (in the theory RCF) to a boolean combination of quantifier-free formulas.
$Th(\mathbb{R})$ is not stable (=one can define an order), a lot of definable sets.

Reals $\mathbb{R}$

- $Th(\mathbb{R}) = Th(F)$ iff F is a real closed field.
- $Th(\mathbb{R})$ is decidable.

A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square.

Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin) (given a psd polynomial $f \in \mathbb{R}[x_l, \ldots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?)

Elimination of quantifiers (to equations): every formula is logically equivalent (in the theory RCF) to a boolean combination of quantifier-free formulas.

$Th(\mathbb{R})$ is not stable (=one can define an order), a lot of definable sets.

Reals $\mathbb{R}$

- $Th(\mathbb{R}) = Th(F)$ iff F is a real closed field.
- $Th(\mathbb{R})$ is decidable.

A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square.

Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin) (given a psd polynomial $f \in \mathbb{R}[x_l, \ldots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?)

Elimination of quantifiers (to equations): every formula is logically equivalent (in the theory RCF) to a boolean combination of quantifier-free formulas.

$Th(\mathbb{R})$ is not stable (=one can define an order), a lot of definable sets.

Tarski's Type Problems: Reals, 30s and later

Reals $\mathbb{R}$

- $Th(\mathbb{R}) = Th(F)$ iff F is a real closed field.
- $Th(\mathbb{R})$ is decidable.

A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square.

Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin) (given a psd polynomial $f \in \mathbb{R}[x_l, \ldots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?)

Elimination of quantifiers (to equations): every formula is logically equivalent (in the theory RCF) to a boolean combination of quantifier-free formulas.
$Th(\mathbb{R})$ is not stable (=one can define an order), a lot of definable sets.

$Th(\mathbb{Z}^n) \neq Th(\mathbb{Z}^m)$, if $n \neq m$.
Schmelev $Th(\mathbb{Z}^n)$ is decidable.

Baur and Monk Every formula is equivalent to a boolean combination of positive primitive formulas. The same is true for module in the language with scalars in the signature.

Building blocks for definable sets are subgroups.

Theorem [Kharlampovich and Myasnikov (1998-2006), independently Sela (2001-2006)]

$Th(F_n) = Th(F_m), m, n > 1.$

Theorem [Kharlampovich and Myasnikov (1998-2006)]

The elementary theory $Th(F)$ of a free group F even with constants from F in the language is decidable.

Solutions to Tarski's problems

Theorem [Kharlampovich and Myasnikov (1998-2006), independently Sela (2001-2006)]

$Th(F_n) = Th(F_m), m, n > 1.$

Theorem [Kharlampovich and Myasnikov (1998-2006)]

The elementary theory $Th(F)$ of a free group F even with constants from F in the language is decidable.

Examples of sentences in the theory of F: (Vaught's identity)
$\forall x \forall y \forall z (x^2 y^2 z^2 = 1 \rightarrow$
$([x, y] = 1 \& [x, z] = 1 \& [y, z] = 1))$

(Torsion free) $\forall x (x^n = 1 \rightarrow x = 1)$

(Commutation transitivity)
$\forall x \forall y \forall z ((x \neq 1 \& y \neq 1 \& z \neq 1 \& [x, y] = 1$
$\& [x, z] = 1) \rightarrow [y, z] = 1)$
CT doesn't hold in $F_2 \times F_2$.

Examples

(CSA) $\forall x \forall y([x, x^y] = 1 \to [x, y] = 1)$

$\forall x, y \exists z(xy = yx \to (x = z^2 \vee y = z^2 \vee xy = z^2))$
not true in a free abelian group of rank ≥ 2.
This implies that if a group G is $\forall\exists$ equivalent to F, then it does not have non-cyclic abelian subgroups.

F has Magnus' property, namely, for any n, m the following sentence is true:
$\forall x \forall y(\exists z_1, \ldots, z_{m+n}(x = \prod_{i=1}^{n} z_i^{-1} y^{\pm 1} z_i \wedge y = \prod_{i=n+1}^{m+n} z_i^{-1} x^{\pm 1} z_i) \to \exists z(x = z^{-1} y^{\pm 1} z))$

Let F be a free group with finite basis. We consider formulas in the language L_A that contains generators of F as constants. Notice that in the language L_A every finite system of equations is equivalent to one equation (this is Malcev's result) and every finite disjunction of equations is equivalent to one equation (this is attributed to Gurevich).

