SYMMETRIC POWERS OF SYMMETRIC BILINEAR FORMS

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Abstract. We study symmetric powers of classes of symmetric bilinear forms in the Witt-Grothendieck ring of a field of characteristic not equal to 2, and derive their basic properties and compute their classical invariants. We relate these to earlier results on exterior powers of such forms.

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1. Introduction

Throughout this paper, K will be a field of characteristic different from 2.

Given a finite-dimensional K-vector space V and a non-negative integer k, denote by $\bigwedge^k V$ the k-fold exterior power of V, and by $S^k V$ the k-fold symmetric power of V. It is well-known that the ring of isomorphism classes of finite-dimensional K-vector spaces under direct sum and tensor product is a λ-ring, with the exterior powers acting as the λ-operations (associated to elementary symmetric polynomials) and the symmetric powers being the "dual" s-operations (associated to complete homogeneous polynomials). The exterior and symmetric powers are in fact functors on the category of finite-dimensional K-vector spaces and K-linear maps, special cases of the Schur functors which arise in representation theory (see, for example, [2] or [3]). A natural question to ask is whether we may define such "Schur powers" of symmetric bilinear and quadratic forms as well.

Let φ be a symmetric bilinear form on an n-dimensional vector space V over K. Bourbaki (see [1, Ch. 9, eqn. (37)]) defined on $\bigwedge^k V$ the k-fold exterior power $\bigwedge^k \varphi$ of φ by:

$$\bigwedge^k \varphi(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det(\varphi(v_i, w_j)) \quad \text{for all } v_1, \dots, v_k, w_1, \dots, w_k \in V$$

for $1 \leq k \leq n$; $\bigwedge^k \varphi :=$ the zero form for $k > n$; and $\bigwedge^0 \varphi := \langle 1 \rangle$. $\bigwedge^k \varphi$ is again a bilinear form and is symmetric if φ is symmetric. Serre remarked on exterior powers (for integral forms) in [14]. In [11] we established the basic facts about exterior powers of a symmetric bilinear form, including formulas for their classical invariants. There we obtained a diagonalisation for the k-fold exterior power of the form $\varphi = \langle a_1, \dots, a_n \rangle$ as

$$\bigwedge^k \varphi = \perp_{1 \leq i_1 < \cdots < i_k \leq n} \langle a_{i_1} \cdots a_{i_k} \rangle,$$

which is just the k^{th} elementary symmetric polynomial in the n one-dimensional summands $\langle a_1 \rangle, \dots, \langle a_n \rangle$ of φ. We also saw that the k-fold exterior power $\bigwedge^k$ is well-defined

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on elements of the Witt-Grothendieck ring $\widehat{W}(K)$ but not on the Witt ring $W(K)$. Further, we showed that $\widehat{W}(K)$ is a λ-ring with the exterior powers as the λ-operations, and hence derived annihilating polynomials in $W(K)$.

Given a symmetric bilinear form φ of dimension n, there are two approaches to extending the concept of k-fold exterior power of φ, giving two classes of definition:

(A) those based on the Bourbaki determinant approach;
(B) those based on evaluating other symmetric polynomials at the one-dimensional forms $\langle a_1\rangle, \ldots, \langle a_n\rangle$.

The first class has the advantage of being computable for any given symmetric bilinear form, (not necessarily diagonal) *before* passing to $\widehat{W}(K)$. Also, it is coordinate-free in a way that the second class is not. It may even be defined over rings where we do not have a Diagonalisation theorem. The main disadvantages of this class of definition are that it does not work in $\widehat{W}(K)$ in characteristic different from zero, and in any case is not consistent with the λ-ring structure on $\widehat{W}(K)$ (that is, it does not satisfy a certain identity which symmetric powers in a λ-ring must satisfy — see Remark 3.8).

The second class has the advantages of being independent of field characteristic and consistent with the λ-ring structure on $\widehat{W}(K)$. However, it requires the form to be a sum of degree-one elements, i. e. a diagonalised form, and so is dependent on choice of coordinates, and less intrinsic.

In characteristic other than 2, we may use the one-one correspondence between quadratic forms and symmetric bilinear forms to define corresponding powers for quadratic forms.

In this article we give these definitions for the symmetric powers and examine their classical invariants and their relation to the exterior powers. This material formed part of the author's Ph. D. thesis [9]. In a later paper [10], we will explore the different ways (both classes (A) and (B)) in which, for any partition π of a positive integer k, we may define a "Schur power" of a form φ.

2. Notation and background

For $x \in \mathbb{R}$, $\lceil x \rceil$ will denote the greatest integer less than or equal to x. We will denote by S_k the symmetric group on k symbols. All vector spaces will be finite-dimensional.

For the meaning of such terms as bilinear and quadratic forms, isometry, the Witt-Grothendieck ring, etc., see, for example, [13]. For the definitions of k-fold exterior power and symmetric power of a finite-dimensional vector space see, for example, [3] or [2]. For λ-ring terminology, see [6].

Unless otherwise stated, by "form" we will mean "symmetric bilinear form". Juxtaposition will denote—according to context—a product of forms $\varphi\psi = \varphi \otimes \psi = \varphi \cdot \psi$, or a scalar multiple of a form: if $\varphi = \langle a_1, \ldots, a_n\rangle$, then $\lambda\varphi = \langle \lambda a_1, \ldots, \lambda a_n\rangle$. We will use a cross to denote an integer times a form: $n \times \varphi$ means the orthogonal sum of φ with itself n times.

We recall the following well-known result, similar to that for exterior powers:

Proposition 2.1. *Let V and W be vector spaces over K and let k be a positive integer. Then*

$$S^k(V \oplus W) = \bigoplus_{i+j=k} S^i V \otimes S^j W.$$

Definition 2.2. The **permanent** of a $k \times k$ matrix $A = (a_{ij})$ with entries in a commutative ring is

$$\operatorname{per}(A) = \sum_{\sigma \in S_k} a_{1\sigma(1)} \cdots a_{k\sigma(k)}.$$

Remark 2.3. The permanent is a *generalised matrix function* — like the determinant, except that the trivial character of S_k, rather than the sign character, is used to weight the summands. It shares some properties of the determinant — e. g. multiplying a row or column of the matrix by a scalar multiplies the permanent by that scalar — but is *not* multiplicative. For more details, see [12, Chapter 7].

The next two easily-proven results give properties of the permanent which are useful for us. By the **direct sum** of two square matrices A and B, we will mean the square matrix $A \oplus B := \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, where O stands for the appropriately sized zero matrices.

Lemma 2.4. *Let A and B be square matrices, possibly of different sizes. Then*

$$\operatorname{per}(A \oplus B) = (\operatorname{per} A)(\operatorname{per} B).$$

Lemma 2.5. *The permanent of the $k \times k$ matrix $\begin{pmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{pmatrix}$ is $k!a^k$.*

3. Factorial symmetric powers of symmetric bilinear forms

We first look at the class (A) definition of k-fold symmetric power of a bilinear form. This definition appears in [5] (at least for $k = 2$).

