On the Existence of Normal Maximal Subgroups in Division Rings *

S. Akbari, M. Mahdavi-Hezavehi †

Abstract

Let D be a division ring with centre F. Denote by D^* the multiplicative group of D. The relation between valuations on D and maximal subgroups of D^* is investigated. In the finite dimensional case, it is shown that F^* has a maximal subgroup if $Br(F)$ is nontrivial provided that the characteristic of F is zero. It is also proved that if F is a local or an algebraic number field, then D^* contains a maximal subgroup that is normal in D^*. It should be observed that every maximal subgroup of D^* contains either D' or F^*, and normal maximal subgroups of D^* contain D', whereas maximal subgroups of D^* do not necessarily contain F^*. It is then conjectured that the multiplicative group of any noncommutative division ring has a maximal subgroup.

Let D be a division algebra of finite dimension over its centre F. Denote by D' the commutator subgroup of the multiplicative group $D^* = D - \{0\}$. For any field F, we use the notation $Br(F)$ for the Brauer group of F. In [1], [2] and [5], the structure of maximal subgroups and finitely generated subnormal subgroups of D^* is investigated and it is shown how these subgroups sit in D^* with respect to F^* and D'. The aim of this note is to show that the existence of maximal subgroups of F^* is essential to study those of D^*. In fact, it is shown that if F^* has no maximal subgroups, then $Br(F)$ is trivial. In this connection, we observe that the multiplicative group of an algebraically closed field has no maximal subgroups whereas there exist fields that have no maximal subgroups

* *Key words:* valuation, division ring, maximal subgroup.

† AMS (2000) *Subject classification* : 16K20, 16K40

but are not algebraically closed. As another example, $\mathbf{R}^*$, the multiplicative group of real numbers of $\mathbf{R}$, has only one maximal subgroup which is associated to the absolute value of $\mathbf{R}$ and there is only one non-commutative division algebra of finite dimension over $\mathbf{R}$. The situation is much better for the field of rational numbers $\mathbf{Q}$. It is shown later that $\mathbf{Q}^*$ has infinitely many maximal subgroups which are associated to the valuations of $\mathbf{Q}$ and it is well known that $Br(\mathbf{Q})$ is infinite. We shall try to establish a connection between valuations on a field F and maximal subgroups of F^* and D^*, where D is a division algebra with centre F. To be more precise, we characterize all maximal subgroups of the field of rational number $\mathbf{Q}^*$ with respect to set of all valuations on $\mathbf{Q}$. As for maximal subgroups of D^*, it is proved that if M is a maximal subgroup of D^* not containing F^*, then $Z(M)$ is a maximal subgroup of F^*. Furthermore, assume that D is of finite dimension over F and m is a maximal subgroup of F^* containing $Z(D')$. It is shown that D^* contains a maximal subgroup M containing m that is normal in D^*. Using these results, we prove if F is a field with a Krull valuation whose value group contains a maximal subgroup, and D is a division algebra of finite dimension over its centre F, then D^* contains a maximal subgroup M which is normal in D^*. We then apply these results to division algebras over algebraic number fields and local fields to show that in these cases D^* contains maximal subgroups which are normal in D^*. In contrast, we shall show that the multiplicative group of the real quaternions contains no normal maximal subgroups. It is generally believed that for any division ring D, D^* has a maximal subgroup. In this connection, it is also proved that if D is finite dimensional over its centre F and F admits a discrete valuation, then D^* contains a maximal subgroup. Finally, it is proved that, under certain mild conditions, each non-zero element of a division algebra D is contained in a maximal subgroup of D^*. We begin the material of this note with the determination of maximal subgroups of multiplicative groups of usual number systems in the following

LEMMA 1. *Let* $\mathbf{Q}$, $\mathbf{R}$, *and* $\mathbf{C}$ *denote the field of rational, real, and complex numbers, respectively. Then we have*
(a) *For any natural number* $r \geq 3$, *consider the epimorphism* f_r *from* Q^* *onto*