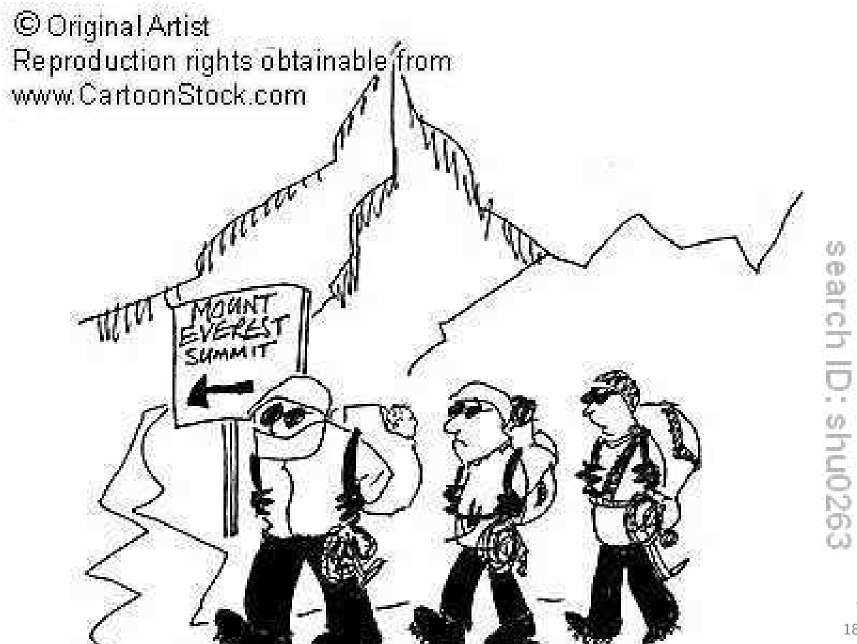
Theorem

(Sela,Kh,Miasn) Every formula in the theory of F is equivalent to the boolean combination of AE-formulas.
Every definable subset of F is defined by some boolean combination of formulas

$$\exists X \forall Y (\vee_{i=1}^{k} (U_i(P, X) = 1 \wedge V_i(P, X, Y) \neq 1)), \tag{1}$$

where X, Y, P are tuples of variables.

Quantifire Elimination



Theorem

(Sela) Every formula in the theory of a non-elementary torsion-free hyperbolic group G is equivalent to the boolean combination of AE-formulas. The theory is stable.

For a free and for a torsion-free hyperbolic group G a more precise result about quantifier elimination holds.

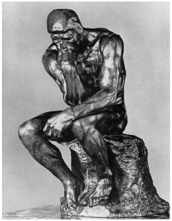
Theorem

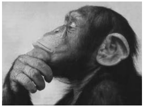
Every definable subset of F is defined by some boolean combination of formulas

$$\exists X \forall Y (U(P, X) = 1 \wedge V(P, X, Y) \neq 1), \qquad (2)$$

where X, Y, P are tuples of variables.

Now we really have to concentrate:





Definition

A *piece* of a word $u \in F$ is a non-trivial subword that appears in two different ways.

Example ab is a piece in $abcb^{-1}a^{-1}$.

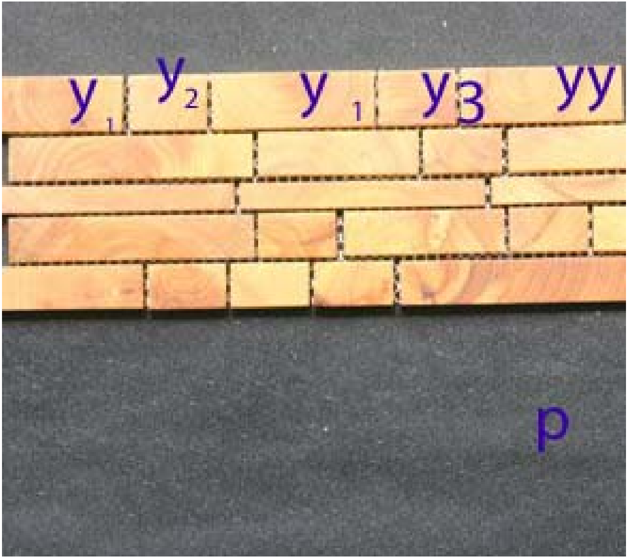
Definition

A proper subset P of F admits parametrization if it is a set of all words p that satisfy a given system of equations (with coefficients) without cancellations in the form

$$p \circeq w_t(y_1, \ldots, y_n), t = 1, \ldots, k, \tag{3}$$

where for all $i = 1, \ldots, n$, $y_i \neq 1$, each y_i appears at least twice in the system and each variable y_i in w_1 is a piece of p.

The empty set and one-element subsets of F admit parametrization.


y 1
y 2
y 1
y 3
yy
p

Definition

A finite union of sets admitting parametrization will be called a *multipattern*. A subset of a multipattern will be called *a sub-multipattern*

Definition

(BF) A subset P of F is *negligible* if there exists $\epsilon > 0$ such that all but finitely many $p \in P$ have a piece such that

$$\frac{length(piece)}{length(p)} \geq \epsilon.$$

A complement of a negligible subset is *co-negligible*.