Definition 3.1. Let V be a vector space of dimension n over K. Let $\varphi : V \times V \longrightarrow K$ be a bilinear form, and let k be a positive integer. We define the **k-fold factorial symmetric power of** φ,

$$\boldsymbol{S}^k\varphi : S^k V \times S^k V \longrightarrow K$$

by

$$\boldsymbol{S}^k\varphi(x_1 \cdots x_k, y_1 \cdots y_k) = \operatorname{per}\big(\varphi(x_i, y_j)\big)_{1 \le i,j \le k},$$

where $\cdot$ is the multiplication in the symmetric algebra of V. We define $\boldsymbol{S}^0\varphi$ to be $\langle 1 \rangle$, the identity form of dimension 1. Clearly $\boldsymbol{S}^1\varphi = \varphi$.

$\boldsymbol{S}^k\varphi$ is easily seen to be a bilinear form and is symmetric if φ is symmetric. If q is the quadratic form associated to φ, we write $\boldsymbol{S}^k q$ for the quadratic form associated to $\boldsymbol{S}^k\varphi$.

Remark 3.2. A diagonalisation and the determinant of this form may then be found in a similar manner to those of the exterior powers as done in [11]; we will see, however, that they involve the field elements 2!, 3!, etc, hence the term “factorial symmetric power”.

Lemma 3.3. *Let $\{v_1, \ldots, v_n\}$ be a set of pairwise orthogonal vectors with respect to a symmetric bilinear form φ, and let $(u_1, \ldots, u_k)$ and $(w_1, \ldots, w_k)$ be two ordered k-tuples with $u_i, w_j \in \{v_1, \ldots, v_n\}$ for each $i, j \in \{1, \ldots, k\}$. Suppose that for some $r \in \{1, \ldots, n\}$ the number of u_i which are equal to v_r differs from the number of w_j which are equal to v_r. Then for each $\sigma \in S_k$, there exists $i \in \{1, \ldots, k\}$ such that $\varphi(u_i, w_{\sigma(i)}) = 0$.*

Proof. Without loss of generality, suppose that $u_{i_1} = \cdots = u_{i_s} = v_r$, $u_{i_{s+1}} \neq v_r$, $\ldots$, $u_{i_k} \neq v_r$ and that $w_{j_1} = \cdots = w_{j_s} = w_{j_{s+1}} = v_r$. Let $\sigma \in S_k$.

If $i \in \{i_1, \ldots, i_s\}$ and $\sigma(i) \notin \{j_1, \ldots, j_{s+1}\}$, we are done, since in this case $\varphi(u_i, w_{\sigma(i)}) = 0$.

So suppose that for each $i \in \{i_1, \ldots, i_s\}$ we have $\sigma(i) \in \{j_1, \ldots, j_{s+1}\}$. Then by injectivity of the permutation σ, we may choose m such that

$$\sigma^{-1}(j_m) \notin \{i_1, \ldots, i_s\}.$$

Setting $i = \sigma^{-1}(j_m)$, we have $u_i \neq v_r$ and $w_{\sigma(i)} = w_{j_m} = v_r$, so $\varphi(u_i, w_{\sigma(i)}) = 0$ as required. $\square$

Remark 3.4. Consider the symmetric power $\boldsymbol{S}^k\varphi$ defined by

$$\boldsymbol{S}^k\varphi(u_1 \cdots u_k, w_1 \cdots w_k) = \operatorname{per}(\varphi(u_i, w_j)).$$

Lemma 3.3 says that if we choose the u_i, w_j from the pairwise orthogonal set $\{v_1, \ldots, v_n\}$, and for some r the number of u_i which are equal to v_r differs from the number of w_j which are equal to v_r, we will have

$$\boldsymbol{S}^k\varphi(u_1 \cdots u_k, w_1 \cdots w_k) = 0.$$

For,

$$\operatorname{per}(\varphi(u_i, w_j)) = \sum_{\sigma \in S_k} \varphi(u_1, w_{\sigma(1)}) \cdots \varphi(u_k, w_{\sigma(k)})$$

and in each summand in the sum over S_k, at least one of the multiplicands will be 0 by the last lemma.

Proposition 3.5. *Let V be a vector space of dimension n over K and let $\varphi = \langle a_1, \ldots, a_n\rangle$ be a diagonalisation of a symmetric bilinear form on V. Let k be a positive integer. Then $\boldsymbol{S}^k\varphi$ is a symmetric bilinear form of dimension $\binom{n+k-1}{k}$ and has a diagonalisation of the form*

$$\boldsymbol{S}^k\varphi = \mathop{\perp}_{\substack{1 \leq i_1 < \cdots < i_l \leq n \\ k_{i_1} + \cdots + k_{i_l} = k}} \langle k_{i_1}! a_{i_1}^{k_{i_1}} \cdots k_{i_l}! a_{i_l}^{k_{i_l}}\rangle.$$

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis for V, with $\varphi(v_i, v_i) = a_i$ and $\varphi(v_i, v_j) = 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$. Let k be a positive integer. Since a basis for S^kV is

$$\{v_{i_1} \cdots v_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\} = \{v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}} : k_{i_1} + \cdots + k_{i_l} = k\}$$

we have immediately that the form $\boldsymbol{S}^k\varphi$ has dimension $\binom{n+k-1}{k}$.

Without loss of generality, we may restrict attention to basis elements of S^kV, by bilinearity of $\boldsymbol{S}^k\varphi$. Let $v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}}$ and $v_{j_1}^{p_{j_1}} \cdots v_{j_m}^{p_{j_m}}$ be two basis elements of S^kV, and consider $\boldsymbol{S}^k\varphi(v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}}, v_{j_1}^{p_{j_1}} \cdots v_{j_m}^{p_{j_m}})$.

If $v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}}$ and $v_{j_1}^{p_{j_1}} \cdots v_{j_m}^{p_{j_m}}$ are not the same element of S^kV, then by Remark 3.4 the permanent $\boldsymbol{S}^k\varphi(v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}}, v_{j_1}^{p_{j_1}} \cdots v_{j_m}^{p_{j_m}})$ will be 0.

Thus it suffices to consider $\boldsymbol{S}^k\varphi(v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}}, v_{i_1}^{k_{i_1}} \cdots v_{i_l}^{k_{i_l}})$. By orthogonality of $\{v_1, \ldots, v_n\}$ we get that the matrix $(\varphi(v_{i_r}, v_{j_s}))$ is a direct sum of matrices,

$$(\varphi(v_{i_r}, v_{j_s})) \;=\; \begin{pmatrix} a_{i_1} & \cdots & a_{i_1} \\ \vdots & \ddots & \vdots \\ a_{i_1} & \cdots & a_{i_1} \end{pmatrix} \oplus \begin{pmatrix} a_{i_2} & \cdots & a_{i_2} \\ \vdots & \ddots & \vdots \\ a_{i_2} & \cdots & a_{i_2} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a_{i_l} & \cdots & a_{i_l} \\ \vdots & \ddots & \vdots \\ a_{i_l} & \cdots & a_{i_l} \end{pmatrix}$$

Then by Lemmata 2.4 and 2.5, $\operatorname{per}(\varphi(v_{i_r}, v_{j_s})) = k_{i_1}!a_{i_1}^{k_{i_1}} \cdots k_{i_l}!a_{i_l}^{k_{i_l}}$. Hence $\boldsymbol{S}^k\varphi$ is represented by a diagonal matrix of size $\binom{n+k-1}{k} \times \binom{n+k-1}{k}$, with entries as claimed. □

Remark 3.6. Observe that if $\varphi = \langle a_1, \ldots, a_n\rangle$ and $\operatorname{char} K \leq k$, then the first term $\langle k!a_1^k\rangle$ in the diagonalisation of $\boldsymbol{S}^k\varphi$ will be 0. Thus $\det \boldsymbol{S}^k\varphi$ will be 0 and so $\boldsymbol{S}^k\varphi$ will be singular regardless of whether or not φ is singular. This is why we are effectively restricted to characteristic 0 when considering factorial symmetric powers.