$G_r := \bigoplus_{i=1}^{+\infty} Z_r^{(i)}$ *given by the rule* $f_r(x) = (\overline{\nu_{p_i}(x)})$*, where* $Z_r^{(i)} \cong Z_r$ *the ring of integers modulo* r*, for each* i*,* p_i *is the* i*-th prime number and* ν_{p_i} *is the* p_i*-adic valuation on* Q *and* $\overline{\nu_{p_i}(x)}$ *is the remainder of* $\nu_{p_i}(x)$ *modulo* r*. Suppose further that* f_2 *is an epimorphism from* Q^* *onto* $G_2 := Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} Z_2^{(i)})$ *, given by the rule* $f_2(x) = (sgn\, x, (\overline{\nu_{p_i}(x)}))$*, where* $sgn\, x$ *denotes the sign of* x*. If* M *is a maximal subgroup of* Q^**, then there exists a prime number* q *and a maximal subspace* W *of* G_q *(*G_q *is a vector space over* Z_q*) such that* $M = f_q^{-1}(W)$*. Conversely, for any prime* q *and any maximal subspace* W *of* G_q*,* $f_q^{-1}(W)$ *is a maximal subgroup of* Q^**.*

(b) $\mathbf{R}^*$ *has only one maximal subgroup.*

(c) *If* F *is an algebraically closed field, then* F^* *contains no maximal subgroup. In particular,* $\mathbf{C}^*$ *has no maximal subgroup.*

PROOF. (a) It is clearly seen that the map θ from Q^* onto

$$G := Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} Z^{(i)}),$$

where $Z^{(i)} \cong Z$, the ring of integers, for each i, given by

$$\theta(x) = (sgn\, x, \nu_{p_1}(x), \nu_{p_2}(x), \ldots),$$

is a group isomorphism. Therefore, there is a $1-1$ correspondence between maximal subgroups of Q^* and G. Let M be a maximal subgroup of Q^* and denote the corresponding maximal subgroup of G by G_M. Thus, there is a prime number q such that $G/G_M \cong Z_q$ and we have $qG \subseteq G_M$. Now two cases can be considered.

Case 1. $q \geq 3$. In this case $qG \subseteq G_M$ implies that $Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)}) \subset G_M$. Thus, $G_M / Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)})$ is a maximal subgroup of $G / Z_2 \bigoplus (\bigoplus_{i=1}^{+\infty} qZ^{(i)}) \cong \bigoplus_{i=1}^{+\infty} Z_q^{(i)}$, where $Z_q^{(i)} \cong Z_q$ for each i. This implies that there is an epimorphism from Q^* onto $\bigoplus_{i=1}^{+\infty} Z_q^{(i)}$, given by $f_q(x) = (\overline{\nu_{p_i}(x)})$. Thus, there exists a maximal subspace W of $\bigoplus_{i=1}^{+\infty} Z_q^{(i)}$ such that $M = f_q^{-1}(W)$.

Case 2. $q = 2$. In this case $qG \subseteq G_M$ yields that $\bigoplus_{i=1}^{+\infty} 2Z^{(i)} \subseteq G_M$ and hence

$G_M / \bigoplus_{i=1}^{+\infty} 2Z$ is isomorphic to a maximal subspace of G_2. Therefore, there exists a maximal subgroup of G_2 say W, such that $M = f_2^{-1}(W)$. The other side of the theorem is clear.

We remark that if we put $W =< e_i \mid i \geq 2 >$, where e_i is the vector whose i-th component is 1 and other components are zero, then we find $Q^+ = f_2^{-1}(W)$.

(b) Assume that M is maximal subgroup of $\mathbf{R}^*$. Then, we must have $\mathbf{R}^*/M \cong Z_p$ for some prime number p. Thus, for each $a \in \mathbf{R}^*$ we have $a^p \in M$. Now, take the equation $x^p = b$ over $\mathbf{R}$. We know that if p is odd, this equation has a solution in $\mathbf{R}$. This means that $a = (a^{1/p})^p \in M$, i.e., $\mathbf{R} = M$. So the only choice of p is $p = 2$. Thus, $M = \mathbf{R}^+$ is the only maximal subgroup of $\mathbf{R}^*$.

(c) Let F be algebraically closed and M be a maximal subgroup of F^*. Then, we have $F^*/M \cong Z_p$ for some prime p. Take an element $x \in F^*$. Since $x^{1/p}$ exists in F^* for any prime p, we conclude that $x = (x^{1/p})^p \in M$, i.e., $F^* = M$ which completes the proof.