Bestvina and Feighn stated that in the language without constants every definable set in F is either negligible or co-negligible. It is obvious that

1) Subsets of negligible sets are negligible.

2) Finite sets are negligible.

[BF] 3) A set S containing a coset of a non-abelian subgroup G of F cannot be negligible

4) A proper non-abelian subgroup of F is neither negligible nor co-negligible.

5) The set of primitive elements of F is neither negligible nor co-negligible if $rank(F) > 2$.

Proof.

3) If $x, y \in G$ and $[x, y] \neq 1$, then the infinite set $\{xyxy^2x \dots xy^ix,\ i \in \mathbb{N}\}$ is not negligible .
Statement 4) follows from 3).
5) Let a, b, c be three elements in the basis of F and denote $F_2 = F(a, b)$ The set of primitive elements contains cF_2, and the complement contains $< [a, b], c^{-1}[a, b]c >$.

□

Lemma

A set P that admits parametrization is negligible. A sub-multipattern is negligible.

Proof.

Suppose P admits parametrization. Let m be the length of word w_1 (as a word in variables y_i's and constants). The set P is negligible with $\epsilon = 1/m$. □

Theorem

Suppose an E-set P is not the whole group F and is defined by the formula

$$\psi(p) = \exists Y U(Y, p) = 1,$$

then it is a multipattern.

Cut Equations

Cut Equations

Cut Equations

Definition

A cut equation $\Pi = (\mathcal{E}, M, X, f_M, f_X)$ consists of a set of intervals $\mathcal{E}$, a set of variables M, a set of parameters X, and two labeling functions

$$f_X : \mathcal{E} \to F[X], \quad f_M : \mathcal{E} \to F[M].$$

For an interval $\sigma \in \mathcal{E}$ the image $f_M(\sigma) = f_M(\sigma)(M)$ is a reduced word in variables $M^{\pm 1}$ and constants from F, we call it a *partition* of $f_X(\sigma)$.

Corollary

Suppose a set P is defined by the formula

$$\psi_1(p) = \exists Y \forall Z (U(Y, p) = 1 \wedge V(Y, Z, p) \neq 1).$$

If the positive formula $\psi(p) = \exists Y (U(Y, p) = 1$ does not define the whole group F, then P is a sub-multipattern, otherwise it is a co-multipattern.

Proof.

If $\psi(p)$ does not define the whole group F, then $\psi_1(p)$ is a sub-multipattern.

Suppose now that $\psi(p)$ defines the whole group. Then $\psi_1(p)$ is equivalent to $\psi_2(p) = \exists Y \forall Z V(Y, Z, p) \neq 1$. Suppose it defines a non-empty set. Consider $\neg\psi_2(p) = \forall Y \exists Z V(Y, Z, p) = 1$.

Lemma

Formula

$$\theta(p) = \forall Y \exists Z V(Y, Z, p) = 1$$

in F in the language L_A is equivalent a positive E-formula $\exists X U(p, X) = 1$.

In this case P is co-multipattern. □

Solution to Malcev's (1965) problem.

Theorem

For every definable subset P of F, P or its complement $\neg P$ is a sub-multipattern.

Corollary

(B,F) Every definable subset of F in the language with constants (and, therefore, in the language without constants) is either negligible or co-negligible.

Corollary

Proper non-abelian subgroups of F are not definable.

Corollary

The set of primitive elements of F is not definable if $\mathrm{rank}(F) > 2$.

Definition

Suppose a torsion-free non-elementary hyperbolic group G is generated by the set A, therefore G is a quotient of the free group $F(A)$. A proper subset P of G admits parametrization if P is the image of the set $\tilde{P}$ in $F(A)$ that admits parametrization in $F(A)$ and there exist constants λ and D such that for each $p \in P$ there is a pre-image $\tilde{p} \in \tilde{P}$ such that the path corresponding to $\tilde{p}$ in the Cayley graph of G is λ-quasigeodesic in D neighborhood of the geodesic p.
The empty set and one-element subsets of G admit parametrization.
A finite union of sets admitting parametrization will be called a *multipattern*. A subset of a multipattern will be called *a sub-multipattern*

Theorem

For every definable subset P of non-elementary torsion free hyperbolic group G, P or its complement $\neg P$ is a sub-multipattern.

Negligible subsets are negligible

Definition

Recall that in complexity theory $T \subseteq F(X)$ is called generic if

$$\rho_n(T) = \frac{|T \cap B_n(X)|}{|B_n|} \to 1, \ \textit{if} \ \ n \to \infty,$$

where $B_n(X)$ is the ball of radius n in the Cayley graph of $F(X)$. A set is negligible is its complement is generic.

Theorem

Negligible sets are negligible .

Thanks!