From [11, Proposition 4.1], a diagonalisation of the k-fold exterior power $\bigwedge^k\varphi$ of a symmetric bilinear form $\varphi = \langle a_1, \ldots, a_n\rangle$ is $\bigwedge^k\varphi = \bigperp_{1\leq i_1<\cdots<i_k\leq n}\langle a_{i_1} \cdots a_{i_k}\rangle$. By choosing each of the $k_{i_j} = 1$ in the orthogonal sum in the statement of Proposition 3.5, we pick out the summands of the type $\langle a_{i_1} \cdots a_{i_k}\rangle$. Thus $\bigwedge^k\varphi$ is a subform of $\boldsymbol{S}^k\varphi$.

Many of the remarks in [11, §4] about $\bigwedge^k\varphi$ hold for $\boldsymbol{S}^k\varphi$ also. If φ is a hyperbolic form over a formally real field then $\boldsymbol{S}^k\varphi$ may or may not be hyperbolic, since (by [11, Remark 4.2]) its subform $\bigwedge^k\varphi$ may or may not be hyperbolic: if $\bigwedge^k\varphi$ has an anisotropic part, so has $\boldsymbol{S}^k\varphi$. Also, for example, $\boldsymbol{S}^3\langle a, b\rangle = \langle 3a^3, 3b^3, 2a^2b, 2b^2a\rangle$ by Proposition 3.5, and so for the hyperbolic form $\langle 1, -1\rangle$ we have $\boldsymbol{S}^3\langle 1, -1\rangle = \langle 3, -3, -2, 2\rangle$, which is hyperbolic.

Non-isometric forms *may* have isometric k-fold symmetric powers. For example, let $\varphi = \langle 1, 2, 3\rangle$, $\psi = \langle 1, 6, 3\rangle$ over a field K in which 3 is not a square, so by comparison of determinants φ and ψ are not isometric. Now $\bigwedge^2\varphi = \langle 2, 3, 6\rangle \simeq \langle 6, 3, 2\rangle = \bigwedge^2\psi$, and so

$$\boldsymbol{S}^2\varphi = \textstyle\bigwedge^2\varphi \perp \langle 2\cdot 1^2, 2\cdot 2^2, 2\cdot 3^2\rangle \simeq \bigwedge^2\psi \perp \langle 2\cdot 1^2, 2\cdot 6^2, 2\cdot 3^2\rangle = \boldsymbol{S}^2\psi.$$

If φ is an isotropic form then $\boldsymbol{S}^k\varphi$ will also be isotropic, since its subform $\bigwedge^k\varphi$ is isotropic, as seen in [11, Remark 4.3].

The same argument as in [11, Remark 4.4] shows that if φ is an anisotropic form then $\boldsymbol{S}^k\varphi$ need not be anisotropic: since $\boldsymbol{S}^k\varphi$ has dimension greater than that of $\bigwedge^k\varphi$, the dimension of $\boldsymbol{S}^k\varphi$ will exceed the u-invariant whenever the dimension of $\bigwedge^k\varphi$ does.

Remark 3.7. Provided we are in characteristic 0, $\boldsymbol{S}^k\varphi$ is well-defined on isometry classes and on elements of the Witt-Grothendieck ring $\widehat{W}(K)$. This is a special case of a result in [10] based on work in [4], but we sketch a proof for $\boldsymbol{S}^k$ now.

Let $\mathcal{V} = \mathcal{V}_K$ be the category of finite-dimensional K-vector spaces, let $\mathcal{C} = \mathcal{C}_K$ be the category of finite-dimensional K-bilinear spaces, and let $\mathcal{D} = \mathcal{D}_K$ be the category whose objects and morphisms are as follows. An object is a pair (V, γ) where V is a finite-dimensional K-vector space and γ is a vector space homomorphism $\gamma : V \longrightarrow V^*$. A morphism in $\mathrm{Hom}_{\mathcal{D}}((V,\gamma),(W,\delta))$ is a vector space homomorphism $f : V \longrightarrow W$ such that $f^* \circ \delta \circ f = \gamma$. Then one easily shows that as categories, $\mathcal{C} \cong \mathcal{D}$ via the functor $\varphi \rightsquigarrow \widehat{\varphi}$ where $\widehat{\varphi} : V \longrightarrow V^*$ is the adjoint of φ. Next, one has that φ is symmetric if and only if the linear map $(\widehat{\varphi})^* : (V^*)^* = V \longrightarrow V^*$ is equal to $\widehat{\varphi}$.

Now call a functor $\mathcal{F} : \mathcal{V} \rightsquigarrow \mathcal{V}$ a **$*$-functor** if there is a (natural) map of functors $h : \mathcal{F} \circ * \longrightarrow * \circ \mathcal{F}$. Call $(\mathcal{F}, h)$ a **symmetric** $*$-functor if for every V, h satisfies $h(V^*) = h(V)^* : \mathcal{F}((V^*)^*) = \mathcal{F}(V) \longrightarrow \mathcal{F}(V^*)^*$. One shows that: a $*$-functor $(\mathcal{F}, h)$ induces a functor $(V,\varphi) \rightsquigarrow (\mathcal{F}_h(V), \mathcal{F}_h(\varphi))$ on $\mathcal{C}$ in a canonical way; if $h(W)$ is an isomorphism for all W and (V,φ) is non-degenerate, then $(\mathcal{F}_h(V), \mathcal{F}_h(\varphi))$ is non-degenerate; and if $(\mathcal{F}, h)$ is a symmetric $*$-functor and (V,φ) is symmetric, then $(\mathcal{F}_h(V), \mathcal{F}_h(\varphi))$ is symmetric. To conclude, one shows that the maps

$$h_{S^k}(V) = h(V) : S^k(V^*) \longrightarrow S^k(V)^*$$

determined by

$$h(V)(\gamma_1 \cdots \gamma_k)(v_1 \cdots v_k) = \sum_{\sigma \in S_k} \gamma_{\sigma(1)}(v_1) \cdots \gamma_{\sigma(k)}(v_k)$$

make S^k into a symmetric $*$-functor and that $h_{S^k}(V)$ is an isomorphism for all finite-dimensional vector spaces V, *provided* K has characteristic 0. The functor induced by S^k on $\mathcal{C}$ is then $\boldsymbol{S}^k$ and $\boldsymbol{S}^k$ is well-defined on elements of $\widehat{W}(K)$.