Now, let $0 \neq [A] \in Br(F)$ be cyclic. It is known that there is a cyclic extension L/F, an automorphism $\sigma \in Gal(L/F)$, and $a \in F$ such that $A \cong (a, L/F, \sigma)$. Now, the map $\theta : F^* \to Br(L/F) \subset Br(F)$, given by the rule $\theta(c) = [c, L/F, \sigma]$, is a nontrivial group homomorphism (cf. Chapter 10 of [3]). This homomorphism is used in the next result to show that F^* has a maximal subgroup.

Lemma 2. *Let F be a field and $0 \neq [A] \in Br(F)$. If A is cyclic, then F^* has a maximal subgroup.*

Proof. Since A is cyclic, there exists a maximal subfield E, say, of A such that E/F is a finite cyclic extension. Now, consider the homomorphism $r_{E/F} : Br(F) \longrightarrow Br(E)$ given by $r_{E/F}(X) = X \otimes_F E$. We have $r_{E/F}(A) = A \otimes_F E$ and since E is a maximal subfield of A we find $r_{E/F}(A) = 0$, that is $A \in Br(E/F)$. Now since E/F is a finite cyclic extension we have $0 \neq Br(E/F) \simeq F^*/N_{E/F}(E^*)$ and if $[E : F] = n$, then we obtain $F^{*n} \subset N_{E/F}(E^*)$ and this implies that $F^* \neq (F^*)^n$. Now the group $F^*/(F^*)^n$ is a nontrivial abelian group of finite exponent and thus, by Baer-Prufer Theorem (cf. [10]), F^* has a maximal subgroup.

It is known that if F is local or global, then every F-central simple algebra is cyclic (cf. [3] or [11]), and so F^* has a maximal subgroup. We also observe that this result also follows easily from the fact that if F has a discrete valuation, then F^* has a maximal subgroup. Therefore, we record this fact as a corollary.

COROLLARY 3. *If F is a local or global field, then F^* has a maximal subgroup.*

COROLLARY 4. *If $Br(F)$ is non-trivial, then either F^* has a maximal subgroup or there exists a primitive p-th root of unity (p is a prime) ω,say, such that $F(\omega)/F$ is a finite cyclic extension and $(F(\omega))^*$ has a maximal subgroup.*

PROOF. Assume that p is a prime and $0 \neq [A] \in Br(F)$ such that $p[A] = 0$. If Char $F = p > 0$, then, by Albert Main Theorem (cf. [3, p. 110]), A is a cyclic F-algebra. Now, by Lemma 2, F^* has maximal subgroup. So we may assume that $CharF \neq p$. If F contains a primitive p-th root of unity, by Merkurjev-Suslin Theorem (cf. [12, p. 236]), A is similar to a tensor product of cyclic F-algebras. Since $[A] \neq 0$ we conclude that at least one of the components in the tensor product is non-trivial. Now, use Lemma 2 to complete the proof of this case. Thus one may assume that F^* does not contain a primitive p-th root of unity. Set $A' = A \otimes_F L$, where $L = F(\omega)$ and w is a primitive p-th root of unity. If $0 = [A'] \in Br(L)$, then $0 \neq Br(L/F) = F^*/N_{L/F}(L^*)$ and so, by Baer-Prufer Theorem, F^* has a maximal subgroup. So suppose that $0 \neq [A'] \in Br(L)$. Then the order of $[A']$ is clearly p and so by Merkurjev-Suslin Theorem there exists a non-trivial cyclic L-algebra. Now, apply Lemma 2 to complete the proof.

To prove our main theorem in this connection, we shall need the following

LEMMA 5. *Suppose $Br(F)$ is nontrivial. Then $Br_p(F)$ is nontrivial for some prime p. Furthermore, there is a cyclic F-algebra under either of the following situations:*
(i) F has characteristic p, or
(ii) F has characteristic $\neq p$ with enough roots of 1.

Proof. Suppose that $0 \neq [A] \in Br_p(F)$. Now, we have $p[A] = 0$. If $Char F = p$, by Albert's Theorem, A is cyclic and the result follows. Now, assume that F has characteristic $\neq p$ with enough roots of 1. By Merkurjev-Suslin Theorem, A is similar to a tensor product of cyclic algebras. Since $0 \neq [A] \in Br_p(F)$ we obtain a non-trivial cyclic algebra as desired.

