From Weil conjectures to Beauville surfaces — via finite simple groups

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Aim: demonstrate how finite group theory helps to solve problems outside finite group theory; and that theorems in finite group theory may rely on methods from outside finite groups.

Thm A: G nonabelian finite simple group, $G \neq SL_2(2^f)$, $PSL_2(7)$
$\Rightarrow \exists\, C \subseteq G$ conjugacy class, $(x,y,z) \in C \times C \times C^{-2}$ with: $xyz = 1$, $G = \langle x, y \rangle$.

1. Application

Conjecture of Peter Neumann (1966):

Thm 1: $1 \neq G \leq GL_n(k) = GL(V)$ irred (k a field) $\Rightarrow \exists\, g \in G$: $\dim C_V(g) \leq \frac{1}{3} \dim V$.

History: Neumann: G solvable $\Rightarrow \leq \frac{7}{18} \dim V$; Segal-Shalev ('99): $\leq \frac{1}{2} \dim V$ using CFSG.

Note: bound is optimal, for $G = SO_3(k) \leq GL_3(k)$ has $\dim C_V(g) \geq 1 = \frac{1}{3} \dim V$ $\forall g \in G$.
For large dimensions, see later.

About proof: Reductions

① may assume k alg. closed, so V absolutely irreducible for G.

② —"— G finitely generated, say $G = \langle g_1, \dots, g_r \rangle$.

③ —"— $G \leq GL_n(R)$, R finitely generated ring (entries of $g_i^{\pm 1}$)

④ —"— $G \leq GL_n(q)$ finite (choose $I \trianglelefteq R$ max. ideal, consider image)

Partial reduction to: G nonab. finite simple, and $\exists\, g \in G$ with all eigenspaces of dimension $\leq \frac{1}{3} \dim V$. (+ arguments for $O_2(G) \neq 1$).

Now use:

Lemma (Scott '77): $1 \neq G = \langle g_1, g_2, g_3 \rangle \leq GL(V)$ irred., $g_1 g_2 g_3 = 1 \Rightarrow$
$\sum_{i=1}^{3} \dim C_V(g_i) \leq \dim V$.

Corollary: In situation of Lemma, assume $g_1 \sim g_2$, $g_3 \sim g_1^{-2} \Rightarrow$
all eigenspaces of g_1 have dimension $\leq \frac{1}{3} \dim V$.

Pf: Let ζ an eigenvalue of g_1, consider $(\zeta^{-1}g_1, \zeta^{-1}g_2, \zeta^{2}g_3)$, satisfies assumptions of Lemma, $C_V(\zeta^{-1}g_1) = (\zeta\text{-eigenspace of } g_1)$, apply Lemma.

□

Thus, for most nonab. finite simple groups, claim on eigenspaces follows from Thm A.

Note: for given G, always choose same C for any representation, any base field.

$$\boxed{\begin{array}{l}\text{Thm 1}': \text{ Let } \varepsilon > 0 \Rightarrow \exists N = N(\varepsilon) \text{ with: } G \text{ finite quasi-simple irred in } GL_n(\mathbb{C}) = GL(V) \\ \text{where } n \geq N \Rightarrow \exists g \in G: \text{ every eigenspace of } g \text{ on } V \text{ has dimension} \leq \varepsilon \cdot \dim V.\end{array}}$$

Idea: $g \in C$ as in Thm. A, $\chi := tr_V|_{\langle g \rangle}$ $\Rightarrow$ by Deligne-Lusztig theory, this is very close to a multiple of the regular character $\Rightarrow$ all eigenspace have similar dimensions. But $o(g) \to \infty$ with $|G| \to \infty$ $\Rightarrow$ eigenspaces small.

□

In positive characteristic, something similar should hold, but would need more information on decomposition numbers...

Ex: $G = A_4 \subseteq GL_3(\mathbb{C}) = GL(V)$ irred., $G(m) := \underbrace{A_4 \times \ldots \times A_4}_{m}$, $V(m) := \underbrace{V \otimes \ldots \otimes V}_{m}$

$\Rightarrow \dim C_{V(m)}(g) \geq \frac{1}{9} \dim V(m)\ \forall g \in G$, but $\dim V(m) \xrightarrow{m} \infty$.

Thus, for non-simple groups, cannot get better than $\frac{1}{9}$, even for large $\dim V$.

2. Application

In classification of (complex irred.) surfaces, most are of "general type"

Catanese (2000) proposes construction of examples of general type surfaces:

C_1, C_2 curves, of genus $g(C_i) \geq 2$, G finite grp acting freely on $C_1 \times C_2$,

$X := (C_1 \times C_2)/G$ will be of general type if "rigid", a Beauville surface.

Conjecture of Bauer-Catanese-Grunewald ('05):

$$\boxed{\text{Thm 2: All finite nonab. simple groups } G \neq A_5 \text{ admit an unmixed Beauville structure.}}$$

"unmixed" means: G acts on $C_1 \times C_2$ stabilizing both factors.

History: Fuertes-Gonzalez-Diez: $G = A_n$

Garion-Larsen-Lubotzky: for $|G|$ sufficiently large } (2010)

GM: Thm 2; Fairbairn-Magaard-Parker: for all quasi-simple G } (2010)

How to translate to group theory?