Remark 3.8. The definition of factorial symmetric powers is not consistent with the λ-ring structure on the Witt-Grothendieck ring, for the following reason. In order for operations s^j in a λ-ring to play the rôle of symmetric powers, they must satisfy the relation (see [6, page 31])

$$\sum_{i+j=k} (-1)^j \lambda^i(x) s^j(x) = 0.$$

Straightforward computation shows that for the example of $\varphi = \langle a, b, c\rangle$, we get

$$\sum_{i+j=3} (-1)^j {\bigwedge}^i \varphi \boldsymbol{S}^j \varphi = \langle 2a, 2b, 2c\rangle - \langle 6a, 6b, 6c\rangle$$

in the Witt-Grothendieck ring. This will be 0 if and only if $\langle 6a, 6b, 6c\rangle$ and $\langle 2a, 2b, 2c\rangle$ are isometric. But $\det\langle 6a, 6b, 6c\rangle = 6abc$, while $\det\langle 2a, 2b, 2c\rangle = 2abc$, and these will not be equal unless 3 is a square in K.

Invariants of factorial symmetric powers. We will work out the determinant and (for $k = 2$) the Hasse invariant of $\boldsymbol{S}^k\varphi$ now, and the signature in Remark 4.15 later.

Proposition 3.9. *Let φ be a symmetric bilinear form of dimension n. Then*

$$\det(\boldsymbol{S}^k\varphi) = D \cdot (\det\varphi)^{\binom{n+k-1}{n}} \quad \textit{where} \quad D = \prod_{\substack{1 \le i_1 < \cdots < i_l \le n \\ k_{i_1} + \cdots + k_{i_l} = k}} k_{i_1}! \cdots k_{i_l}!.$$

In particular, for $k = 2$,

$$\det(\boldsymbol{S}^2\varphi) = 2^n (\det\varphi)^{n+1}.$$

Proof. Write $\varphi = \langle a_1, \dots, a_n \rangle$ in diagonal form, so $\det \varphi = a_1 a_2 \cdots a_n$. From the diagonalisation of $\boldsymbol{S}^k$ in Proposition 3.5, we have that its determinant is

$$\det(\boldsymbol{S}^k\varphi) = \left(\prod_{\substack{1 \le i_1 < \cdots < i_l \le n \\ k_{i_1} + \cdots + k_{i_l} = k}} k_{i_1}! \cdots k_{i_l}! \right) \left(\prod_{\substack{1 \le i_1 < \cdots < i_l \le n \\ k_{i_1} + \cdots + k_{i_l} = k}} a_{i_1}^{k_{i_1}} \cdots a_{i_l}^{k_{i_l}} \right).$$

The second product is symmetric in $a_1, \dots, a_n$, it consists of $\binom{n+k-1}{k}$ terms (the dimension of $\boldsymbol{S}^k\varphi$), and each term is itself a product of k of the a_i. So altogether we have

$$\frac{(n+k-1)!\,k}{k!\,(n-1)!} = \frac{(n+k-1)!}{(k-1)!\,(n-1)!}$$

of the scalars a_i, multiplied together. Since there are n of the a_i, and each occurs equally often, we have that each occurs

$$\frac{1}{n} \frac{(n+k-1)!}{(k-1)!\,(n-1)!} = \frac{(n+k-1)!}{(k-1)!\,(n)!} = \binom{n+k-1}{n}$$

times, giving

$$\begin{aligned} \det(\boldsymbol{S}^k\varphi) &= D \cdot (a_1 a_2 \cdots a_n)^{\binom{n+k-1}{n}} \\ &= D \cdot (\det \varphi)^{\binom{n+k-1}{n}} \end{aligned}$$

where we write

$$D := \prod_{\substack{1 \le i_1 < \cdots < i_l \le n \\ k_{i_1} + \cdots + k_{i_l} = k}} k_{i_1}! \cdots k_{i_l}!.$$

When $k = 2$, $2!$ occurs n times and $1!$ occurs $\binom{n}{2}$ times, giving $D = 2! \cdots 2! 1! \cdots 1! = 2^n$ and $D \cdot (\det \varphi)^{\binom{n+k-1}{n}} = D \cdot (\det \varphi)^{\binom{n+1}{n}} = 2^n (\det \varphi)^{n+1}$. This completes the proof. □

Proposition 3.10. *Let φ be a form of dimension n with determinant d. Then*

$$s(\boldsymbol{S}^2\varphi) = s(\textstyle\bigwedge^2\varphi) = s(\varphi)^n (d, -1)^{(n-1)(n-2)/2}.$$

Proof. Write $\varphi = \langle a_1, \dots, a_n \rangle$ in diagonal form. From Proposition 3.5,

$$\begin{aligned} \boldsymbol{S}^2\varphi &= \left(\perp_{1 \le i_1 < i_2 \le n} \langle a_{i_1} a_{i_2} \rangle \right) \perp \langle 2a_1^2, \dots, 2a_n^2 \rangle \\ &= \textstyle\bigwedge^2 \varphi \perp n \times \langle 2 \rangle. \end{aligned}$$

Now the Hasse invariant of $n \times \langle 2 \rangle$ is $\prod_{i<j} (2,2) = (2,2)^{n(n-1)/2}$ and its determinant is 2^n. Also, by [11, Proposition 5.1], $\det(\bigwedge^2\varphi) = d^{n-1}$. By properties of the Hasse invariant,

$$\begin{aligned} s(\boldsymbol{S}^2\varphi) &= s(\textstyle\bigwedge^2\varphi)(2,2)^{n(n-1)/2}(2^n, \det(\textstyle\bigwedge^2\varphi)) \\ &= s(\textstyle\bigwedge^2\varphi)(2,-1)^{n(n-1)/2}(2^n, d^{n-1}) \\ &= s(\textstyle\bigwedge^2\varphi)(2,-1)^{n(n-1)/2}(2, d)^{n(n-1)}. \end{aligned}$$

Now $(2, -1) = (2, 1-2) = 1_{\mathrm{Br}(K)}$ and $n(n-1)$ is an even integer, so this becomes $s(\boldsymbol{S}^2\varphi) = s(\bigwedge^2\varphi)$ and the result follows from [11, Corollary 11.3]. □

Algebraic properties of factorial symmetric powers. As for $\bigwedge^k$ in [11], we examine how $\boldsymbol{S}^k$ behaves with respect to scalar multiplication and orthogonal sum of forms.

Repeated application of the property of a permanent that multiplying a row or column of the matrix by a scalar multiplies the permanent by that scalar gives us that $\boldsymbol{S}^k$ is k-homogeneous with respect to scalar multiplication, that is,

$$\boldsymbol{S}^k(\alpha\varphi) = \alpha^k \boldsymbol{S}^k\varphi = \begin{cases} \alpha\boldsymbol{S}^k\varphi & \text{if } k \text{ is odd;} \\ \boldsymbol{S}^k\varphi & \text{if } k \text{ is even.} \end{cases}$$

Proposition 3.11. *Let φ and ψ be symmetric bilinear forms over K and let k be a positive integer. Then*

$$\boldsymbol{S}^k(\varphi \perp \psi) = \bigperp_{i+j=k} \boldsymbol{S}^i\varphi \cdot \boldsymbol{S}^j\psi.$$

Proof. Let φ, ψ and k be as in the statement. Suppose that φ acts on a vector space V of dimension n, and ψ acts on a vector space W of dimension m. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be orthogonal bases for V and W with respect to φ and ψ respectively.