We are now in a position to prove

Theorem 6. *Assume that $Br(F)$ is non-trivial. Then $Br_p(F)$ is nontrivial for some prime p. Furthermore, F^* has a maximal subgroup under either of the following situations:*
(i) F has characteristic zero, or
(ii) F has characteristic p, or
(iii) F has characteristic $\neq p$ with enough roots of 1.

Equivalently, if F has the above conditions and F^* is divisible, then $Br(F)$ is trivial.

Proof. Assume that $Char F = 0$. If F^* has no maximal subgroups, then F^* is divisible. Now, by Lemma 3 of [9], F^* contains all roots of unity which contradicts Corollary 4. If $Char F = p > 0$, by Lemma 5, there exists a cyclic F-algebra B such that $0 \neq [B] \in Br_p(F)$. Now, by Lemma 2 the result follows. Finally, assume that F has characteristic $\neq p$ with enough roots of 1. Again, using Lemma 5 and Lemma 2 as above completes the proof of the theorem.

We observe that the converse of Theorem 6 is not true in general. For let F be algebraically closed and consider $F(x)$. Then we know that $F(x)$ has a discrete valuation and so $F(x)^*$ has a maximal subgroup whereas, by Tsen-Lang Theorem (cf. [12, p. 211]), $Br(F(x))$ is trivial.

It is not known that if F^* is divisible, then $Br(F)$ is trivial without any condition on the characteristic of F.

Remark. Here we establish a connection between the existence of a maximal subgroup of F^* and certain cohomology groups and modules. Let L/F be a finite Galois extension and the 0-th cohomology group of L is nonzero, i. e., $H^0(G, L^*) \neq \{0\}$, where $G = Gal(L/F)$. Therefore, we conclude that $F^*/N_{L/F}(L^*)$ is nonzero which implies that F^* has a maximal subgroup. Now, assume that F_s denotes the separable closure of the field F. Let us put

$G_m = F_s^*$. It is known that the cohomology of the module G_m is important, for example we have $H^0(F, G_m) = F^*$, and by Hilbert's Theorem 90 we also have $H^1(F, G_m) = 0$ and $H^2(F, G_m) \cong Br(F)$. Suppose that μ_n is the group of all n-th roots of unity and n is prime to the characteristic of F. Now, we have the exact sequence,

$$1 \to \mu_n \to G_m \xrightarrow{n} G_m \to 1,$$

which is referred to as the Kummer sequence and n denotes the endomorphism $x \to x^n$. The corresponding cohomology sequence is also called the Kummer sequence, which is as follows:

$$1 \to \mu_n(F) \to F^* \xrightarrow{n} F^* \to H^1(F, \mu_n) \to 1$$

$$1 \to H^2(F, \mu_n) \to Br(F) \xrightarrow{n} Br(F).$$

Thus, we find the isomorphisms $H^1(F, \mu_n) \cong \frac{F^*}{F^{*n}}$ and $H^2(F, \mu_n) \cong_n Br(F)$. If $\mu_n \subset F^*$, we obtain $H^2(F, \mu_n \otimes \mu_n) \cong_n Br(F) \otimes \mu_n(F)$. Therefore, if Char $F = 0$ and $H^2(F, \mu_n \otimes \mu_n) \neq 0$, then F^* has a maximal subgroup. Now, let F be a field that has no maximal subgroup. Then F^* is divisible, and thus $F^{*^p} = F^*$, where p is the characteristic of F, and so F is perfect. Since we have $F^* = F^{*q}$ for all $q \neq p$ we obtain $H^1(F, \mu_q) = 1$. Conversely, if F is perfect and we have $H^1(F, \mu_q) = 1$ for all $q \neq p$, then F^* is divisible.