Riemann-existence theorem: $G = \langle g_1, \dots, g_r \rangle$ finite grp, $g_1 \cdots g_r = 1 \Rightarrow$
$\exists\, C \to \mathbb{P}^1$ G-covering of curves, ramified at r points, non-trivial stabilizers only in $\bigcup_{i=1}^{r} \langle g_i \rangle^G$. "rigid" (in above sense) iff $r = 3$.

Thus, given two such realizations $G = \langle g_1, g_2, g_3 \rangle = \langle h_1, h_2, h_3 \rangle$, action on $C_1 \times C_2$ is free iff $\bigcup_{i=1}^{3} \langle g_i \rangle^G \cap \bigcup_{i=1}^{3} \langle h_i \rangle^G = \{1\}$.

To prove Thm 2, find suitable generating systems for simple groups. One is given by Thm A, and there $\bigcup_{i=1}^{3} \langle g_i \rangle^G = \langle g_1 \rangle^G$. For the second choose generators with coprime orders.

3. On proof of Thm A

G finite group, $C \subseteq G$ class, $x \in C$, then

$$|\{(y,z) \in C \times C^{-2}\}| = \frac{|C| \cdot |C^{-2}|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)^2 \chi(z)}{\chi(1)} = \frac{|C| \cdot |C^{-2}|}{|G|} \Big(1 + \underbrace{\sum_{\chi \neq 1_G} \cdots}_{\varepsilon}\Big).$$

If $|\varepsilon| < 1 \Rightarrow \exists$ triples $(x,y,z) \in C \times C \times C^{-2}$.

If $|\varepsilon| < \frac{1}{2} \Rightarrow \exists \geq \frac{|C| \cdot |C^{-2}|}{2 \cdot |G|}$ triples, hopefully enough to generate G.

Basically, one needs to consider groups of Lie type.

Choose $x \in G$ contained in few maximal subgroups (e.g.: generating a Coxeter torus).

Obtain explicit list of overgroups of $\langle x \rangle$ in all types.

Then estimate above structure constant using DL-theory:

Lusztig: $\mathrm{Irr}(G) = \bigsqcup_{s \in G^*} \mathcal{E}(G,s)$, and:

$\chi \in \mathcal{E}(G,s)$ non-zero on $x \in G$ semisimple $\Rightarrow \exists\, T \ni x$ max torus with $T^* \subseteq C_{G^*}(s)$.

For us, will have $\langle x \rangle = T$, so $x \in C^\circ_{G^*}(s)^*$, whence essentially $s \in T^*$.

So only few Lusztig families will contribute to structure constant.

Also need: $x \in G$ regular semisimple $\Rightarrow |\chi(x)| \leq |W|$ for all $\chi \in \mathrm{Irr}(G)$ where W = Weyl group of G.

All these later results rest on properties of ℓ-adic cohomology, so the Weil conjectures

Ex: $G = E_8(q)$, choose $x \in G$ with $|\langle x \rangle| = \Phi_{30}(q) \Rightarrow$ only $N_G(x) \geqslant \langle x \rangle$ maximal containing x, so only 30 triples in $N_G(x)$; $\frac{|C|^2}{|G|} \approx q^{232}$ generating ones.

In classical groups, in general many overgroups (e.g. extension field groups).

4. Variations on Thm A.

Relax some of the constraints on triples in Thm A, to get:

Thm B: G nonab. finite simple $\Rightarrow \exists\, C_1, C_2 \subseteq G$ classes with $C_1 C_2 \cup \{1\} = G$.

(Moreover, when $G \neq PSL_2(7), PSL_2(17)$, both with elts of order prime to 6).

History: classical groups $\neq O^+_{4n}(q)$: M.-Saxl-Weigel ('94).

Larsen-Shalev-Tiep ('10): slightly weaker result.

Idea: compute structure constant $n(C_1, C_2, C_3)$ for any class C_3 (rather: estimate)

Corollary: G finite nonab. simple $\Rightarrow$ every $g \in G$ is a product of two mth powers, for m any prime power, or power of 6.

(compare to the talk of Aner Shalev)

We also show Thompson's conjecture for various series of small rank groups.

Thm C: G finite nonab. simple $\Rightarrow \exists\, C, D \subseteq G$ classes with: $G = \langle c, d\rangle \quad \forall (c,d) \in C \times D$.

History: Kantor-Lubotzky-Shalev ('11): for $|G|$ large enough.

Pf. for $G = E_8(q)$: choose $x \in G$ as above, $|\langle x\rangle| = \Phi_{30}(q) \Rightarrow \exists!$ maximal subgroup $M \geq \langle x\rangle$.

Take $C = [x]$, $D = [y]$ for any derangement in perm. action of G on G/M.

□

We get a slightly stronger, Aut(G)-invariant statement, which yields:

Corollary: $\mathcal{X}$ a family of finite groups closed under subgroups, quotients, extensions.

Then: G belongs to $\mathcal{X} \iff \forall x, y \in G\ \exists g \in G$ with $\langle x, y^g\rangle \in \mathcal{X}$.

(compare to the talk of Marcel Herzog)