Let $U = V \oplus W$, so U is the underlying space for $\varphi \perp \psi$, with basis

$$\{v_1, \ldots, v_n, w_1, \ldots, w_m\}.$$

By Proposition 2.1, this result is true for the underlying vector spaces and we now show it is also true of the forms.

Let $\varphi(v_i, v_i) = a_i$ for $i = 1, \ldots, n$, and $\psi(w_j, w_j) = a_{n+j}$ for $j = 1, \ldots, m$.

Let $\langle k_{i_1}!a_{i_1}^{k_{i_1}} \cdots k_{i_r}!a_{i_r}^{k_{i_r}} k_{i_{r+1}}!a_{i_{r+1}}^{k_{i_{r+1}}} \cdots k_{i_{r+s}}!a_{i_{r+s}}^{k_{i_{r+s}}}\rangle$ be a one-dimensional summand in the diagonalisation of $\boldsymbol{S}^k(\varphi \perp \psi)$, where $1 \le i_1 < \cdots < i_r \le n < i_{s+1} < \cdots < i_{r+s} \le n+m$. Let $i = k_{i_1} + \cdots + k_{i_r}$ and $j = k_{i_{r+1}} + \cdots + k_{i_{r+s}}$. Then $\langle k_{i_1}!a_{i_1}^{k_{i_1}} \cdots k_{i_r}!a_{i_r}^{k_{i_r}}\rangle$ is a one-dimensional summand in the diagonalisation of $\boldsymbol{S}^i(\varphi)$, while $\langle k_{i_{r+1}}!a_{i_{r+1}}^{k_{i_{r+1}}} \cdots k_{i_{r+s}}!a_{i_{r+s}}^{k_{i_{r+s}}}\rangle$ is a one-dimensional summand in the diagonalisation of $\boldsymbol{S}^j(\psi)$. Thus

$$\langle k_{i_1}!a_{i_1}^{k_{i_1}} \cdots k_{i_r}!a_{i_r}^{k_{i_r}} \cdots k_{i_{r+s}}!a_{i_{r+s}}^{k_{i_{r+s}}}\rangle = \langle k_{i_1}!a_{i_1}^{k_{i_1}} \cdots k_{i_r}!a_{i_r}^{k_{i_r}}\rangle\langle k_{i_{r+1}}!a_{i_{r+1}}^{k_{i_{r+1}}} \cdots k_{i_{r+s}}!a_{i_{r+s}}^{k_{i_{r+s}}}\rangle$$

is a one-dimensional summand in the diagonalisation of $\boldsymbol{S}^i\varphi \cdot \boldsymbol{S}^j\psi$. Each such summand of $\boldsymbol{S}^k(\varphi \perp \psi)$ is thus a summand of $\boldsymbol{S}^i\varphi \cdot \boldsymbol{S}^j\psi$ for some i and j with $i + j = k$ and by dimension count, there is a one-one correspondence between one-dimensional summands of $\boldsymbol{S}^k(\varphi \perp \psi)$ and one-dimensional summands of $\bigperp_{i+j=k} \boldsymbol{S}^i\varphi \cdot \boldsymbol{S}^j\psi$. This completes the proof. $\square$

Remark 3.12. We may view the fact that $\boldsymbol{S}^k(\varphi \perp \psi) = \bigperp_{i+j=k} \boldsymbol{S}^i\varphi\cdot\boldsymbol{S}^j\psi$ as giving another pre-λ-ring structure on $\widehat{W}(K)$, incompatible with the first. In particular, the dimension map dim is not an augmentation with respect to this pre-λ-ring structure, as it does not commute with the λ-operations: if φ has dimension n, then $\dim \boldsymbol{S}^k(\varphi) = \binom{n+k-1}{k}$, but in $\mathbb{Z}$, $\lambda^k(n) = \binom{n}{k}$. Moreover, with respect to this pre-λ-ring structure, no element φ of $\widehat{W}(K)$ has finite degree.

4. Non-factorial symmetric powers of symmetric bilinear forms

We now look at the class (B) definition of symmetric power. Recall that the k^{th} **complete homogeneous polynomial** in indeterminates $X_1, \ldots, X_n$ is

$$H_k = \sum_{1 \le i_1 \le \cdots \le i_k \le n} X_{i_1} \cdots X_{i_k} = \sum_{k_{i_1} + \cdots + k_{i_l} = k} X_{i_1}^{k_{i_1}} \cdots X_{i_l}^{k_{i_l}}$$

consisting of all distinct monomials of degree k in the $X_1, \ldots, X_n$.

Definition 4.1. Given a diagonalisation $\langle a_1, \ldots, a_n \rangle$ of a symmetric bilinear form φ of dimension n, we define the **k-fold non-factorial symmetric power of φ**,

$$\begin{aligned} S^k\langle a_1, \ldots, a_n \rangle &:= H_k(\langle a_1 \rangle, \ldots, \langle a_n \rangle) \\ &= \perp_{1 \le i_1 \le \cdots \le i_k \le n} \langle a_{i_1} \cdots a_{i_k} \rangle \\ &= \perp_{k_{i_1} + \cdots + k_{i_l} = k} \langle a_{i_1}^{k_{i_1}} \cdots a_{i_l}^{k_{i_l}} \rangle \end{aligned}$$

Remark 4.2. This definition is the same as the characterisation of $\boldsymbol{S}^k\langle a_1, \ldots, a_n \rangle$ given in Proposition 3.5, except for factorials: hence "non-factorial". $S^k\varphi$ is defined over fields of arbitrary characteristic, unlike $\boldsymbol{S}^k\varphi$ (φ is regular $\iff$ no a_i is 0 $\iff$ $S^k\varphi$ is regular).

Since the k^{th} elementary symmetric polynomial is homogeneous of degree k, every term in it occurs in the complete homogeneous polynomial H_k. Thus $\bigwedge^k\varphi$ is a subform of $S^k\varphi$.

By the Fundamental Theorem of symmetric functions, the complete homogeneous polynomial H_k, being a symmetric function, has a unique expression as a polynomial in the elementary symmetric polynomials. In our context, the elementary symmetric polynomial E_i is the exterior power $\bigwedge^i$, which we know to be well-defined on the Witt-Grothendieck ring from [11]. Thus S^k is well-defined on the Witt-Grothendieck ring.

S^k, unlike $\boldsymbol{S}^k$, is consistent with the λ-ring structure on the Witt-Grothendieck ring, since (see [6], [8]) the elementary symmetric polynomials E_i and the complete homogeneous polynomials H_j satisfy precisely the relation

$$\sum_{i+j=k} (-1)^j E_i H_j = 0.$$

Many of the remarks in [11, §4] about $\bigwedge^k\varphi$ hold for $S^k\varphi$ also. As with $\boldsymbol{S}^k\varphi$, if φ is a hyperbolic form over a formally real field then $S^k\varphi$ may or may not be hyperbolic, since its subform $\bigwedge^k\varphi$ may or may not be hyperbolic. And, for the hyperbolic form $\langle 1, -1 \rangle$, $S^3\langle 1, -1 \rangle = \langle 1, -1, -1, 1 \rangle$ is also hyperbolic.