Now, we turn to study maximal subgroups of D^* and show how normal maximal subgroups of D^* are related to maximal subgroups of F^* and valuations on F. We recall that if D is a division ring with center F, then D is called of *type* 2 if for any two elements $a, b \in D$, the F-algebra $F[a, b]$ is finite dimensional. To state our next result, we need some more preparations. Denote by $G(D)$ the group D^*/F^*D'. When D is algebraic over its centre F, $G(D)$ is torsion (cf. [6]). Some algebraic properties of $G(D)$ are investigated in [4] and [8]. We continue our study with

THEOREM 7. *Let D be a division ring of type 2 with centre F and v be a discrete valuation on F. If there exists a natural number m such that $G(D)$ has no element of order m, then D^* has a maximal subgroup.*

PROOF. First we show that there exists a non-zero homomorphism of D^* into the additive group of rational numbers $\mathbf{Q}$. By assumption there is a prime

p and natural number r such that $G(D)$ has no element of order p^r. Now define a function $w : D^* \longrightarrow \mathbf{Q}$ by $w(a) = \frac{1}{[F(a):F]}v(N_{F(a)/F}(a))$. We claim that w is a homomorphism. To see this, suppose that S is a subalgebra of finite dimension over F and $a \in S$, we prove that $w(a) = \frac{1}{[S:F]}v(N_{S/F}(a))$. Put $[F(a) : F] = n$ and $[S : F] = s$, it follows that $n|s$ and $N_{S/F}(a) = (N_{F(a)/F}(a))^{s/n}$. This implies that $\frac{1}{s}\nu(N_{S/F}(a)) = \frac{1}{n}N_{F(a)/F}(a) = w(a)$. Thus if $a, b \in D^*$ and $F(a, b) = S$, then we find $[S : F] < \infty$ since D is of type 2. Now since $N_{S/F}(ab) = N_{S/F}(a)N_{S/F}(b)$ we have $w(ab) = w(a) + w(b)$. Thus w is a homomorphism as claimed. Now, since $G(D)$ is torsion for any $x \in D^*$ there exists $n(x) > 0$ such that $x^{n(x)} = tc$, where $t \in F$ and $c \in D'$, and p^r does not divide $n(x)$. So we obtain $w(x) = \frac{1}{n(x)}w(t)$, and this implies that $\frac{1}{p^r}$ does not belong to the image of w, $Im(w)$, i.e., $Im(w) \neq \mathbf{Q}$. We now claim that $Im(w)$ is not divisible. Since otherwise, assume that $h/q \in Q\backslash Im(w)$. It is easily seen that $w|F = v$ and so $Im(w) \neq 0$. Now if $0 \neq u \in Im(w)$, then we have $uh \in Im(w)$ and the equation $qux = uh$ has a solution in $Im(w)$ which is a contradiction. Consequently, $Im(w)$ is not divisible and so $Im(w)$ has a maximal subgroup and therefore D^* has a maximal subgroup.

Corollary 8. *Let D be a finite dimensional division algebra over its centre F. If F has a discrete valuation, then D^* has a maximal subgroup.*

In the next result we deal with maximal subgroups of D^* which do not contain F^*, and the theorem also shows how maximal subgroups of D^* arise from those of F^*.

Theorem 9. *Let D be a division ring with centre $Z(D) = F$. Then we have the following*
(a) *If M is a maximal subgroup of D^* not containing F^*, then $Z(M)$ is a maximal subgroup of F^*.*
(b) *Assume that D is of finite index n over F and m is a maximal subgroup of F^* containing $Z(D')$. Then D^* has a maximal subgroup M containing m that is normal in D^*.*

Proof. (a) We have $D^* = F^*M$ and thus $D' = M'$. This shows that M is normal in D^*. Now, by a result of [8], we have $Z(M) = F^* \cap M$ and so $Z_p \cong D^*/M = F^*M/M \cong F^*/Z(M)$, for some prime number p. This implies that $Z(M)$ is maximal in F^*.

(b) Assume that m is a maximal subgroup of F^* which contains $Z(D')$. Thus $F^*/m \cong Z_p$ for some prime number p. Consider the normal subgroup mD' of D^*. If $mD' = D^*$, then we obtain $m = mZ(D') = F^*$ which is a contradiction. So, take the nontrivial group D^*/mD'. We know that D^*/F^*D' is torsion of a bounded exponent dividing the index of D over F (cf. [7] or [8]). Now, since $F^*/m \cong Z_p$ we conclude that the group D^*/mD' is torsion of a bounded exponent. Therefore, by Baer-Prufer theorem (cf. [10]), D^*/mD' is isomorphic to a direct product of cyclic groups Z_{r_i}, where r_i divides the index n for all i. In this way, we may obtain a maximal subgroup N of D^* containing mD' and thus the result follows.