Again, non-isometric forms *may* have isometric k-fold symmetric powers. Let $\varphi = \langle 1, 2, 3 \rangle$, $\psi = \langle 1, 6, 3 \rangle$ be as in Remark 3.6 over a field K in which 3 is not a square. Then φ and ψ are not isometric, but $\bigwedge^2\varphi$ and $\bigwedge^2\psi$ *are* isometric. So

$$S^2\varphi \;=\; \textstyle\bigwedge^2\varphi \perp \langle 1^2, 2^2, 3^2 \rangle \;\simeq\; \bigwedge^2\psi \perp \langle 1^2, 6^2, 3^2 \rangle \;=\; S^2\psi.$$

If φ is an isotropic form then $S^k\varphi$ will also be isotropic, since its subform $\bigwedge^k\varphi$ is isotropic.

If φ is an anisotropic form then $S^k\varphi$ need not be anisotropic: since $\dim S^k\varphi > \dim \bigwedge^k\varphi$, we have that $\dim S^k\varphi > u(K)$, the u-invariant of K, whenever $\dim \bigwedge^k\varphi > u(K)$.

Invariants of non-factorial symmetric powers.

Proposition 4.3. *Let φ be a symmetric bilinear form of dimension n, and let k be a positive integer. Then*

$$\det(S^k\varphi) = (\det\varphi)^{\binom{n+k-1}{n}}.$$

In particular, for $k = 2$, $\det(S^2\varphi) = (\det\varphi)^{n+1}$.

Proof. Write $\varphi = \langle a_1, \dots, a_n\rangle$, so $\det\varphi = a_1a_2\cdots a_n$. Then

$$S^k\langle a_1, \dots, a_n\rangle = H_k(\langle a_1\rangle, \dots, \langle a_k\rangle) = \perp_{1\le i_1\le\cdots\le i_k\le n} \langle a_{i_1}\cdots a_{i_k}\rangle$$

in diagonal form and so its determinant is

$$\det(S^k\langle a_1, \dots, a_n\rangle) = \prod_{1\le i_1\le\cdots\le i_k\le n} a_{i_1}\cdots a_{i_k}.$$

By the proof of Proposition 3.9, this is just $(\det\varphi)^{\binom{n+k-1}{n}}$, and the result follows. □

Proposition 4.4. *Let φ be an n-dimensional symmetric bilinear form over K and let k be a positive integer. Then*

$$S^k\varphi = \perp_{i=0}^{\lceil k/2\rceil} \left(\left(\binom{n+i-1}{i} \times \langle 1\rangle\right) \cdot \bigwedge^{k-2i}\varphi\right) = \perp_{i=0}^{\lceil k/2\rceil} \binom{n+i-1}{i} \times \bigwedge^{k-2i}\varphi.$$

Proof. Let $\varphi = \langle a_1, \dots, a_n\rangle$ and let $\psi = \langle a_1^2, \dots, a_n^2\rangle$. Let $\langle a_{i_1}^{k_{i_1}}\cdots a_{i_l}^{k_{i_l}}\rangle$ be an arbitrary one-dimensional summand in the diagonalisation of $S^k\varphi$ in Definition 4.1.

Write each $k_{i_j} = 2r_{i_j} + \varepsilon_{i_j}$ where $r_{i_j} = \lceil k_{i_j}/2\rceil$ and ε_{i_j} is 0 or 1 according as to whether k_{i_j} is even or odd, respectively. Then

$$a_{i_1}^{k_{i_1}}\cdots a_{i_l}^{k_{i_l}} = a_{i_1}^{2r_{i_1}}\cdots a_{i_l}^{2r_{i_l}} \cdot a_{i_1}^{\varepsilon_{i_1}}\cdots a_{i_l}^{\varepsilon_{i_l}}$$

so

$$\langle a_{i_1}^{k_{i_1}}\cdots a_{i_l}^{k_{i_l}}\rangle = \langle a_{i_1}^{2r_{i_1}}\cdots a_{i_l}^{2r_{i_l}}\rangle \cdot \langle a_{i_1}^{\varepsilon_{i_1}}\cdots a_{i_l}^{\varepsilon_{i_l}}\rangle.$$

Suppose that $\sum_j r_{i_j} = i$. Then $\langle a_{i_1}^{2r_{i_1}}\cdots a_{i_l}^{2r_{i_l}}\rangle = \langle (a_{i_1}^2)^{r_{i_1}}\cdots(a_{i_l}^2)^{r_{i_l}}\rangle$ is a one-dimensional summand of $S^i\psi$ and $\langle a_{i_1}^{\varepsilon_{i_1}}\cdots a_{i_l}^{\varepsilon_{i_l}}\rangle$ is a one-dimensional summand of $\bigwedge^{k-2i}\varphi$, each uniquely determined by the integers $k_{i_1}, \dots, k_{i_l}$ describing $\langle a_{i_1}^{k_{i_1}}\cdots a_{i_l}^{k_{i_l}}\rangle$. Conversely, given two such terms $\langle (a_{i_1}^2)^{r_{i_1}}\cdots(a_{i_l}^2)^{r_{i_l}}\rangle$ from $S^i\psi$ and $\langle a_{i_1}^{\varepsilon_{i_1}}\cdots a_{i_l}^{\varepsilon_{i_l}}\rangle$ from $\bigwedge^{k-2i}\varphi$, we can recover the k_{i_j}. Hence, as forms,

$$S^k\varphi = \perp_{k_{i_1}+\cdots+k_{i_l}=k} a_{i_1}^{k_{i_1}}\cdots a_{i_l}^{k_{i_l}} = \perp_{i=0}^{\lceil k/2\rceil} S^i\psi \cdot \bigwedge^{k-2i}\varphi.$$

Finally, since $\psi = \langle a_1^2, \dots, a_n^2\rangle \simeq n \times \langle 1\rangle$, the dimension formula for symmetric powers gives $S^i\psi \simeq \binom{n+i-1}{i} \times \langle 1\rangle$ and the result follows. □

Remark 4.5. Let K be a field with ordering $\mathcal{P}$. Since the signature is a ring homomorphism, Proposition 4.4 allows us to calculate the signature of $S^k\varphi$ in terms of signatures of various $\bigwedge^i(\varphi)$, which we can compute from [11, Proposition 10.1]. For example, we have the next two results. We first need some notation. We define, for positive integers k and n, a polynomial $D_n^k(t) \in \mathbb{Z}[t]$ as the $k \times k$ determinant

$$D_n^k(t) := \det \begin{pmatrix} t & 1 & & & \\ n & t & 2 & & 0 \\ t & \ddots & \ddots & \ddots & \\ \vdots & \ddots & n & t & k-1 \\ & \dots & t & n & t \end{pmatrix}.$$

(In the notation of [11, Definition 9.13], this would be $D_n^{k-1}(t)$.)