The next result shows how valuations on the centre F of a division ring D enable one to obtain maximal subgroups of D^* which are normal.

Corollary 10. *Let F be a field with a Krull valuation whose value group contains a maximal subgroup. Assume that D is a division algebra of finite dimension over its centre F. Then D^* contains a maximal subgroup M which is normal in D^*.*

Proof. Let v be a Krull valuation on F whose value group Γ, say, has a maximal subgroup. Then, we have $F^*/U \cong \Gamma$, where U is the group of units of the valuation. Since Γ contains a maximal subgroup , the isomorphism above induces a maximal subgroup L of F^* containing U. We know that $Z(D')$ is torsion. Thus $Z(D') \subset L$. Now, Theorem 9 completes the proof.

Corollary 11. *Let F be an algebraic number field, and assume that D is an F -central division algebra. Then D^* contains a maximal subgroup M which is normal in D^*.*

Proof. It is known that the p-adic valuation of $\mathbf{Q}$ extends to a discrete valuation on F. Now, using Corollary 10 completes the proof.

Given a local field F, one may easily check that F has a discrete valuation. Now, using Corollary 10 again, we obtain the following

Corollary 12. *Let F be a local field, and assume that D is an F -central division algebra. Then D^* contains a maximal subgroup M which is normal in*

D^*.

In contrast to above results, we may observe that not any multiplicative group of a division algebra contains normal maximal subgroups as the following result shows.

THEOREM 13. *Let D be the real quaternion division algebra. Then D^* contains no normal maximal subgroups.*

PROOF. Let M be a maximal subgroup of D^* which is normal in D^*. Since M is maximal we have $D^*/M \cong Z_p$ for some prime number p. Take an element $x \in D \backslash \mathbf{R}$, then $\mathbf{R}[x] \cong \mathbf{C}$. Therefore, every element in D^* has a p-th root and so $D^* = M$ which is a contradiction and so the result follows.

Generally, in view of the above results, one is tempted to state the following

CONJECTURE. *Let D be a non-commutative division ring. Then D^* contains a maximal subgroup.*

Let F be a field. It is not true in general that each element $a \in F^*$ is contained in a maximal subgroup of F^* even if F^* has maximal subgroups. For example, by Lemma 1, $\mathbf{R}^+$ is the only maximal subgroup of $\mathbf{R}$ which does not contain negative real numbers. But for finite dimensional division algebras the situation is different as the following result shows. To state the theorem, we observe that when D is of finite dimension over its centre F, $G(D)$ is torsion of a bounded exponent dividing the index of D over F (cf. [8]).

THEOREM 14. *Let D be a division algebra of finite dimension over its centre F. Then we have the following*
(a) *If $G(D)$ is not cyclic, then each element $x \in D^*$ is contained in a normal maximal subgroup of D^*.*
(b) *If $G(D)$ is cyclic and non-trivial, and F^* contains a maximal subgroup containing $Z(D')$, then each element $x \in D^*$ is contained in a normal maximal subgroup of D^*.*

PROOF. (a) We know, by the Lemma of [6], that $G(D)$ is torsion of a bounded exponent dividing the index m of D over F. Since $G(D)$ is not trivial, by Baer-Prufer Theorem (cf. [10]), we have $G(D) \cong Z_{r_1} \times Z_{r_2} \times \cdots$, where $r_i|m$ for each i. Therefore, there are maximal subgroups M of D^* which contain F^*D'. Thus, if $x \in F^*D'$, then we obtain $x \in M$ and the result follows. So, we may assume that $x \notin F^*D'$. Put $H = F^*D' < x >$. Since $D' \subset H$ we conclude that H is normal in D^*. If $F^*D' < x >= D^*$, then we have

$$G(D) = \frac{F^*D' < x >}{F^*D'} \cong \frac{< x >}{F^*D' \cap < x >} \cong Z_t,$$