Corollary 4.6. *Let φ be an n-dimensional symmetric bilinear form over a field K, having signature r with respect to the ordering $\mathcal{P}$ of K, and let k be a positive integer. Then*

$$\operatorname{sign}_{\mathcal{P}}(S^k\varphi) = \sum_{i=0}^{\lceil k/2 \rceil} \binom{n+i-1}{i} \operatorname{sign}_{\mathcal{P}}(\textstyle\bigwedge^{k-2i}\varphi) = \displaystyle\sum_{i=0}^{\lceil k/2 \rceil} \binom{n+i-1}{i} \frac{D_n^{k-2i-1}(r)}{(k-2i)!}.$$

Proof. The first equality follows from Proposition 4.4 and the fact that $\operatorname{sign}_{\mathcal{P}}$ is a ring homomorphism. The second equality follows from [11, Proposition 10.1]. $\square$

Corollary 4.7. *Let φ be a symmetric bilinear form of signature 0 over an ordered field K and let k be an odd positive integer. Then the symmetric power $S^k\varphi$ has signature 0.*

Proof. Let $\operatorname{sign}_{\mathcal{P}}(\varphi) = 0$. If k is odd, so is each $k-2i$ for $i = 0, \dots, \lceil k/2 \rceil$. Then by [11, Corollary 10.5], each $\operatorname{sign}_{\mathcal{P}}(\bigwedge^{k-2i}\varphi) = 0$. The result follows from Corollary 4.6. $\square$

Remark 4.8. As a dual to the determinant formula in [11, Remark 9.4], there is the formula

$$k! \times H_k = \det \begin{pmatrix} P_1 & -1 & & & \\ P_2 & P_1 & -2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ P_{k-1} & P_{k-2} & \ddots & P_1 & -k+1 \\ P_k & P_{k-1} & \dots & P_2 & P_1 \end{pmatrix}$$

(see [8, 8, page 20]) where the P_j are the power sums in $X_1, \dots, X_n$, that is, the Adams operations Ψ_j in the λ-ring $\widehat{W}(K)$. By [11, Proposition 9.20],

$$\Psi^j(\varphi) = \begin{cases} n \times \langle 1 \rangle, & \text{for } j \text{ even,} \\ \varphi, & \text{for } j \text{ odd,} \end{cases}$$

so

$$k! \times S^k(\varphi) = \det \begin{pmatrix} \varphi & -1 & & & \\ n & \varphi & -2 & & 0 \\ \varphi & n & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \varphi & -k+1 \\ & & & n & \varphi \end{pmatrix}.$$

Definition 4.9. We define a polynomial $C_n^k(t) \in \mathbb{Z}[t]$ as the $k \times k$ determinant

$$C_n^k(t) := \det \begin{pmatrix} t & -1 & & & \\ n & t & -2 & & 0 \\ t & \ddots & \ddots & \ddots & \\ \vdots & \ddots & n & t & -k+1 \\ & \dots & t & n & t \end{pmatrix} = \operatorname{per} \begin{pmatrix} t & 1 & & & \\ n & t & 2 & & 0 \\ t & \ddots & \ddots & \ddots & \\ \vdots & \ddots & n & t & k-1 \\ & \dots & t & n & t \end{pmatrix}.$$

Clearly, $C_n^k(t)$ is of degree k, and $k! \times S^k(\varphi) = C_n^k(\varphi)$.

Lemma 4.10. *Let n, k be positive integers. Then the polynomial $C_n^k(t)$ is monic and is even (respectively odd) according as k is even (respectively odd).*

Proof. We first establish the recurrence relation

$$C_n^k(t) = tC_n^{k-1}(t) + (k-1)(n+k-2)C_n^{k-2}(t) \qquad (*)$$

by expanding the determinant in Definition 4.9 along the rightmost column.

We now proceed by complete induction on k. Fix n. Clearly $C_n^1(t) = t$, $C_n^2(t) = t^2 + n$, so the result is true for $k = 1$ and $k = 2$. Suppose the result is true for $1, 2, \dots, k-1$. Comparing degrees in $(*)$ shows that

$$\text{leading term of } C_n^k(t) = t(\text{leading term of } C_n^{k-1}(t))$$

so $C_n^k(t)$ is monic by the induction hypothesis. Moreover, $C_n^{k-1}(t)$ has opposite parity to $C_n^{k-2}(t)$ by the induction hypothesis, so the parity of $tC_n^{k-1}(t)$ is equal to the parity of $C_n^{k-2}(t)$. Thus the parity of $k-2$ is the parity of the right hand side of $(*)$, so it is the parity of $C_n^k(t)$, completing the proof. □

Remark 4.11. This gives an alternative proof of Corollary 4.7. Let k be an odd positive integer, and let φ be a symmetric bilinear form of signature 0 with respect to the ordering $\mathcal{P}$ of K. By Lemma 4.10, C_n^k is odd, so $C_n^k(0) = 0$. Since $\operatorname{sign}_{\mathcal{P}} : \widehat{W}(K) \longrightarrow \mathbb{Z}$ is a ring-homomorphism, $\operatorname{sign}_{\mathcal{P}} S^k\varphi = C_n^k(\operatorname{sign}_{\mathcal{P}} \varphi)/k! = C_n^k(0)/k! = 0$.

Proposition 4.12. *Suppose that $k = 2l$ is an even positive integer, and φ is a symmetric bilinear form with $\operatorname{sign}_{\mathcal{P}} \varphi = 0$ and dimension $n = 2m$. Then*

$$\operatorname{sign}_{\mathcal{P}}(S^k\varphi) = \binom{m+l-1}{l}.$$

Proof. As above, $\operatorname{sign}_{\mathcal{P}}(S^k\varphi) = C_n^k(\operatorname{sign}_{\mathcal{P}}\varphi)/k! = C_n^k(0)/k!$. Setting $t = 0$ in the recurrence relation $(*)$ in Lemma 4.10, we get

$$\begin{aligned} C_n^k(0) &= (k-1)(n+k-2)C_n^{k-2}(0) \\ &= \cdots \\ &= (k-1)(n+k-2)(k-3)(n+k-4)\cdots 3(n+2)1(n) \\ &= \prod_{j=0}^{l-1}(2j+1)(n+2j), \end{aligned}$$

Now write $k! = (2l)! = \prod_{j=0}^{l-1}(2j+1)(2j+2)$.

Then

$$\begin{aligned} \operatorname{sign}(S^k\varphi) &= \prod_{j=0}^{l-1}\frac{(2j+1)(2m+2j)}{(2j+1)(2j+2)} \\ &= \prod_{j=0}^{l-1}\frac{m+j}{j+1} \\ &= \binom{m+l-1}{l} \end{aligned}$$

which completes the proof. □

Corollary 4.13. *Let φ be a hyperbolic form and let k be a positive integer. Then $S^k\varphi$ is hyperbolic if and only if k is odd.*

Corollary 4.14. *Let φ be a hyperbolic form of dimension $n = 2m$, so $\varphi = m \times \langle 1, -1\rangle$ and let $k = 2l$ be an even positive integer. Then*

$$S^k\varphi = \binom{m+l-1}{l} \times \langle 1\rangle \perp \frac{1}{2}\left(\binom{n+k-1}{k} - \binom{m+l-1}{l}\right) \times \langle 1, -1\rangle.$$

Remark 4.15. $\boldsymbol{S}^k\varphi$ differs from $S^k\varphi$ only in that its one-dimensional summands are multiplied by factorials $1!, 2!, \ldots \in K$. Each such factorial is a positive integer times the field identity and so is positive with respect to any ordering of an ordered field. Multiplication by such a factorial will not change the sign of any summand $\langle a_{i_1}^{k_{i_1}} \cdots a_{i_l}^{k_{i_l}}\rangle$ of $S^k\varphi$ and so all of the above results about the signature of $S^k\varphi$ apply to $\boldsymbol{S}^k\varphi$ as well.