since $G(D)$ is torsion. This contradicts our assumption that $G(D)$ is not cyclic. Therefore, $F^*D' < x > \neq D^*$ and $D^*/F^*D' < x >$ is an abelian torsion group of bounded exponent. Thus, By Baer-Prufer Theorem, we obtain $D^*/F^*D' < x > \cong Z_{t_1} \times Z_{t_2} \times \cdots$, where $t_i|m$ for all i. Now, take a maximal subgroup L, say, of $Z_{t_1} \times Z_{t_2} \times \cdots$ and consider the inverse image M, say, of L under the indicated isomorphism. Then, M is a maximal subgroup of D^* which contains x and so the result follows.
(b) By the argument used in part (a), it is enough to consider $F^*D' < x >= D^*$, where $x \notin F^*D'$. Now, we have $D^*/D' < x > \cong F^*/F^* \cap D' < x >$. If $D^* = D' < x >$, then $D^*/D' \cong < x > /D' \cap < x >$. The cyclic group $< x > /D' \cap < x >$ can not be finite since otherwise D^*/D' would be torsion. This is not possible by Proposition 1 of [7]. Thus, we must have $< x > /D' \cap < x > \cong Z \cong D^*/D'$. But then we obtain $D^{(1)}/D' \cong nZ$ for some $n \in Z$, where $D^{(1)}$ is the group of reduced norm 1 elements. This is not possible either since $D^{(1)}/D'$ is a torsion group. Therefore, $D^* \neq D' < x >$. Now, since $Z(D')(F^* \cap < x >) = (F^* \cap D')(F^* \cap < x >) \subset F^* \cap D' < x >$ and F^* contains a maximal subgroup L, say, containing $Z(D')$, by Theorem 9, we conclude that there is a maximal subgroup M in D^* containing $D' < x >$. So $x \in M$ which completes the proof.

It is believed that the condition in Theorem 14 for $G(D)$ to be trivial is superfluous. In fact, it is a conjecture in [4] that $G(D)$ is rarely trivial and it only happens for the real quaternions. So, Theorem 14 applies to a wide range of division algebras. For examples in which all the conditions of Theorem 14 are satisfied, see [4].

The authors thank the referee for his constructive comments. They are also indebted to the Research Council of Sharif University of Technology for support. The second author would also like to thank Professor J. P. Tignol for fruitful discussions about the Remark while he was visiting the Universite Catholique de Louvain in June 2000.

References

[1] Akbari, S., Mahdavi-Hezavehi, M., *Normal Subgroups of $GL_n(D)$ Are not Finitely Generated*, Proc. Amer. Math. Soc., Vol. 128, No. 6, 1827-1832 (2000).

[2] Akbari, S., Mahdavi-Hezavehi, M., Mahmudi, M. G., *Maximal Subgroups of $GL_1(D)$*, Journal of Algebra, 217, 422-433 (1999).

[3] Draxl P. K., Skew Fields, Cambridge University Press, (1983).

[4] Hazrat, R., Mahdavi-Hezavehi, M., Mirzaii, B., *Reduced K-Theory and the Group $G(D) = D^*/F^*D'$*, Algebraic K-Theory and Its Applications, Proc. of the Workshop and Symposium, World Scientific Pub., 403-409 (1999).

[5] Mahdavi-Hezavehi, M., Mahmudi, M. G., Yasamin, S., *Finitely Generated Subnormal Subgroups of $GL_n(D)$ Are Central*, Journal of Algebra, 225, 517-521 (2000).

[6] Mahdavi-Hezavehi, M., *Extending Valuations to Algebraic Division Algebras*, Comm. Algebra, Vol.22(11), 4373-4378 (1994).

[7] Mahdavi-Hezavehi, M., *Determinant-like Functions for Matrices over Finite Dimensional Division Algebras*, Algebraic K-Theory and Its Applications, Proc. of the Workshop and Symposium, World Scientific Pub., 489-502 (1999).

[8] Mahdavi-Hezavehi, M., *Commutator Subgroups of Finite Dimensional Division Algebras*, Rev. Roumaine Math. Pures Appl., 43(9-10), 853-867 (1998).

[9] May, W., *Multiplicative Groups of Fields*, Proc. London Math. Soc. Vol. 3, No. 21, 295-306 (1972).

[10] Robinson, Derek J. S., *A Course in the Theory of Groups*, GTM, No. 80, Springer-Verlag, (1982).

[11] Rowen L., *Ring Theory*, Volume II, Academic Press, INC, (1988).

S. Akbari, s_akbari@sharif.edu
M. Mahdavi-Hezavehi, mahdavih@sharif.edu
Department of Mathematical Sciences
Sharif University of Technology
P.O.Box 11365-9415
Tehran, Iran