Remark 4.16. Proposition 4.4 does not hold for factorial symmetric powers, as may be seen from the simple example of $\varphi = \langle a, b, c\rangle$ and $\boldsymbol{S}^3\varphi = \langle abc, 2a, 2b, 2c, 2a, 2b, 2c, 6a, 6b, 6c\rangle$. However, another form can be used instead of the $\binom{n+i-1}{i} \times \langle 1\rangle$ in Proposition 4.4.

Proposition 4.17. *Let φ be an n-dimensional symmetric bilinear form over K and let k be a positive integer. Then*

$$\boldsymbol{S}^k\varphi = \coprod_{i=0}^{\lceil k/2\rceil} \vartheta_i \cdot \bigwedge\nolimits^{k-2i}\varphi$$

where for each i from 0 to $\lceil k/2\rceil$, ϑ_i is the $\binom{n+i-1}{i}$-dimensional form given by

$$\vartheta_i = \coprod \langle (2i_1+1)!(2i_2+1)!\cdots(2i_{k-2i}+1)!(2i_{k-2i+1})!\cdots(2i_n)!\rangle,$$

the orthogonal sum being over all $i_1, \dots, i_n$ with $i_1 + \cdots + i_n = i$.

Proof. This proof proceeds in a similar way to the proof of Proposition 4.4, and is left as a straightforward exercise. Of course, $\vartheta_0 = \langle 1 \rangle$. $\square$

Example 4.18. Some examples of the forms ϑ_i are as follows.

(a) From the proof of Proposition 3.10, $\boldsymbol{S}^2\varphi = \bigwedge^2\varphi \perp n \times \langle 2 \rangle$. Here $k = 2$, and $i = 1$ corresponds to $n \times \langle 2 \rangle = \langle 2!, \dots, 2! \rangle \times \langle 1 \rangle$, which is

$$\Big(\mathop{\perp}_{j} \langle (2\cdot 0)! \cdots \underbrace{(2\cdot 1)}_{j^{\text{th}}} \cdots !(2\cdot 0)! \rangle\Big) \times \langle 1 \rangle = \Big(\mathop{\perp}_{i_1+\cdots+i_n=1} \langle (2i_1)! \cdots (2i_n)! \rangle\Big) \times \textstyle\bigwedge^0\varphi.$$

Thus

$$\boldsymbol{S}^2\varphi = \textstyle\bigwedge^2\varphi \perp \Big(\mathop{\perp}_{i_1+\cdots+i_n=1} \langle (2i_1)! \cdots (2i_n)! \rangle\Big) \times \textstyle\bigwedge^0\varphi$$

which is in the form of Proposition 4.17 above.

(b) For $k = 4$ and $i = 1$ we get

$$\vartheta_1 = \mathop{\perp}_{i_1+\cdots+i_n=1} \langle (2i_1+1)!(2i_2+1)!(2i_3)! \cdots (2i_n)! \rangle = 2 \times \langle 6 \rangle \perp (n-2) \times \langle 2 \rangle.$$

The following result is [11, Theorem 11.2].

Theorem 4.19. *Let (V, φ) be a quadratic space of dimension n and determinant d, and let k be a positive integer. Then the Hasse invariant of $\bigwedge^k\varphi$ is*

$$s(\textstyle\bigwedge^k\varphi) = s(\varphi)^g (d, -1)^e$$

where

$$g = \binom{n-2}{k-1}, \qquad e = \binom{\binom{n-1}{k-1}}{2}.$$

Remark 4.20. The Hasse invariants of $S^k\varphi$ and $\boldsymbol{S}^k\varphi$ are alike but not so much so that we can treat them simultaneously. Our approach is: we know the Hasse invariant of an exterior power, so we can write a symmetric power in terms of exterior powers and thence (for small k) get formulae for the Hasse invariant of a symmetric power.

Proposition 4.21. *Let φ be a form of dimension n with determinant d. Then*

$$s(S^2\varphi) = s(\textstyle\bigwedge^2\varphi) = s(\varphi)^n (d, -1)^{(n-1)(n-2)/2}$$

and so $s(S^2\varphi) = s(\boldsymbol{S}^2\varphi)$.

Proof. From Proposition 4.4, $S^2\varphi = \bigwedge^2\varphi \perp n \times \bigwedge^0\varphi = \bigwedge^2\varphi \perp n \times \langle 1 \rangle$. From the definition of the Hasse invariant, it is clear that adding copies of the identity form does not alter the Hasse invariant, and so $s(S^2\varphi) = s(\bigwedge^2\varphi)$. $\square$

Proposition 4.22. *Let φ be a form of dimension n with determinant d. Then*

$$s(S^3\varphi) = s(\varphi)^e (d, -1)^f$$

where $e = \binom{n-2}{2} + n = \frac{1}{2}(n^2 - 3n + 6)$ and $f = \binom{\binom{n-1}{2}}{2} + (n-1)\binom{n}{2}$.

Proof. From Proposition 4.4, $S^3\varphi = \bigwedge^3\varphi \perp n \times \bigwedge^1\varphi = \bigwedge^3\varphi \perp n \times \varphi$. One computes $s(\bigwedge^3\varphi)$ using [11, Theorem 11.2],

$$s(\bigwedge^3\varphi) = s(\varphi)^{\binom{n-2}{2}}(d,-1)^{\binom{\binom{n-1}{2}}{2}}.$$

It can easily be shown that $s\,(n \times \varphi) = s(\varphi)^n(d,-1)^{\binom{n}{2}}$. Also, $\det(n \times \varphi) = d^n$ and, by [11, Proposition 5.1], $\det(\bigwedge^3\varphi) = d^{\binom{n-1}{2}}$. The result follows from a computation, using a standard formula for the Hasse invariant of a product of two forms φ_1, φ_2 of dimensions n_1 and n_2 respectively, and with determinants d_1 and d_2 respectively:

$$s(\varphi_1 \otimes \varphi_2) = s(\varphi_1)^{n_1}s(\varphi_2)^{n_2}(d_1,d_2)^{n_1n_2-1}(d_1,d_1)^{\binom{n_2}{2}}(d_2,d_2)^{\binom{n_1}{2}}.$$

(See, for example, [7, Proposition 9]). □

Algebraic properties of non-factorial symmetric powers. Since H_k is a homogeneous polynomial, S^k is k-homogeneous with respect to scalar multiplication. The next result is true in any λ-ring with operations s^k arising from the H_k (see [6, page 47]).

Proposition 4.23. *Let φ and ψ be symmetric bilinear forms over K and let k be a positive integer. Then*

$$S^k(\varphi \perp \psi) = \mathop{\perp}_{i+j=k} S^i\varphi \cdot S^j\psi.$$

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